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Edition

Acing AP Calculus

AB and BC

Student
Classroom
Edition

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Practice Tests

Hundreds of
Examples and
Exercises

Definitions,
Theorems, and
Key Formulas

ePUB

Acing
AP Calculus
AB and BC

by Thomas Hyun

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About the Calculus AB and Calculus BC Exams

The AP exams in calculus test your understanding of basic concepts in calculus, as well as its methodology and applications. The material covered by the Calculus AB exam is roughly equivalent to a one-semester introductory college course in calculus. The Calculus BC exam is an extension of the AB material, adding on more advanced concepts such as improper integrals, series, logistic curves, and parametric and polar functions. It is important to note that both exams require a similar depth of understanding to the extent that they cover the same topics. Students who take the BC exam also receive a subscore that represents their knowledge of the AB material.

Using this Book

This book helps students review and master calculus concepts in the most concise and straightforward manner possible. Each concept is presented through a succinct definition followed by an example problem demonstrating its application. Every chapter includes exercise sets followed by step-by-step solutions to each problem. Sections dedicated to Calculus BC material are denoted by the $\boxed{BC}$ symbol. Note that Chapters 8 and 9 are exclusively for the BC exam.

Students taking the AP exam may choose to cover one chapter every day or two in the month before the real exam. Alternatively, students and teachers may find it helpful to work through each chapter after finishing the corresponding section in their calculus course at school. Even students who choose not to take the AP exam will find this textbook a useful supplement to their calculus courses. At the end of the book are four full-length practice tests, two each for the AB and BC exams. Practice tests are also accompanied by full-length solutions.

A Note on Graphing Calculators

The calculus AP exams consist of a multiple-choice and a free-response section, with each section including one part that requires use of a graphing calculator and one during which no electronic devices are permitted. While calculators cannot substitute for the necessary depth of understanding or provide any shortcuts where students are required to show their work, the testmakers who develop the AP calculus exam recognize that a graphing calculator is an integral part of the course. Therefore students should become comfortable with their graphing calculators through regular use. Check the AP website at apcentral.collegeboard.com for more details on restrictions on calculators.

Successful completion of an AP exam represents a high level of achievement. Students using this book as either a companion or as a review to their calculus courses should be confident of their ability to master the necessary calculus concepts.

| PART A |

Math Review

Chapter 1
Limits and Continuity

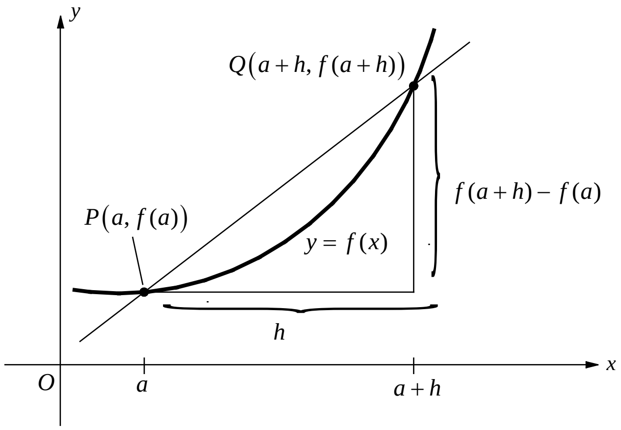
1.1 Rates of Change

Definition of Average Rate of Change

The expression

$$\boldsymbol{\frac{f(a+h)-f(a)}{h}}$$

is called the **difference quotient** for f at a and represents the **average rate of change** of $y=f(x)$ from a to $a+h$.



Geometrically, the rate of change of f from a to $a+h$ is the **slope** of the secant line through the points $P(a, f(a))$ and $Q(a+h, f(a+h))$.

If $f(t)$ is the position function of a particle that is moving on a straight line, then in the time interval from $t=a$ to $t=a+h$, the change in position is $f(a+h)-f(a)$, and the average velocity of the particle over the time interval is

$$\textbf{average velocity} = \frac{\textbf{displacement}}{\textbf{time}} = \boldsymbol{\frac{f(a+h)-f(a)}{h}}.$$

Example 1 □ The displacement of a particle moving in a straight line is given by the equation of motion $f(t)=t^3-4t+3$. Find the average velocity of the particle over the interval $0 \le t \le 4$.

Solution □ Average velocity $= \dfrac{f(4)-f(0)}{4-0} = \dfrac{(64-16+3)-3}{4-0}$

$$=12$$

Exercises - Rate of Change

Multiple Choice Questions

1. The traffic flow at a particular intersection is modeled by the function f defined by $f(t) = 25 + 6\cos(\frac{x}{3})$ for $0 \le t \le 120$. What is the average rate of change of the traffic flow over the time interval $30 \le t \le 40$.

(A) 0.743 (B) 0.851 (C) 0.935 (D) 1.176

---

2. The rate of change of the altitude of a hot air balloon rising from the ground is given by $y(t) = t^3 - 3t^2 + 3t$ for $0 \le t \le 10$. What is the average rate of change in altitude of the balloon over the time interval $0 \le t \le 10$.

(A) 56 (B) 73 (C) 85 (D) 94

Free Response Questions

t (sec)	0	10	20	30	40	50	60	70	80	90
$f(t)$ (ft/sec)	0	28	43	67	82	85	74	58	42	35

3. The table above shows the velocity of a car moving on a straight road. The car's velocity v is measured in feet per second.

(a) Find the average velocity of the car from $t = 60$ to $t = 90$.

(b) The instantaneous rate of change of f (See Ch. 2.1 for an explanation of instantaneous rate of change) with respect to x at $x = a$ can be approximated by finding the average rate of change of f near $x = a$. Approximate the instantaneous rate of change of f at $x = 40$ using two points, $x = 30$ and $x = 50$.

1.2 The Limit of a Function and One Sided Limits

Definition of Limit

The statement

$$\lim_{x\to c} f(x) = L$$

means f approaches the **limit** L as x approaches c.
Which is read "the limit of $f(x)$, as x approaches c, equals L."

Basic Limits

1. If f is the constant function $f(x)=k$, then for any value of c,

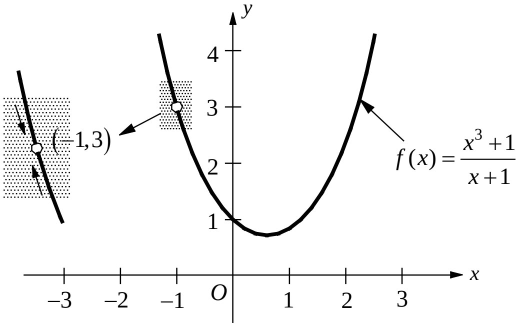
$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k.$$

2. If f is the polynomial function $f(x)=x^n$, then for any value of c,

$$\lim_{x\to c} f(x) = \lim_{x\to c} x^n = c^n.$$

Finding Limits Graphically

Consider the graph of the function $f(x)=\dfrac{x^3+1}{x+1}$. The given function is defined for all real numbers x except $x=-1$. The graph of f is a parabola with the point $(-1,3)$ removed as shown below. Even though $f(-1)$ is not defined, we can make the value of $f(x)$ as close to 3 as we want by choosing an x close enough to -1.



Although $f(x)$ is not defined when $x=-1$, the limit of $f(x)$ as x approaches -1 is 3, because the definition of a limit says that we consider values of x that are close to c, but not equal to c.

One Sided Limits

The **right-hand limit** means that x approaches c from values greater than c.
This limit is denoted as

$$\lim_{x\to c^+} f(x) = L.$$

The **left-hand limit** means that x approaches c from values less than c.
This limit is denoted as

$$\lim_{x\to c^-} f(x) = L.$$

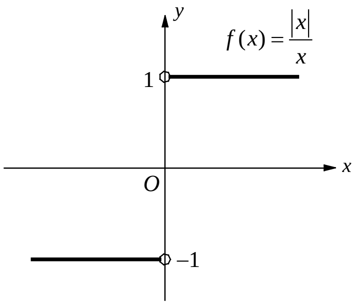
The Existence of a Limit

The limit of $f(x)$ as x approaches c is L if and only if

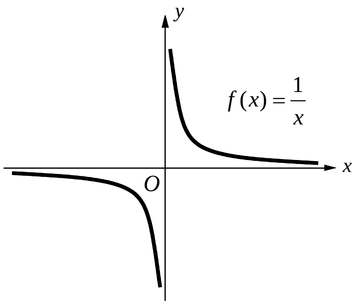
$$\lim_{x\to c^+} f(x) = L \quad \text{and} \quad \lim_{x\to c^-} f(x) = L.$$

Limits That Fail to Exists

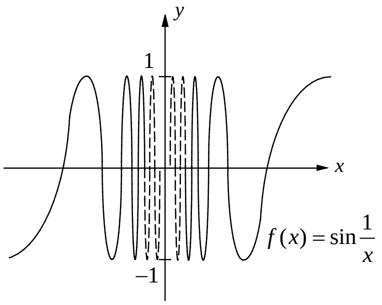
Some limits that fail to exists are illustrated below.



The left limit and the right limit is different. $\lim_{x\to 0}\frac{|x|}{x}=1$, if $x>0$ and $\lim_{x\to 0}\frac{|x|}{x}=-1$, if $x<0$.



As x approaches 0 from the right or the left, $f(x)$ increases or decreases without bound.



The values of $f(x)$ oscillate between -1 and 1 infinitely often as x approaches 0.

Example 1 □ Find the limit.

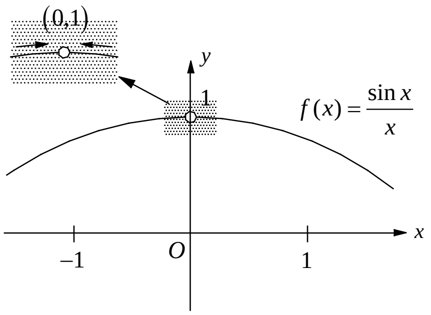
(a) $\lim_{x\to 2}(7)$ (b) $\lim_{x\to -1}(x^3-2x)$ (c) $\lim_{x\to 0}\frac{\sin x}{x}$

Solution □ (a) $\lim_{x\to 2}(7)=7$ $\lim_{x\to c}k=k$

(b) $\lim_{x\to -1}(x^3-2x)=(-1)^3-2(-1)=1$ $\lim_{x\to c}x^n=c^n$

(c) The function $f(x)$ is not defined when $x=0$. Find the Limit Graphically.

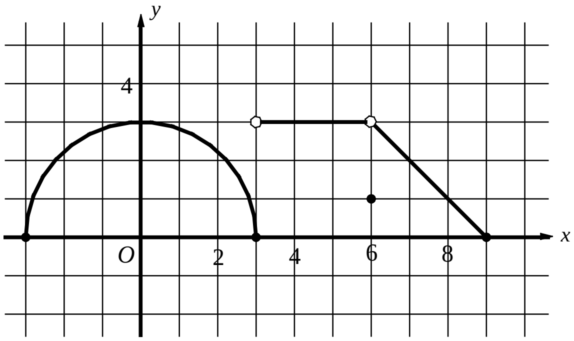
Graph $y=\frac{\sin x}{x}$ using a graphing calculator. The limit of $f(x)=\frac{\sin x}{x}$ as x approaches 0 is 1.



Example 2 □ The graph of the function f is shown in the figure below. Find the limit or value of the function at a given point.

(a) $\lim\limits_{x \to 3^-} f(x)$ (b) $\lim\limits_{x \to 3^+} f(x)$ (c) $\lim\limits_{x \to 3} f(x)$

(d) $\lim\limits_{x \to 6} f(x)$ (e) $f(3)$ (f) $f(6)$



Graph of $y = f(x)$

Solution □ (a) $\lim\limits_{x \to 3^-} f(x) = 0$

(b) $\lim\limits_{x \to 3^+} f(x) = 3$

(c) $\lim\limits_{x \to 3} f(x)$ does not exists since $\lim\limits_{x \to 3^-} f(x) \neq \lim\limits_{x \to 3^+} f(x)$.

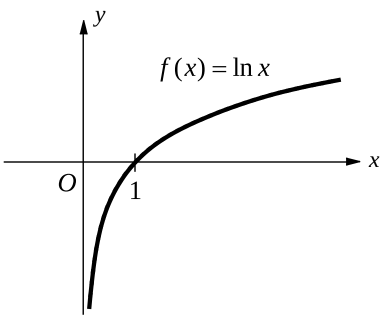
(d) $\lim\limits_{x \to 6} f(x) = 3$, because $\lim\limits_{x \to 6^-} f(x) = 3 = \lim\limits_{x \to 6^+} f(x)$.

(e) $f(3) = 0$

(f) $f(6) = 1$

Example 3 □ Find $\lim\limits_{x \to 0} \ln x$, if it exists.

Solution □ As x approaches to 0, $\ln x$ decreases without bound. Since the value of $f(x)$ does not approach a number, $\lim\limits_{x \to 0} \ln x$ does not exists.



Exercises - The Limit of a Function and One Sided Limits

Multiple Choice Questions

1. $\lim\limits_{x \to \frac{\pi}{6}} \sec^2 x =$

(A) $\frac{3}{4}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{4}{3}$ (D) $\frac{2\sqrt{3}}{3}$

---

2. If $f(x) = \begin{cases} x^2 + 3, & x \neq 1 \\ 1, & x = 1 \end{cases}$, then $\lim\limits_{x \to 1} f(x) =$

(A) 1 (B) 2 (C) 3 (D) 4

---

3. $\lim\limits_{x \to 1} \frac{|x-1|}{1-x} =$

(A) –2 (B) –1 (C) 1 (D) nonexistent

---

4. Let f be a function given by $f(x) = \begin{cases} 3 - x^2, & \text{if } x < 0 \\ 2 - x, & \text{if } 0 \leq x < 2 \\ \sqrt{x-2}, & \text{if } x > 2 \end{cases}$.

Which of the following statements are true about f ?

I. $\lim\limits_{x \to 0} f(x) = 2$

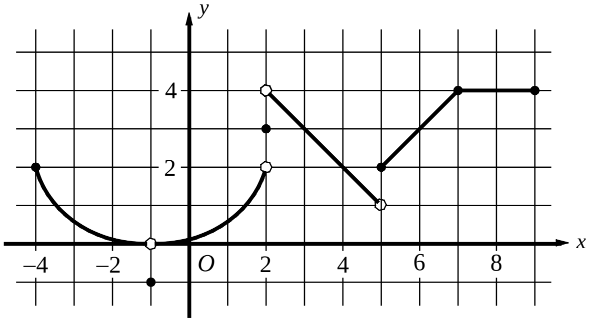
II. $\lim\limits_{x \to 2} f(x) = 0$

III. $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 6} f(x)$

(A) I only (B) II only (C) II and III only (D) I , II, and III

Free Response Questions

Questions 5-11 refer to the following graph.



The figure above shows the graph of $y = f(x)$ on the closed interval $[-4,9]$.

5. Find $\lim\limits_{x\to -1} \cos(f(x))$.

---

6. Find $\lim\limits_{x\to 2^-} f(x)$.

---

7. Find $\lim\limits_{x\to 2^+} f(x)$.

8. Find $\lim_{x\to 2} f(x)$.

---

9. Find $f(2)$.

---

10. Find $\lim_{x\to 5^-} \arctan(f(x))$.

---

11. Find $\lim_{x\to 5^+} \left[x\, f(x)\right]$.

1.3 Calculating Limits Using the Limit Laws

Limit Laws

Let c and k be real numbers and the limits $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist.

Then

1. $\mathbf{\lim\limits_{x\to c}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to c} f(x)\pm\lim\limits_{x\to c} g(x)}$

2. $\mathbf{\lim\limits_{x\to c}\left[f(x)\cdot g(x)\right]=\lim\limits_{x\to c} f(x)\cdot\lim\limits_{x\to c} g(x)}$

3. $\mathbf{\lim\limits_{x\to c}\left[k\,f(x)\right]=k\lim\limits_{x\to c} f(x)}$

4. $\mathbf{\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}}$

5. $\mathbf{\lim\limits_{x\to c}\left[f(x)\right]^{n}=\left[\lim\limits_{x\to c} f(x)\right]^{n}}$

6. $\mathbf{\lim\limits_{x\to c}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to c} f(x)}}$

Special Trigonometric Limits

1. $\mathbf{\lim\limits_{x\to 0}\dfrac{\sin x}{x}}=1$

2. $\mathbf{\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}}=0$

Example 1 □ Find the limits.

(a) $\lim\limits_{x\to 2}\dfrac{x^2+3x-10}{x-2}$ (b) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+3}-2}{x-1}$

Solution □ (a) $\lim\limits_{x\to 2}\dfrac{x^2+3x-10}{x-2}$

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}\,(x+5)}{\cancel{x-2}}$$ Factor and cancel out common factors.

$$=\lim_{x\to 2}(x+5)=7$$

(b) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+3}-2}{x-1}$

$$=\lim_{x\to 1}\frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)}$$ Rationalize the numerator.

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}\,(\sqrt{x+3}+2)}$$ Simplify and cancel out common factors.

$$=\lim_{x\to 1}\frac{1}{(\sqrt{x+3}+2)}$$

$$=\frac{1}{4}$$

Example 2 □ Find the limits.

(a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{3x}$ (b) $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$ (c) $\lim\limits_{x\to\pi/4}\dfrac{\tan x-1}{\sin x-\cos x}$

Solution □ (a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{3x}$

$$=\lim_{x\to 0}\left(\frac{\sin 4x(\frac{4}{3})}{3x(\frac{4}{3})}\right)$$

Multiply the numerator and denominator by $\frac{4}{3}$.

$$=\lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right)(\frac{4}{3})$$

Simplify.

$$=(1)\left(\frac{4}{3}\right)$$

$\lim\limits_{x\to 0}\dfrac{\sin 4x}{4x}=1$

$$=\frac{4}{3}$$

(b) $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=\lim\limits_{x\to 0}\dfrac{\sin x}{\cos x}\cdot\dfrac{1}{x}$

$\tan x=\dfrac{\sin x}{\cos x}$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}$$

$\lim\limits_{x\to c}\left[f(x)\cdot g(x)\right]=\lim\limits_{x\to c}f(x)\cdot\lim\limits_{x\to c}g(x)$

$$=1\cdot 1$$

$\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ and $\lim\limits_{x\to 0}\dfrac{1}{\cos x}=1$

$$=1$$

(c) $\lim\limits_{x\to\pi/4}\dfrac{\tan x-1}{\sin x-\cos x}$

$$=\lim_{x\to\pi/4}\frac{(\sin x/\cos x)-1}{\sin x-\cos x}$$

$\tan x=\dfrac{\sin x}{\cos x}$

$$=\lim_{x\to\pi/4}\frac{\left[(\sin x/\cos x)-1\right]\cos x}{\left[\sin x-\cos x\right]\cos x}$$

$$=\lim_{x\to\pi/4}\frac{\cancel{\left[\sin x-\cos x\right]}}{\cancel{\left[\sin x-\cos x\right]}\ \cos x}$$

Simplify and cancel out common factors.

$$=\lim_{x\to\pi/4}\frac{1}{\cos x}$$

$$=\frac{1}{\cos(\pi/4)}=\sqrt{2}$$

Exercises - Calculating Limits Using the Limit Laws

Multiple Choice Questions

1. $\lim\limits_{x\to\pi/3} \dfrac{\sin(\frac{\pi}{3}-x)}{\frac{\pi}{3}-x} =$

(A) -1 (B) 0 (C) $\dfrac{\sqrt{3}}{2}$ (D) 1

---

2. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 2x} =$

(A) $\dfrac{2}{3}$ (B) 1 (C) $\dfrac{3}{2}$ (D) nonexistent

---

3. $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x} =$

(A) $\dfrac{1}{8}$ (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{2}$ (D) nonexistent

---

4. $\lim\limits_{x\to 1} \dfrac{\sqrt{3+x}-2}{x^3-1} =$

(A) $\dfrac{1}{12}$ (B) $\dfrac{1}{6}$ (C) $\sqrt{3}$ (D) nonexistent

5. $\lim\limits_{\theta\to 0}\dfrac{\theta+\theta\cos\theta}{\sin\theta\cos\theta}=$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2

---

6. $\lim\limits_{x\to 0}\dfrac{\tan 3x}{x}=$

(A) 0 (B) $\frac{1}{3}$ (C) 1 (D) 3

---

7. $\lim\limits_{x\to 3}\dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}=$

(A) $-\frac{1}{9}$ (B) $\frac{1}{9}$ (C) -9 (D) 9

Free Response Questions

8. If $\lim\limits_{x\to 0}\dfrac{\sqrt{2+ax}-\sqrt{2}}{x}=\sqrt{2}$ what is the value of a?

9. Find $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, if $f(x)=\sqrt{2x+1}$.

---

10. Find $\lim_{x\to 0}\frac{f(x)-g(x)}{\sqrt{g(x)+7}}$, if $\lim_{x\to 0} f(x)=2$ and $\lim_{x\to 0} g(x)=-3$.

---

11. Find $\lim_{x\to\sqrt{3}} g(x)$, if $\lim_{x\to\sqrt{3}}\frac{1}{x^2+g(x)}=\frac{1}{5}$.

1.4 Properties of Continuity and Intermediate Value Theorem

Definition of Continuity

A function f is **continuous at c** if the following three conditions are met.

1. $f(c)$ is defined
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

Intermediate Value Theorem

If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c)=k$.

Specifically, if f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a,b]$.

Example 1 □ For what values of a is $f(x)=\begin{cases} x^2, & x\le 1 \\ ax+2, & 1<x\le 3 \end{cases}$ continuous at $x=1$?

Solution □ For f to be continuous at $x=1$, $\lim_{x\to 1} f(x)$ must equal $f(1)$.

1. $f(1)=(1)^2=1$
2. $\lim_{x\to 1} f(x)$ exists if $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)$.

   $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x^2=1$

   $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(ax+2)=a+2$

   $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x) \Rightarrow 1=a+2 \Rightarrow a=-1$
3. $\lim_{x\to 1} f(x)=1=f(1)$

   Therefore $f(x)$ is continuous at $x=1$ if $a=-1$.

Example 2 □ Let f be a function given by $f(x)=x^3-4x+2$. Use the Intermediate Value Theorem to show that there is a root of the equation on $[0,1]$.

Solution □ $f(x)$ is continuous on $[0,1]$ and $f(0)=2>0$ and $f(1)=-1<0$.
By the Intermediate Value Theorem, $f(x)=0$ for at least one value c between 0 and 1.
$f(1)=-1<f(c)=0<f(0)=2$

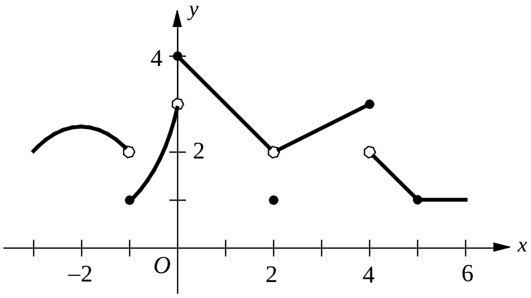
Using a graphing calculator, we find that $c\approx 0.539$, which is between 0 and 1.

Exercises - Properties of Continuity and Intermediate Value Theorem

Multiple Choice Questions

1. Let f be a function defined by $f(x)=\begin{cases}\dfrac{x^2-a^2}{x-a}, & \text{if } x\neq a\\ 4, & \text{if } x=a\end{cases}$. If f is continuous for all real numbers x, what is the value of a?

(A) $\dfrac{1}{2}$ (B) 0 (C) 1 (D) 2



2. The graph of a function f is shown above. If $\lim\limits_{x\to a} f(x)$ exists and f is not continuous at $x=a$, then $a=$

(A) −1 (B) 0 (C) 2 (D) 4

---

3. If $f(x)=\begin{cases}\dfrac{\sqrt{3x-1}-\sqrt{2x}}{x-1}, & \text{for } x\neq 1\\ a, & \text{for } x=1\end{cases}$, and if f is continuous at $x=1$, then $a=$

(A) $\dfrac{1}{4}$ (B) $\dfrac{\sqrt{2}}{4}$ (C) $\sqrt{2}$ (D) 2

4. Let f be a continuous function on the closed interval $[-2,7]$. If $f(-2)=5$ and $f(7)=-3$, then the Intermediate Value Theorem guarantees that

(A) $f'(c)=0$ for at least one c between -2 and 7

(B) $f'(c)=0$ for at least one c between -3 and 5

(C) $f(c)=0$ for at least one c between -3 and 5

(D) $f(c)=0$ for at least one c between -2 and 7

Free Response Questions

5. Let g be a function defined by $g(x)=\begin{cases} \dfrac{\pi \sin x}{x}, & \text{if } x<0 \\ a-bx, & \text{if } 0\le x<1. \\ \arctan x, & \text{if } x\ge 1 \end{cases}$

If g is continuous for all real numbers x, what are the values of a and b?

6. Evaluate $\lim_{a \to 0} \frac{-1+\sqrt{1+a}}{a}$.

---

7. What is the value of a , if $\lim_{x \to 0} \frac{\sqrt{ax+9}-3}{x} = 1$?

1.5 Limits and Asymptotes

The statement $\lim_{x \to c} f(x) = \infty$ means that $f(x)$ **approaches infinity as** x **approaches** c.

The statement $\lim_{x \to c} f(x) = -\infty$ means that $f(x)$ **approaches negative infinity as** x **approaches** c.

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a vertical asymptote of the graph of f.

The graph of rational function given by $y = \dfrac{f(x)}{g(x)}$ has a **vertical asymptote** at $x = c$ if $g(c) = 0$ and $f(c) \neq 0$.

Definition of Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

The statement $\lim_{x \to \infty} f(x) = L$ means that $f(x)$ has the **limit** L **as** x **approaches infinity**.

The statement $\lim_{x \to -\infty} f(x) = L$ means that $f(x)$ has the **limit** L **as** x **approaches negative infinity**.

Example 1 □ Find all vertical asymptotes of the graph of each function.

(a) $f(x) = \dfrac{x}{x^2 - 1}$ (b) $f(x) = \dfrac{x^2 - 4x - 5}{x^2 - x - 2}$

Solution □ (a) $f(x) = \dfrac{x}{x^2 - 1}$

$= \dfrac{x}{(x+1)(x-1)}$ Factor the denominator.

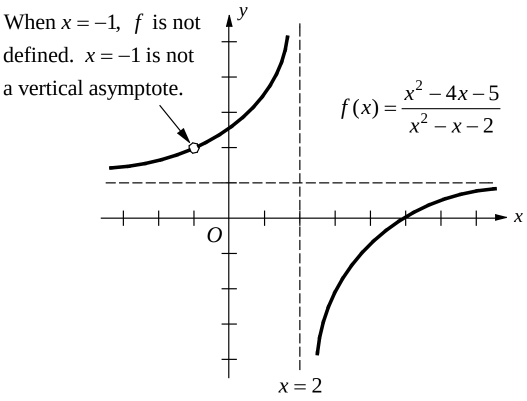
The denominator is 0 at $x = -1$ and $x = 1$. The numerator is not 0 at these two points. There are two vertical asymptotes, $x = -1$ and $x = 1$.

(b) $f(x) = \dfrac{x^2 - 4x - 5}{x^2 - x - 2}$

$= \dfrac{\cancel{(x+1)}(x-5)}{\cancel{(x+1)}(x-2)}$ Factor and cancel out common factors.

$= \dfrac{x-5}{x-2}$

At all values other than $x = -1$, the graph of f coincides with the graph of $y = (x-5)/(x-2)$.
So, $x = 2$ is the only vertical asymptote. At $x = -1$ the graph is not continuous.


When $x = -1$, f is not defined. $x = -1$ is not a vertical asymptote.

Example 2 □ Find the limit.

(a) $\lim\limits_{x\to\infty} \frac{x^3 - 4x^2 + 7}{2x^3 - 3x - 5}$ (b) $\lim\limits_{x\to\infty} \frac{\sqrt{4x^2 + 6x}}{3x - 2}$

Solution □ (a) $\lim\limits_{x\to\infty} \frac{(x^3 - 4x^2 + 7)/x^3}{(2x^3 - 3x - 5)/x^3}$ Divide the numerator and denominator by x^3.

$$= \lim_{x\to\infty} \frac{1 - 4/x + 7/x^3}{2 - 3/x^2 - 5/x^3} = \frac{1 - 0 + 0}{2 - 0 - 0} = \frac{1}{2}$$

(b) $\lim\limits_{x\to\infty} \frac{\left(\sqrt{4x^2 + 6x}\right)/x}{(3x-2)/x}$ Divide the numerator and denominator by x.

$$= \lim_{x\to\infty} \frac{\sqrt{4x^2/x^2 + 6x/x^2}}{3x/x - 2/x}$$ For $x > 0$, $\sqrt{x^2} = x$.

$$= \frac{\sqrt{4 + 6/x}}{3 - 2/x} = \frac{\sqrt{4+0}}{3-0} = \frac{2}{3}$$

Example 3 □ Find the horizontal asymptotes of the graph of the function $f(x) = \frac{\sqrt[3]{2x^3 - 9}}{x}$.

Solution □ $\lim\limits_{x\to\infty} \frac{\sqrt[3]{2x^3 - 9}}{x}$

$$= \lim_{x\to\infty} \frac{(\sqrt[3]{2x^3 - 9})/x}{x/x}$$ Divide the numerator and denominator by x.

$$= \lim_{x\to\infty} \frac{\sqrt[3]{2x^3/x^3 - 9/x^3}}{1}$$

$$= \frac{\sqrt[3]{2-0}}{1} = \sqrt[3]{2}$$ The line $y = \sqrt[3]{2}$ is a horizontal asymptote.

Exercises - Limits and Asymptotes

Multiple Choice Questions

1. $\lim_{x\to\infty} \dfrac{3+2x^2-x^4}{3x^4-5} =$

(A) -2 (B) $-\dfrac{1}{3}$ (C) $\dfrac{1}{5}$ (D) 1

---

2. What is $\lim_{x\to -\infty} \dfrac{x^3+x-8}{2x^3+3x-1} =$

(A) $-\dfrac{1}{2}$ (B) 0 (C) $\dfrac{1}{2}$ (D) 2

---

3. Which of the following lines is an asymptote of the graph of $f(x)=\dfrac{x^2+5x+6}{x^2-x-12}$?

I. $x=-3$
II. $x=4$
III. $y=1$

(A) II only (B) III only (C) II and III only (D) I, II, and III

4. If the horizontal line $y=1$ is an asymptote for the graph of the function f, which of the following statements must be true?

(A) $\lim_{x\to\infty} f(x)=1$

(B) $\lim_{x\to 1} f(x)=\infty$

(C) $f(1)$ is undefined

(D) $f(x)=1$ for all x

---

5. If $x=1$ is the vertical asymptote and $y=-3$ is the horizontal asymptote for the graph of the function f, which of the following could be the equation of the curve?

(A) $f(x)=\dfrac{-3x^2}{x-1}$

(B) $f(x)=\dfrac{-3(x-1)}{x+3}$

(C) $f(x)=\dfrac{-3(x^2-1)}{x-1}$

(D) $f(x)=\dfrac{-3(x^2-1)}{(x-1)^2}$

---

6. What are all horizontal asymptotes of the graph of $y=\dfrac{6+3e^x}{3-3e^x}$ in the xy- plane?

(A) $y=-1$ only

(B) $y=2$ only

(C) $y=-1$ and $y=2$

(D) $y=0$ and $y=2$

Free Response Questions

7. Let $f(x) = \dfrac{3x-1}{x^3-8}$.

(a) Find the vertical asymptote(s) of f. Show the work that leads to your answer.

(b) Find the horizontal asymptote(s) of f. Show the work that leads to your answer.

---

8. Let $f(x) = \dfrac{\sin x}{x^2+2x}$.

(a) Find the vertical asymptote(s) of f. Show the work that leads to your answer.

(b) Find the horizontal asymptote(s) of f. Show the work that leads to your answer.

Chapter 2
Differentiation

2.1 Definition of Derivatives and the Power Rule

Definition of Derivatives

The **derivative** of a function f at x, denoted by $f'(x)$ is

$$\boldsymbol{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

if the limit exists.

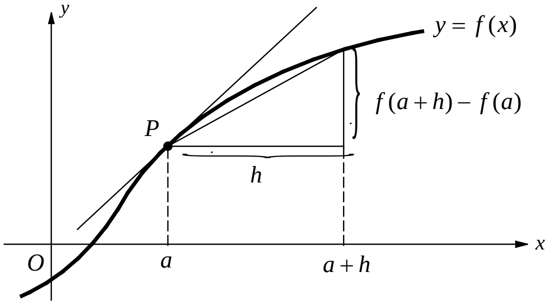
If we replace $x = a$ in the above equation, the derivative of a function f at a number a, denoted by $f'(a)$, is

$$\boldsymbol{f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}.$$

If we write $x = a + h$, then $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore, an equivalent way of stating the definition of derivative is

$$\boldsymbol{f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}}.$$

The process of finding the derivative of a function is called **differentiation**. In addition to $f'(x)$, which is read as "f prime of x," other notations such as $\frac{dy}{dx}$, y', and $\frac{d}{dx}\left[f(x)\right]$ are used to denote the derivative of $y = f(x)$.



Geometrically, the limit is the slope of the curve at *P*.

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$= \text{slope of curve at } P$$
$$= \text{slope of tangent at } P$$

Differentiability Implies Continuity

A function f is **differentiable** at a only if $f'(a)$ exists. If a function is differentiable at $x = a$, then f is continuous at $x = a$. Continuity, however, does not imply differentiability.

The derivative $f'(a)$ is the **instantaneous rate of change** of $f(x)$ with respect to x when $x = a$.

One-Sided Derivatives

The **left-hand derivative** of f at a is defined by

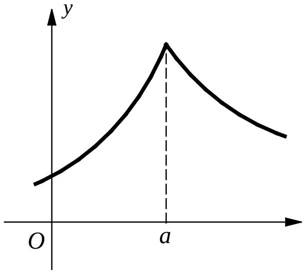
$$f'(a^-) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}, \text{ if the limit exist.}$$

The **right-hand derivative** of f at a is defined by

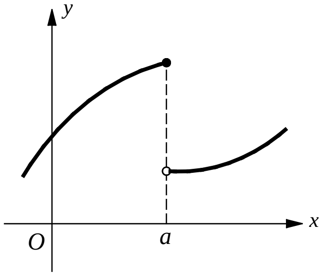
$$f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}, \text{ if the limit exist.}$$

$\boldsymbol{f'(a)}$ **exists** if and only if these one sided derivatives exist and are equal.

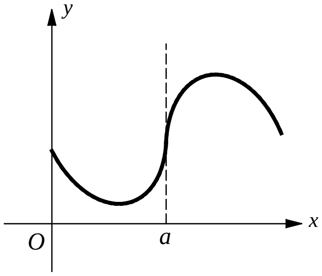
There are three possible ways for f not to be differentiable at $x = a$.



The graph has a corner.



The graph is not continuous.



The graph has a vertical tangent.

Basic Differentiation Rules

The Constant Rule

The derivative of a constant function is 0. $\quad \boldsymbol{\frac{d}{dx}}[\boldsymbol{c}] = 0$

The Power Rule

If n is any real number, then $\quad \boldsymbol{\frac{d}{dx}\left[x^n\right] = nx^{n-1}}$.

Example 1 □ What is $\lim_{h\to 0} \frac{(3+h)^4 - 81}{h}$?

Solution □ $\lim_{h\to 0} \frac{(3+h)^4 - 81}{h} = \lim_{h\to 0} \frac{(3+h)^4 - 3^4}{h} = f'(3)$, where $f(x) = x^4$ and $a = 3$.

$f'(x) = 4x^3$ Power rule

$f'(3) = 4(3)^3 = 108$

Example 2 □ Find the derivative of $f(x) = x^3 - 2x + \frac{1}{x} + 5$ at $x = 2$.

Solution □

$$f(x) = x^3 - 2x + x^{-1} + 5 \qquad 1/x = x^{-1}$$

$$f'(x) = 3x^{3-1} - 2x^{1-1} + (-1)x^{-1-1} + 0 \qquad \text{The power rule and the constant rule}$$

$$= 3x^2 - 2x^0 - x^{-2} = 3x^2 - 2 - \frac{1}{x^2}$$

$$f'(2) = 3(2)^2 - 2 - \frac{1}{(2)^2} = \frac{39}{4}$$

Example 3 □ Find a function f and a number a such that

$$f'(a) = \lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2}}{h}.$$

Solution □

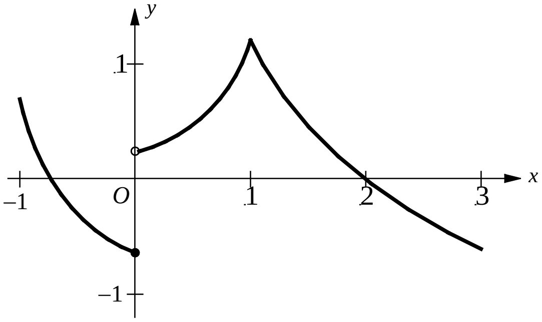
$$f'(a) = \lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

From the above equation we can conclude that

$f(a+h) = \sin\left(\frac{\pi}{6} + h\right)$ and $f(a) = \frac{1}{2} = \sin(\frac{\pi}{6})$.

So $a = \frac{\pi}{6}$ and $f(x) = \sin x$.

Example 4 □ The graph of f is shown in the figure below. For what values of x, $-1 < x < 3$, is f not differentiable?



Solution □ f is not differentiable at $x = 0$ since the graph is discontinuous at $x = 0$.

f is not differentiable at $x = 1$ since the graph has a corner, where the one-sided derivatives differ. (At $x = 1$, $f'(1^-) > 0$ and $f'(1^+) < 0$.)

Exercises - Definition of Derivatives and the Power Rule

Multiple Choice Questions

1. $\lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h}=$

(A) $\frac{1}{12}$ (B) $\frac{1}{4}$ (C) $\frac{\sqrt[3]{2}}{2}$ (D) $\sqrt[3]{2}$ (E) 2

---

2. $\lim_{h\to 0}\frac{(2+h)^5-32}{h}$ is

(A) $f'(5)$, where $f(x)=x^2$

(B) $f'(2)$, where $f(x)=x^5$

(C) $f'(5)$, where $f(x)=2^x$

(D) $f'(2)$, where $f(x)=2^x$

$$f(x)=\begin{cases}1-2x, & \text{if } x\le 1\\ -x^2, & \text{if } x>1\end{cases}$$

3. Let f be the function given above. Which of the following must be true?

I. $\lim_{x\to 1} f(x)$ exists.

II. f is continuous at $x=1$.

III. f is differentiable at $x=1$.

(A) I only

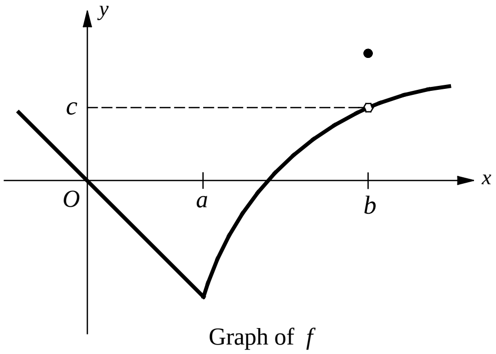
(B) I and II only

(C) II and III only

(D) I, II, and III

4. What is the instantaneous rate of change at $x = -1$ of the function $f(x) = -\sqrt[3]{x^2}$?

(A) $-\frac{2}{3}$ (B) $-\frac{1}{3}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$



Graph of f

5. The graph of a function f is shown in the figure above. Which of the following statements must be false?

(A) $f(x)$ is defined for $0 \le x \le b$.

(B) $f(b)$ exists.

(C) $f'(b)$ exists.

(D) $\lim_{x \to a^-} f'(x)$ exists.

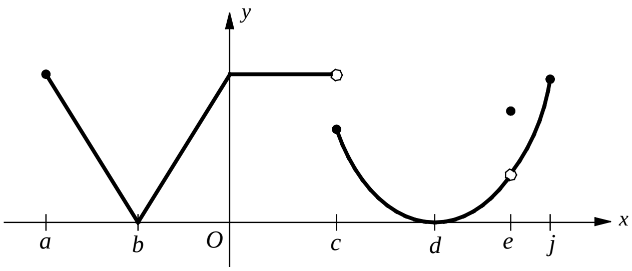
6. If f is a differentiable function, then $f'(1)$ is given by which of the following?

I. $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

II. $\lim_{x \to 1} \frac{f(x) - f(1)}{x-1}$

III. $\lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$

(A) I only (B) II only (C) I and II only (D) I and III only



Graph of f

7. The graph of a function f is shown in the figure above. At how many points in the interval $a < x < j$ is f' not defined?

(A) 3 (B) 4 (C) 5 (D) 6

Free Response Questions

8. Let f be the function defined by $f(x) = \begin{cases} mx^2 - 2 & \text{if } x \le 1 \\ k\sqrt{x} & \text{if } x > 1 \end{cases}$. If f is differentiable at $x = 1$, what are the values of k and m?

9. Let f be a function that is differentiable throughout its domain and that has the following properties.

(1) $f(x+y) = f(x) + x^3y - xy^3 - f(y)$

(2) $\lim_{x \to 0} \frac{f(x)}{x} = 1$

Use the definition of the derivative to show that $f'(x) = x^3 - 1$.

---

10. Let f be the function defined by

$$f(x) = \begin{cases} x+2 & \text{for } x \le 0 \\ \frac{1}{2}(x+2)^2 & \text{for } x > 0. \end{cases}$$

(a) Find the left-hand derivative of f at $x = 0$.

(b) Find the right-hand derivative of f at $x = 0$.

(c) Is the function f differentiable at $x = 0$? Explain why or why not.

(d) Suppose the function g is defined by

$$g(x) = \begin{cases} x+2 & \text{for } x \le 0 \\ a(x+b)^2 & \text{for } x > 0, \end{cases}$$

where a and b are constants. If g is differentiable at $x = 0$, what are the values of a and b?

2.2 The Product and Quotient Rules, and Higher Derivatives

The Product Rule

If f and g are both differentiable, then $\boldsymbol{\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)}$.

The Quotient Rule

If f and g are both differentiable, then $\boldsymbol{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}}$.

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own. The **second derivative** f'' is the derivative of f' and the **third derivative** f''' is the derivative of the second derivative.

In general, the *n*th derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. Higher derivatives are denoted as follows.

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$
$\vdots$			
***n*th derivative**:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$

Example 1 □ Differentiate the function $f(x) = (x^3 - 7)(x^2 - 4x)$.

Solution □ $f'(x) = (x^3 - 7)\frac{d}{dx}(x^2 - 4x) + (x^2 - 4x)\frac{d}{dx}(x^3 - 7)$ The product rule

$= (x^3 - 7)(2x - 4) + (x^2 - 4x)(3x^2)$

$= 5x^4 - 16x^3 - 14x + 28$

Example 2 □ Differentiate the function $f(x) = \dfrac{3x^2 - x}{\sqrt{x+1}}$.

Solution □ $f'(x) = \dfrac{\sqrt{x+1} \cdot \dfrac{d}{dx}(3x^2 - x) - (3x^2 - x) \cdot \dfrac{d}{dx}(\sqrt{x+1})}{(\sqrt{x+1})^2}$ The quotient rule

$$= \frac{(\sqrt{x+1})(6x-1) - (3x^2 - x) \cdot \dfrac{1}{2\sqrt{x+1}}}{(x+1)}$$

$$= \frac{\left[(\sqrt{x+1})(6x-1) - (3x^2 - x) \cdot \dfrac{1}{2\sqrt{x+1}}\right] \cdot 2\sqrt{x+1}}{(x+1) \cdot 2\sqrt{x+1}}$$

Multiply the numerator and denominator by $2\sqrt{x+1}$.

$$= \frac{9x^2 + 11x - 2}{2(x+1)^{3/2}}$$

Example 3 □ If $f(x) = \dfrac{1}{6}x^3 + 24\sqrt{x}$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f'''(9)$.

Solution □ $f'(x) = \dfrac{1}{6} \cdot 3x^2 + 24 \cdot \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2}x^2 + 12x^{-1/2}$

$$f''(x) = \frac{1}{2} \cdot 2x + 12(-\frac{1}{2}x^{-3/2}) = x - 6x^{-3/2}$$

$$f'''(x) = 1 - 6(-\frac{3}{2})x^{-5/2} = 1 + \frac{9}{x^2\sqrt{x}}$$

$$f'''(9) = 1 + \frac{9}{9^2\sqrt{9}} = 1 + \frac{1}{27} = \frac{28}{27}$$

Exercises - The Product and Quotient Rules and Higher Derivatives

Multiple Choice Questions

1. If $f(x)=(x^3-2x+5)(x^{-2}+x^{-1})$, then $f'(1)=$

(A) -10 (B) -6 (C) $-\dfrac{9}{2}$ (D) $\dfrac{7}{2}$

---

2. If $f(x)=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$ then $f'(x)=$

(A) $\dfrac{\sqrt{x}}{(\sqrt{x}+1)^2}$

(B) $\dfrac{x}{(\sqrt{x}+1)^2}$

(C) $\dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

(D) $\dfrac{\sqrt{x}-1}{\sqrt{x}(\sqrt{x}+1)^2}$

---

3. If $g(2)=3$ and $g'(2)=-1$, what is the value of $\dfrac{d}{dx}\left(\dfrac{g(x)}{x^2}\right)$ at $x=2$?

(A) -3 (B) -1 (C) 0 (D) 2

4. If $f(x)=\dfrac{x}{x-\dfrac{a}{x}}$ and $f'(1)=\dfrac{1}{2}$, what is the value of a?

(A) $-\dfrac{5}{2}$ (B) -1 (C) $\dfrac{1}{2}$ (D) 2

---

5. If $y=4\sqrt{x}-16\sqrt[4]{x}$, then $y''=$

(A) $\sqrt[4]{x}-3$ (B) $-3\sqrt{x}+3$ (C) $\dfrac{-\sqrt[4]{x}+3}{x\sqrt[4]{x^3}}$ (D) $\dfrac{\sqrt{x}-3}{x\sqrt[4]{x}}$

---

6. If $y=x^2\cdot f(x)$, then $y''=$

(A) $x^2 f''(x)+x f'(x)+2f(x)$

(B) $x^2 f''(x)+x f'(x)+f(x)$

(C) $x^2 f''(x)+2x f'(x)+f(x)$

(D) $x^2 f''(x)+4x f'(x)+2f(x)$

---

7. Let $f(x)=\dfrac{1}{2}x^6-10x^3+12x$. What is the value of $f(x)$, when $f'''(x)=0$?

(A) $-\dfrac{23}{4}$ (B) $-\dfrac{3}{2}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{5}{2}$

Free Response Questions

8. Let $h(x) = x \cdot f(x) \cdot g(x)$. Find $h'(1)$, if $f(1) = -2$, $g(1) = 3$, $f'(1) = 1$, and $g'(1) = \frac{1}{2}$.

9. Let $g(x) = \frac{x}{\sqrt{x} - 1}$. Find $g''(4)$.

2.3 The Chain Rule and the Composite Functions

The Chain Rule

If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then $y = f\left(g(x)\right)$ is differentiable and

$$\boldsymbol{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

or, equivalently,

$$\boldsymbol{\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'(x)}\,.$$

The Power Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\boldsymbol{\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}}$$

or, equivalently,

$$\boldsymbol{\frac{d}{dx}\left[g(x)\right]^n = n\left[g(x)\right]^{n-1} \cdot g'(x)}\,.$$

Example 1 □ Find $\dfrac{dy}{dx}$ for $y = \sqrt{x^4 - 2x + 5}$.

Solution □
$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(x^4 - 2x + 5)^{-1/2}\frac{d}{dx}(x^4 - 2x + 5) && \text{Power Chain Rule}\\ &= \frac{1}{2\sqrt{x^4 - 2x + 5}} \cdot (4x^3 - 2)\\ &= \frac{2x^3 - 1}{\sqrt{x^4 - 2x + 5}}\end{aligned}$$

Example 2 □ Find $h''(x)$ if $h(x) = f(x^3)$.

Solution □
$$\begin{aligned}h'(x) &= f'(x^3)\frac{d}{dx}(x^3) && \text{Chain Rule}\\ &= f'(x^3)\cdot(3x^2)\\ h''(x) &= f'(x^3)\frac{d}{dx}(3x^2) + (3x^2)\frac{d}{dx}f'(x^3) && \text{Product Rule}\\ &= f'(x^3)(6x) + (3x^2)f''(x^3)\frac{d}{dx}(x^3) && \text{Chain Rule}\\ &= f'(x^3)(6x) + (3x^2)f''(x^3)(3x^2)\\ &= 6x\,f'(x^3) + 9x^4 f''(x^3)\end{aligned}$$

Exercises - The Chain Rule and the Composite Functions

Multiple Choice Questions

1. If $f(x)=\sqrt{x+\sqrt{x}}$, then $f'(x)=$

(A) $\dfrac{1}{2\sqrt{x+\sqrt{x}}}$ (B) $\dfrac{\sqrt{x}+1}{2\sqrt{x+\sqrt{x}}}$ (C) $\dfrac{2\sqrt{x}}{4\sqrt{x+\sqrt{x}}}$ (D) $\dfrac{2\sqrt{x}+1}{4\sqrt{x^2+x\sqrt{x}}}$

---

2. If $f(x)=(x^2-3x)^{3/2}$, then $f'(4)=$

(A) $\dfrac{15}{2}$ (B) 9 (C) $\dfrac{21}{2}$ (D) 15

---

3. If f , g , and h are functions that is everywhere differentiable, then the derivative of $\dfrac{f}{g\cdot h}$ is

(A) $\dfrac{g\ h f'-f\ g'\ h'}{g\ h}$

(B) $\dfrac{g\ h f'-f\ g\ h'-f\ h g'}{g\ h}$

(C) $\dfrac{g\ h f'-f\ g\ h'-f\ g'h}{g^2h^2}$

(D) $\dfrac{g\ h f'-f\ g\ h'+f\ h g'}{g^2h^2}$

4. If $f(x) = (3 - \sqrt{x})^{-1}$, then $f''(4) =$

(A) $\dfrac{3}{32}$ (B) $\dfrac{3}{16}$ (C) $\dfrac{3}{4}$ (D) $\dfrac{9}{4}$

Free Response Questions

Questions 5-9 refer to the following table.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	1	–1
2	–2	1	–1	3
3	1	4	2	3
4	5	2	1	–2

The table above gives values of f, f', g, and g' at selected values of x.

5. Find $h'(1)$, if $h(x) = f(g(x))$.

---

6. Find $h'(2)$, if $h(x) = x\,f(x^2)$.

---

7. Find $h'(3)$, if $h(x) = \dfrac{f(x)}{\sqrt{g(x)}}$.

8. Find $h'(2)$, if $h(x)=\left[f(2x)\right]^2$.

---

9. Find $h'(1)$, if $h(x)=(x^9+f(x))^{-2}$.

---

10. Let f and g be differentiable functions such that $f(g(x))=2x$ and $f'(x)=1+\left[f(x)\right]^2$.

(a) Show that $g'(x)=\dfrac{2}{f'(g(x))}$.

(b) Show that $g'(x)=\dfrac{2}{1+4x^2}$.

2.4 Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\left(\sin x\right)=\cos x \qquad \frac{d}{dx}\left(\cos x\right)=-\sin x$$

$$\frac{d}{dx}\left(\tan x\right)=\sec^2 x \qquad \frac{d}{dx}\left(\cot x\right)=-\csc^2 x$$

$$\frac{d}{dx}\left(\sec x\right)=\sec x\tan x \qquad \frac{d}{dx}\left(\csc x\right)=-\csc x\cot x$$

Note: $\sec^2 x$ means $(\sec x)^2$.

Example 1 □ Find $\frac{dy}{dx}$ for $y=x^2\sin x+2x\cos x$.

Solution □ $$\frac{dy}{dx}=x^2\cos x+\sin x\cdot 2x+2\left(x(-\sin x)+\cos x\cdot 1\right)$$

$$=x^2\cos x+2x\sin x-2x\sin x+2\cos x$$

$$=x^2\cos x+2\cos x$$

Example 2 □ Find y' if $y=\tan^2(x^3)$.

Solution □ $y=\tan^2(x^3)=\left[\tan(x^3)\right]^2$ $\tan^2 x=(\tan x)^2$

$y'=2\left[\tan(x^3)\right]\frac{d}{dx}\left(\tan(x^3)\right)$ Power Chain Rule

$=2\tan(x^3)\sec^2(x^3)\frac{d}{dx}(x^3)$ Chain Rule

$=2\tan(x^3)\sec^2(x^3)(3x^2)$ Power Rule

$=6x^2\tan(x^3)\sec^2(x^3)$

Exercises - Derivatives of Trigonometric Functions

Multiple Choice Questions

1. $\lim\limits_{h\to 0}\dfrac{\cos(\frac{\pi}{3}+h)-\frac{1}{2}}{h} =$

(A) $-\frac{1}{2}$ (B) $-\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{\sqrt{3}}{2}$

---

2. $\lim\limits_{h\to 0}\dfrac{\sin 2(x+h)-\sin 2x}{h} =$

(A) $2\sin 2x$ (B) $-2\sin 2x$ (C) $2\cos 2x$ (D) $-2\cos 2x$

---

3. If $f(x)=\sin(\cos 2x)$, then $f'(\frac{\pi}{4}) =$

(A) 0 (B) –1 (C) 1 (D) –2

---

4. If $y = a\sin x + b\cos x$, then $y + y'' =$

(A) 0 (B) $2a\sin x$ (C) $2b\cos x$ (D) $-2a\sin x$

5. $\dfrac{d}{dx}\sec^2(\sqrt{x}) =$

(A) $\dfrac{2\sec(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

(B) $\dfrac{2\sec^2(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

(C) $\dfrac{\sec^2(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

(D) $\dfrac{\sec(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$

---

6. $\dfrac{d}{dx}\left[x^2\cos 2x\right] =$

(A) $-2x\sin 2x$

(B) $2x(-x\sin 2x + \cos 2x)$

(C) $2x(x\sin 2x - \cos 2x)$

(D) $2x(x\sin 2x - \cos 2x)$

---

7. If $f(\theta) = \cos\pi - \dfrac{1}{2\cos\theta} + \dfrac{1}{3\tan\theta}$, then $f'(\dfrac{\pi}{6}) =$

(A) $\dfrac{1}{2}$ (B) 1 (C) $\dfrac{4}{\sqrt{3}}$ (D) $2\sqrt{3}$

Free Response Questions

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	$-1/2$	$3/2$	4	$\sqrt{2}$
$\pi/4$	-2	1	2	3

8. The table above gives values of f, f', g, and g' at selected values of x.

   Find $h'(\frac{\pi}{4})$, if $h(x) = f(x) \cdot g(\tan x)$.

---

9. Find the value of the constants a and b for which the function $f(x) = \begin{cases} \sin x, & x < \pi \\ ax + b, & x \geq \pi \end{cases}$ is differentiable at $x = \pi$.

2.5 Derivatives of Exponential and Logarithmic Functions

The Derivatives of Exponential Function

$$\frac{d}{dx}\left[e^x\right]=e^x \qquad \frac{d}{dx}\left[e^u\right]=e^u\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^x\right]=a^x\ln a \qquad \frac{d}{dx}\left[a^u\right]=a^u(\ln a)\frac{du}{dx}$$

Example 1 □ Find y'' if $y=e^{\cos x}$.

Solution □ $y'=\frac{d}{dx}(e^{\cos x})$

$$=e^{\cos x}\frac{d}{dx}(\cos x) \qquad \frac{d}{dx}\left[e^u\right]=e^u\frac{du}{dx}$$

$$=e^{\cos x}(-\sin x)$$

$$y''=e^{\cos x}\cdot(-\cos x)+e^{\cos x}(-\sin x)\cdot(-\sin x)$$

$$=e^{\cos x}\left[-\cos x+\sin^2 x\right]$$

Example 2 □ Differentiate $y=3^{\sqrt{x^2-x}}$.

Solution □ $\frac{dy}{dx}=3^{\sqrt{x^2-x}}(\ln 3)\frac{d}{dx}(\sqrt{x^2-x})$ $\qquad \frac{d}{dx}\left[a^u\right]=a^u(\ln a)\frac{du}{dx}$

$$=3^{\sqrt{x^2-x}}(\ln 3)(\frac{1}{2\sqrt{x^2-x}})\frac{d}{dx}(x^2-x)$$

$$=\ln 3(3^{\sqrt{x^2-x}})(\frac{2x-1}{2\sqrt{x^2-x}})$$

The Derivatives of Logarithmic Function

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{\ln a}\cdot\frac{1}{x}$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}$$

Properties of Logarithms

$$\ln xy = \ln x + \ln y \qquad \ln\frac{x}{y} = \ln x - \ln y$$

$$\ln x^p = p\ln x \qquad e^{\ln x} = x$$

Example 4 □ Find y' if $y = \frac{\ln x}{x^2}$.

Solution □ $y = \frac{\ln x}{x^2} = \ln x \cdot x^{-2}$

$y' = \ln x \frac{d}{dx}x^{-2} + x^{-2}\frac{d}{dx}\ln x$ Product Rule

$= \ln x(-2x^{-3}) + x^{-2}\cdot\frac{1}{x}$

$= \frac{-2\ln x}{x^3} + \frac{1}{x^2}\cdot\frac{1}{x}$

$= \frac{-2\ln x + 1}{x^3}$

Example 5 □ Find y' if $y = x^{\ln x}$.

Solution □ $y = x^{\ln x}$

$\ln y = \ln(x^{\ln x})$ Take natural log of both sides.

$\ln y = \ln x \cdot \ln x = (\ln x)^2$ $\ln x^p = p\ln x$

$\frac{d}{dx}\ln y = \frac{d}{dx}(\ln x)^2$

$\frac{1}{y}\frac{d}{dx}(y) = 2\ln x\frac{d}{dx}(\ln x)$ Chain Rule

$\frac{1}{y}\cdot y' = 2\ln x\cdot\frac{1}{x}$ $\frac{dy}{dx} = y'$

$y' = y\left[2\ln x\cdot\frac{1}{x}\right]$ Multiply by y on both sides.

$y' = x^{\ln x}\cdot\frac{2\ln x}{x}$

Exercises - Derivatives of Exponential and Logarithmic Functions

Multiple Choice Questions

1. $\lim\limits_{h\to 0}\dfrac{\frac{1}{2}\left[\ln(e+h)-1\right]}{h}$ is

(A) $f'(1)$, where $f(x)=\ln\sqrt{x}$

(B) $f'(1)$, where $f(x)=\ln\sqrt{x+e}$

(C) $f'(e)$, where $f(x)=\ln\sqrt{x}$

(D) $f'(e)$, where $f(x)=\ln(\frac{x}{2})$

---

2. If $f(x)=e^{\tan x}$, then $f'(\frac{\pi}{4})=$

(A) $\frac{e}{2}$ (B) e (C) $2e$ (D) $\frac{e^2}{2}$

---

3. If $y=\ln(\cos x)$, then $y'=$

(A) $-\tan x$ (B) $\tan x$ (C) $-\cot x$ (D) $\csc x$

---

4. If $y=x^x$, then $y'=$

(A) $x^x\ln x$ (B) $x^x(1+\ln x)$ (C) $x^x(x+\ln x)$ (D) $\dfrac{x^x\ln x}{x}$

5. If $y = e^{\sqrt{x^2+1}}$, then $y' =$

(A) $\sqrt{x^2+1}\, e^{\sqrt{x^2+1}}$

(B) $2x\sqrt{x^2+1}\, e^{\sqrt{x^2+1}}$

(C) $\dfrac{e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

(D) $\dfrac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

---

6. If $y = (\sin x)^{1/x}$, then $y' =$

(A) $(\sin x)^{\frac{1}{x}}\left[\dfrac{\ln(\sin x)}{x}\right]$

(B) $(\sin x)^{\frac{1}{x}}\left[\dfrac{x - \ln(\sin x)}{x^2}\right]$

(C) $(\sin x)^{\frac{1}{x}}\left[\dfrac{x \sin x - \ln(\sin x)}{x^2}\right]$

(D) $(\sin x)^{\frac{1}{x}}\left[\dfrac{x \cot x - \ln(\sin x)}{x^2}\right]$

---

7. If $f(x) = \ln\left[\sec(\ln x)\right]$, then $f'(e) =$

(A) $\dfrac{\cos 1}{e}$ (B) $\dfrac{\sin 1}{e}$ (C) $\dfrac{\tan 1}{e}$ (D) $\dfrac{\cot 1}{e}$

8. If $y = x^{\ln\sqrt{x}}$, then $y' =$

(A) $\dfrac{x^{\ln\sqrt{x}}\ln x}{2x}$

(B) $\dfrac{x^{\ln\sqrt{x}}\ln x}{x}$

(C) $\dfrac{2x^{\ln\sqrt{x}}\ln x}{x}$

(D) $\dfrac{x^{\ln\sqrt{x}}(1+\ln x)}{x}$

Free Response Questions

9. Let $f(x) = xe^x$ and $f^{(n)}(x)$ be the nth derivative of f with respect to x. If $f^{(10)}(x) = (x+n)\,e^x$, what is the value of n?

---

10. Let f and h be twice differentiable functions such that $h(x) = e^{f(x)}$. If $h''(x) = e^{f(x)}\left[1+x^2\right]$, then $f'(x) =$

2.6 Tangent lines and Normal lines

The **slope of the tangent line** to the graph of f at the point $P(a, f(a))$ is the number

$$\boldsymbol{m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$$

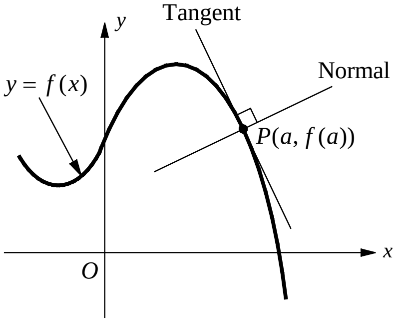
if the limit exists.

Recall from the definition of the derivative that this limit is just $f'(a)$.

In **point slope form** the equation of the tangent line is

$$\boldsymbol{y - f(a) = f'(a)(x-a)}$$

The **normal line** to the graph of f at the point $P(a, f(a))$ is the line that passes through P and is perpendicular to the tangent line to the graph of f at P.



Since the tangent line and the normal line are perpendicular, their slopes are negative reciprocal of each other. Therefore $m_1 \cdot m_2 = -1$.

Example 1 □ Write the equations of the tangent line and normal line to the graph of $y = x - \frac{x^2}{10}$ at the point $(4, \frac{12}{5})$.

Solution □ $y' = 1 - \frac{2x}{10} = 1 - \frac{x}{5}$

At the point $(4, \frac{12}{5})$, the slope is

$y'\big|_{x=4} = 1 - \frac{4}{5} = \frac{1}{5}$ and the equation of tangent line at this point is

$y - \frac{12}{5} = \frac{1}{5}(x-4)$ or $y = \frac{1}{5}x + \frac{8}{5}$.

At the point $(4, \frac{12}{5})$, the slope of the normal line is negative reciprocal of $1/5$, or -5.

So the equation of normal line at this point is

$y - \frac{12}{5} = -5(x-4)$ or $y = -5x + 22.4$.

Exercises - Tangent Lines and Normal Lines

Multiple Choice Questions

1. The equation of the line tangent to the graph of $y = x\sqrt{3+x^2}$ at the point $(1,2)$ is

(A) $y = \frac{3}{2}x - \frac{1}{2}$ (B) $y = 2x + \frac{1}{2}$ (C) $y = \frac{5}{2}x - \frac{1}{2}$ (D) $y = \frac{5}{2}x + \frac{1}{2}$

---

2. Which of the following is an equation of the line tangent to the graph of $f(x) = x^2 - x$ at the point where $f'(x) = 3$?

(A) $y = 3x - 2$

(B) $y = 3x + 2$

(C) $y = 3x - 4$

(D) $y = 3x + 4$

---

3. A curve has slope $2x + x^{-2}$ at each point (x, y) on the curve. Which of the following is an equation for this curve if it passes through the point $(1,3)$?

(A) $y = 2x^2 + \frac{1}{x}$

(B) $y = x^2 - \frac{1}{x} + 3$

(C) $y = x^2 + \frac{1}{x} + 1$

(D) $y = x^2 - \frac{2}{x^2} + 4$

4. An equation of the line normal to the graph of $y = \tan x$, at the point $(\frac{\pi}{6}, \frac{1}{\sqrt{3}})$ is

(A) $y - \frac{1}{\sqrt{3}} = -\frac{1}{4}(x - \frac{\pi}{6})$

(B) $y - \frac{1}{\sqrt{3}} = \frac{1}{4}(x - \frac{\pi}{6})$

(C) $y - \frac{1}{\sqrt{3}} = -\frac{3}{4}(x - \frac{\pi}{6})$

(D) $y - \frac{1}{\sqrt{3}} = \frac{3}{4}(x - \frac{\pi}{6})$

---

5. If $2x + 3y = 4$ is an equation of the line normal to the graph of f at the point $(-1, 2)$, then $f'(-1) =$

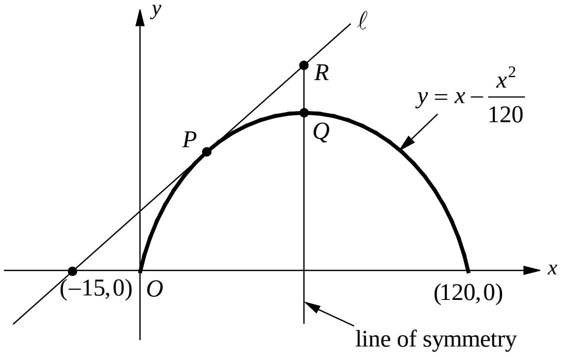
(A) $-\frac{2}{3}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) $\frac{3}{2}$

---

6. If $2x - y = k$ is an equation of the line normal to the graph of $f(x) = x^4 - x$, then $k =$

(A) $\frac{23}{16}$ (B) $\frac{13}{18}$ (C) $\frac{15}{16}$ (D) $\frac{9}{8}$

Free Response Questions



7. Line ℓ is tangent to the graph of $y = x - \frac{x^2}{120}$ at the point P and intersects x-axis at $(-15, 0)$ as shown in the figure above.

   (a) Find the x-coordinates of point P.

   (b) Write an equation for line ℓ.

   (c) If the line of symmetry for the curve $y = x - \frac{x^2}{120}$ intersects line ℓ at point R, what is the length of $\overline{QR}$?

2.7 Implicit Differentiation

The functions that we have met so far have been described by an equation, of the form $y = f(x)$, that expresses y explicitly in terms of x for example, $y = \frac{1}{x^2+1}$ or $y = x + \cos x$. Some functions, however, are defined **implicitly** by a relation between x and y such as $x^2 + 4y^2 - 9 = 0$ or $\sin xy = x^2 + 1$. In some cases it is possible to solve such an equation for y as an explicit function of x. But when we are unable to solve for y as a function of x, we may still be able to find $\frac{dy}{dx}$ by using the method of **implicit differentiation**.

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Solve for dy/dx.

The **tangent line is horizontal** if $dy/dx = 0$.
The **tangent line is vertical** when the denominator in the expression for dy/dx is 0.

Example 1 □ Find dy/dx if $y^2 = x^2 - \cos xy$

Solution □

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\cos xy)$$

$$2y\frac{dy}{dx} = 2x + (\sin xy)\frac{d}{dx}(xy)$$

Differentiate both sides with respect to x treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\sin xy)\left(x\frac{dy}{dx} + y\right)$$

Treat xy as a product.

$$2y\frac{dy}{dx} - (\sin xy)\left(x\frac{dy}{dx}\right) = 2x + (\sin xy)\,y$$

Collect terms with dy/dx.

$$(2y - x\sin xy)\frac{dy}{dx} = 2x + y\sin xy$$

Factor out dy/dx.

$$\frac{dy}{dx} = \frac{2x + y\sin xy}{2y - x\sin xy}$$

Solve for dy/dx by dividing.

Example 2 □ Consider the curve defined by $x^3 + y^3 = 4xy + 1$.

(a) Find dy/dx.

(b) Write an equation for the line tangent to the curve at the point $(2,1)$.

Solution □ (a) $3x^2 + 3y^2 \dfrac{dy}{dx} = 4(x\dfrac{dy}{dx} + 1 \cdot y)$ Differentiate both sides.

$$3x^2 + 3y^2 \frac{dy}{dx} = 4x\frac{dy}{dx} + 4y$$

$$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$$

(b) $\left.\dfrac{dy}{dx}\right|_{(2,1)} = \dfrac{4(1) - 3(2)^2}{3(1)^2 - 4(2)} = \dfrac{8}{5}$

The tangent line is $y - 1 = \dfrac{8}{5}(x - 2)$ or $y = \dfrac{8}{5}x - \dfrac{11}{5}$.

Example 3 □ Consider the curve given by $x^3 + y^3 - 6xy = 0$.

(a) Find dy/dx.

(b) Find the *x*-coordinates of each point on the curve where the tangent line is horizontal.

(c) Find the *y*-coordinates of each point on the curve where the tangent line is vertical.

Solution □ (a) $3x^2 + 3y^2 \dfrac{dy}{dx} - 6(x\dfrac{dy}{dx} + 1 \cdot y) = 0$ Differentiate both sides.

$$3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

(b) The tangent line is horizontal when $2y - x^2 = 0$.

$2y - x^2 = 0$ gives $y = 1/2\, x^2$.

Substitute $y = 1/2\, x^2$ to the given equation.

$$x^3 + (\frac{1}{2}x^2)^3 - 6x(\frac{1}{2}x^2) = 0 \;\Rightarrow\; \frac{1}{8}x^6 - 2x^3 = 0$$

$$\Rightarrow\; \frac{1}{8}x^3(x^3 - 16) = 0 \;\Rightarrow\; x = 0 \text{ or } x = \sqrt[3]{16}$$

(c) The tangent line is vertical when $y^2 - 2x = 0$.

$y^2 - 2x = 0$ gives $x = 1/2\, y^2$.

Substitute $x = 1/2\, y^2$ to the given equation.

$$(\frac{1}{2}y^2)^3 + y^3 - 6(\frac{1}{2}y^2)y = 0 \;\Rightarrow\; \frac{1}{8}y^6 - 2y^3 = 0$$

$$\Rightarrow\; \frac{1}{8}y^3(y^3 - 16) = 0 \;\Rightarrow\; y = 0 \text{ or } y = \sqrt[3]{16}$$

Exercises - Implicit Differentiation

Multiple Choice Questions

1. If $3xy + x^2 - 2y^2 = 2$, then the value of $\frac{dy}{dx}$ at the point $(1,1)$ is

(A) 5 (B) $\frac{7}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{7}{2}$

---

2. If $3x^4 - x^2 - y^2 = 0$, then the value of $\frac{dy}{dx}$ at the point $(1, \sqrt{2})$ is

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{3\sqrt{2}}{2}$ (C) $\frac{5\sqrt{2}}{2}$ (D) $\frac{7\sqrt{2}}{2}$

---

3. If $x^2 y + 2xy^2 = 5x$, then $\frac{dy}{dx} =$

(A) $\frac{5 - 4xy - 4y}{x^2 + 4xy}$

(B) $\frac{5 - 2xy - 2y^2}{x^2 + 4xy}$

(C) $\frac{5 - 2xy - y^2}{x^2 + 2xy}$

(D) $\frac{5 - xy - 2y}{x^2 - 2xy}$

4. If $xy + \tan(xy) = \pi$, then $\frac{dy}{dx} =$

(A) $-y\sec^2(xy)$ (B) $-y\cos^2(xy)$ (C) $-x\sec^2(xy)$ (D) $-\frac{y}{x}$

---

5. An equation of the line tangent to the graph of $3y^2 - x^3 - xy^2 = 7$ at the point $(1,2)$ is

(A) $y = \frac{3}{4}x - \frac{3}{8}$ (B) $y = \frac{3}{4}x + \frac{1}{2}$ (C) $y = -\frac{7}{8}x + \frac{3}{2}$ (D) $y = \frac{7}{8}x + \frac{9}{8}$

---

6. An equation of the line normal to the graph of $2x^2 + 3y^2 = 5$ at the point $(1,1)$ is

(A) $y = \frac{3}{2}x + 1$ (B) $y = \frac{3}{2}x - \frac{1}{2}$ (C) $y = -\frac{2}{3}x + \frac{5}{3}$ (D) $y = -\frac{2}{3}x + \frac{3}{2}$

---

7. If $x + \sin y = y + 3$, then $\frac{d^2y}{dx^2} =$

(A) $\frac{-\sin y}{(1-\cos y)^2}$ (B) $\frac{-\sin y}{(1+\cos y)^2}$ (C) $\frac{-\sin y}{(1-\cos y)^3}$ (D) $\frac{-\sin y}{(1+\cos y)^3}$

Free Response Questions

8. Consider the curve given by $x^3 - xy + y^2 = 3$.

(a) Find $\frac{dy}{dx}$.

(b) Find all points on the curve whose *x*-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the *x*-coordinate of each point on the curve where the tangent line is horizontal.

9. Consider the curve $x^2 + y^2 - xy = 7$.

(a) Find $\frac{dy}{dx}$.

(b) Find all points on the curve whose *x*-coordinate is 2, and write an equation for the tangent line at each of these points.

(c) Find the *x*-coordinate of each point on the curve where the tangent line is vertical.

2.8 Derivatives of an Inverse Function

The Derivative of an Inverse Function

Let f be a differentiable function whose inverse function f^{-1} is also differentiable. Then, providing that the denominator is not zero,

$$\boldsymbol{(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}} \qquad \text{or} \qquad \boldsymbol{(f^{-1})'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}}\,.$$

Example 1 □ Let $f(x) = x^2 - \dfrac{3}{x}$.

(a) What is the value of $f^{-1}(8)$?

(b) What is the value of $(f^{-1})'(8)$?

Solution □ (a) $f(x) = x^2 - \dfrac{3}{x} = 8$, when $x = 3$.

Since $f(3) = 8$, $f^{-1}(8) = 3$.

(b) $f'(x) = 2x - 3(-1)x^{-2} = 2x + \dfrac{3}{x^2}$

$$f'(3) = 2(3) + \frac{3}{3^2} = \frac{19}{3}$$

$$(f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(3)} \qquad\qquad f^{-1}(8) = 3$$

$$= \frac{1}{19/3} = \frac{3}{19}$$

Example 2 □ If $f(2) = 5$ and $f'(2) = \dfrac{1}{4}$, find $(f^{-1})'(5)$.

Solution □ Since $f(2) = 5$, $f^{-1}(5) = 2$

$$(f^{-1})'(5) = \frac{1}{f'\left(f^{-1}(5)\right)} = \frac{1}{f'(2)} = \frac{1}{1/4} = 4$$

Exercises - Derivatives of an Inverse Function

Multiple Choice Questions

1. Let f and g be functions that are differentiable everywhere. If g is the inverse function of f and if $g(3) = 4$ and $f'(4) = \frac{3}{2}$, then $g'(3) =$

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

---

2. If $f(-3) = 2$ and $f'(-3) = \frac{3}{4}$, then $(f^{-1})'(2) =$

(A) $\frac{1}{2}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $-\frac{3}{4}$

---

3. If $f(x) = x^3 - x + 2$, then $(f^{-1})'(2) =$

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) 4 (D) 6

---

4. If $f(x) = \sin x$, then $(f^{-1})'(\frac{\sqrt{3}}{2}) =$

(A) $\frac{1}{2}$ (B) $\frac{2\sqrt{3}}{3}$ (C) $\sqrt{3}$ (D) 2

5. If $f(x)=1+\ln x$, then $(f^{-1})'(2)=$

(A) $-\frac{1}{e}$ (B) $\frac{1}{e}$ (C) $-e$ (D) e

Free Response Questions

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	3	-2	2	6
0	-2	-1	0	-3
1	0	1	-1	2
2	-1	4	3	-1

6. The functions f and g are differentiable for all real numbers. The table above gives the values of the functions and their first derivatives at selected values of x.

(a) If f^{-1} is the inverse function of f, write an equation for the line tangent to the graph of $y=f^{-1}(x)$ at $x=-1$.

(b) Let h be the function given by $h(x)=f(g(x))$. Find $h(1)$ and $h'(1)$.

(c) Find $(h^{-1})'(3)$, if h^{-1} is the inverse function of h.

2.9 Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

Example 1 □ Differentiate $y = x\tan^{-1} x$.

Solution □ $\frac{dy}{dx} = \frac{d}{dx}(x\tan^{-1} x)$

$$= x\frac{d}{dx}\tan^{-1} x + \tan^{-1} x\frac{d}{dx}(x) \qquad \text{Product Rule}$$

$$= x\cdot\frac{1}{1+x^2} + \tan^{-1} x\cdot 1$$

$$= \frac{x}{1+x^2} + \tan^{-1} x$$

Example 2 □ Differentiate $y = \frac{1}{\cos^{-1} x}$.

Solution □ $y = \frac{1}{\cos^{-1} x} = (\cos^{-1} x)^{-1}$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x)^{-1}$$

$$= -1(\cos^{-1} x)^{-2}\frac{d}{dx}(\cos^{-1} x) \qquad \text{Power Chain Rule}$$

$$= -\frac{1}{(\cos^{-1} x)^2}\cdot -\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{(\cos^{-1} x)^2\sqrt{1-x^2}}$$

Exercises - Derivatives of Inverse Trigonometric Functions

Multiple Choice Questions

1. $\dfrac{d}{dx}(\arcsin x^2) =$

(A) $-\dfrac{2x}{\sqrt{1-x^2}}$ (B) $\dfrac{2x}{\sqrt{x^2-1}}$ (C) $\dfrac{2x}{\sqrt{x^4-1}}$ (D) $\dfrac{2x}{\sqrt{1-x^4}}$

---

2. If $f(x) = \arctan(e^{-x})$, then $f'(-1) =$

(A) $\dfrac{-e}{1+e}$ (B) $\dfrac{e}{1+e}$ (C) $\dfrac{-e}{1+e^2}$ (D) $\dfrac{-1}{1+e^2}$

---

3. If $f(x) = \arctan(\sin x)$, then $f'(\dfrac{\pi}{3}) =$

(A) $\dfrac{2}{7}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{\sqrt{2}}{3}$ (D) $\dfrac{\sqrt{3}}{3}$

4. If $y=\cos(\sin^{-1}x)$, then $y'=$

(A) $-\dfrac{1}{\sqrt{1-x^2}}$ (B) $-\dfrac{x}{\sqrt{1-x^2}}$ (C) $\dfrac{2x}{\sqrt{1-x^2}}$ (D) $-\dfrac{2x}{\sqrt{x^2-1}}$

Free Response Questions

5. Let f be the function given by $f(x)=x^{\tan^{-1}x}$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x=1$.

2.10 Approximating a Derivative

If a function f is defined by a table of values, then the approximation values of its derivatives at b can be obtained from the average rate of change using values that are close to b.

x	$\cdots$	a	$\cdots$	b	$\cdots$	c	$\cdots$
$f(x)$	$\cdots$	$f(a)$	$\cdots$	$f(b)$	$\cdots$	$f(c)$	$\cdots$

For $a < b < c$,

$$f'(b) \approx \frac{f(c)-f(b)}{c-b} \text{ or}$$

$$f'(b) \approx \frac{f(b)-f(a)}{b-a} \text{ or}$$

$$f'(b) \approx \frac{f(c)-f(a)}{c-a}.$$

Example 1 □ The temperature of the water in a coffee cup is a differentiable function F of time t. The table below shows the temperature of coffee in a cup as recorded every 3 minutes over 12minute period.

t	0	3	6	9	12
$F(t)$	205	197	192	186	181

(a) Use data from the table to find an approximation for $F'(6)$?

(b) The rate at which the water temperature decrease for $0 \le t \le 12$ is modeled by $F(t) = 120 + 85e^{-0.03t}$ degrees per minute. Find $F'(6)$ using the given model.

Solution □ (a) $F'(6) \approx \dfrac{F(6)-F(3)}{6-3} = \dfrac{192-197}{3} = -\dfrac{5}{3}\ {}^{\circ}F/\text{min}$ or

$$F'(6) \approx \frac{F(9)-F(6)}{9-6} = \frac{186-192}{3} = -2\ {}^{\circ}F/\text{min} \text{ or}$$

$$F'(6) \approx \frac{F(9)-F(3)}{9-3} = \frac{186-197}{6} = -\frac{11}{6}\ {}^{\circ}F/\text{min}$$

(b) $F'(t) = 0 + 85e^{-0.03t}\dfrac{d}{dx}(-0.03t)$

$$= 85e^{-0.03t}(-0.03) = -2.55e^{-0.03t}$$

$$F'(6) = -2.55e^{-0.03(6)} = -2.55e^{-0.18} \approx -2.129\ {}^{\circ}F/\text{min}$$

Exercises - Approximating a Derivative

Multiple Choice Questions

1. Some values of differentiable function f are shown in the table below. What is the approximation value of $f'(3.5)$?

x	3.0	3.3	3.8	4.2	4.9
$f(x)$	21.8	26.1	32.5	38.2	48.7

(A) 8 (B) 10 (C) 13 (D) 16

Free Response Questions

Month	1	2	3	4	5	6
Temperature	−8	0	25	50	72	88

2. The normal daily maximum temperature F for a certain city is shown in the table above.

(a) Use data in the table to find the average rate of change in temperature from $t=1$ to $t=6$.

(b) Use data in the table to estimate the rate of change in maximum temperature at $t=4$.

(c) The rate at which the maximum temperature changes for $1 \le t \le 6$ is modeled by $F(t) = 40 - 52\sin(\frac{\pi t}{6} - 5)$ degrees per minute. Find $F'(4)$ using the given model.

Chapter 3
Applications of Differentiation

3.1 Related Rates

In a related rates problem, the idea is to compute the rates of change of two or more related variables with respect to time. The procedure involves finding an equation that relates the variables and using the Chain Rule to differentiate both sides with respect to time.

Guidelines For Solving Related Rate Problems

1. Read the problem carefully and draw a diagram if possible.
2. Name the variables and constants. Use t for time.
3. Write an equation that relates the variables whose rates of change are given. If necessary, use the geometry or trigonometry of the situation and write an equation whose rates of change are to be determined.
4. Combine two or more equations to get a single equation that relates the variable.
5. Use the Chain Rule to differentiate both sides of the equation with respect to t.
6. Substitute the given numerical information into the resulting equation and solve for the unknown rate.

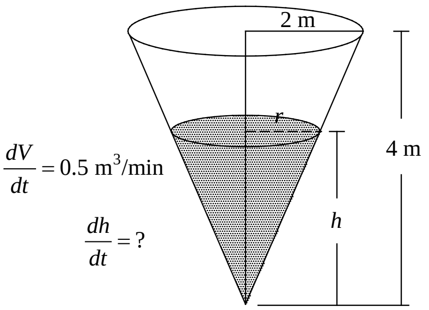
Example 1 □ Water runs into a conical tank at a rate of $0.5\ \text{m}^3/\text{min}$. The tank stands point down and has a height of 4m and a base radius of 2m. How fast is the water level rising when the water is 2.5m deep?

Solution □ Draw a picture and name the variables and constants. Let $V =$ volume of water in the tank, $h =$ depth of water in the tank, and $r =$ radius of the surface of the water at time t.

Volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

In order to eliminate r we use the similar triangle in the figure.

$$\frac{2}{4} = \frac{r}{h} \quad \Rightarrow \quad r = \frac{h}{2}$$



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{h}{2})^2 h = \frac{\pi}{12}h^3$$

Substitute $r = \frac{h}{2}$.

$$\frac{dV}{dt} = \frac{\pi}{12}(3h^2 \frac{dh}{dt}) = \frac{\pi h^2}{4}\frac{dh}{dt}$$

Differentiate V and h with respect to t.

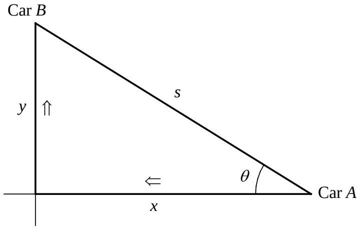
$$0.5 = \frac{\pi(2.5)^2}{4}\frac{dh}{dt}$$

$\frac{dV}{dt} = 0.5$ and $h = 2.5$.

$$\frac{dh}{dt} = \frac{(0.5)4}{\pi(2.5)^2} \approx 0.102$$

So, when $h = 2.5$, the water level is rising at a rate of 0.102 meters per minute.

Example 2 □ Car A is traveling due west toward the intersection at a speed of 45 miles per hour. Car B is traveling due north away from the intersection at a speed of 30 mph. Let x be the distance between Car A and the intersection at time t, and let y be the distance between Car B and the intersection at time t as shown in the figure at the right.



(a) Find the rate of change, in miles per hour, of the distance between the two cars when $x = 32$ miles and $y = 24$ miles.

(b) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 32$ miles and $y = 24$ miles.

Solution □ (a) Let $s =$ the distance between the two cars.

$$s^2 = x^2 + y^2$$

Pythagorean Theorem

$$s^2 = 32^2 + 24^2 = 1600$$

$x = 32$ and $y = 24$.

$$s = \sqrt{1600} = 40$$

$$dx/dt = -45 \text{ mi/h}$$

dx/dt is negative since the car is moving to the left.

$$dy/dt = 30 \text{ mi/h}$$

dy/dt is positive since the car is moving up.

$$\frac{d}{dt}s^2 = \frac{d}{dt}(x^2 + y^2)$$

Differentiate both sides with respect to t.

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Power Chain Rule

$$2(40)\frac{ds}{dt} = 2(32)(-45) + 2(24)(30)$$

Substitution

$$\frac{ds}{dt} = -18 \text{ mph}$$

(b) $\tan\theta = \dfrac{y}{x}$

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(\frac{y}{x})$$

Differentiate both sides with respect to t .

$$\sec^2\theta\frac{d\theta}{dt} = \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$$

$$(\frac{40}{32})^2\frac{d\theta}{dt} = \frac{(32)(30)-(24)(-45)}{(32)^2}$$

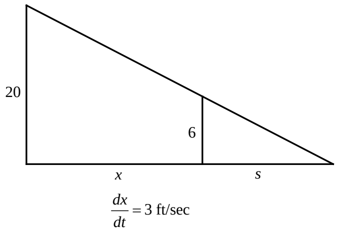
$$\frac{d\theta}{dt} = 1.275 \text{ rad/hr}$$

Example 3 □ A man 6 feet tall walks at a rate of 3 feet per second away from a light that is 20 feet above the ground.

(a) At what rate is the tip of his shadow moving when he is 12 feet from the base of the light.

(b) At what rate is the length of his shadow changing when is 12 feet from the base of the light.

Solution □ Draw a picture and name the variables and constants. Let $x =$ the distance from the base to the man and $s =$ the distance from the man to the tip of his shadow.



By similar triangles, $\dfrac{20}{x+s} = \dfrac{6}{s}$.

$\Rightarrow\ 20s = 6(x+s)\ \Rightarrow\ 14s = 6x$

$\Rightarrow\ s = \dfrac{3}{7}x$

(a) The tip of the shadow moves at a rate of

$$\frac{d}{dt}(x+s)$$

$x+s$ is the distance from the base to the tip of shadow.

$$= \frac{d}{dt}(x+\frac{3}{7}x) = \frac{10}{7}\frac{dx}{dt}$$

$s = \dfrac{3}{7}x$

$$= \frac{10}{7}(3) = \frac{30}{7} \text{ ft/sec}$$

$\dfrac{dx}{dt} = 3$

(b) The length of the shadow changing at a rate of

$$\frac{d}{dt}(s) = \frac{d}{dt}(\frac{3}{7}x) = \frac{3}{7}\frac{dx}{dt}$$

s is the length of shadow.

$$= \frac{3}{7}(3) = \frac{9}{7} \text{ ft/sec}$$

Exercises - Related Rates

Multiple Choice Questions

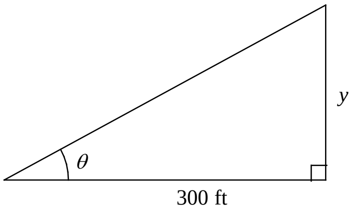
1. The radius of a circle is changing at the rate of $1/\pi$ inches per second. At what rate, in square inches per second, is the circle's area changing when $r = 5$ in ?

(A) $\frac{5}{\pi}$ (B) 10 (C) $\frac{10}{\pi}$ (D) 15

---

2. The volume of a cube is increasing at the rate of $12\ \text{in}^3/\text{min}$. How fast is the surface area increasing, in square inches per minute, when the length of an edge is 20 in?

(A) 1 (B) $\frac{6}{5}$ (C) $\frac{4}{3}$ (D) $\frac{12}{5}$



3. In the figure shown above, a hot air balloon rising straight up from the ground is tracked by a television camera 300 ft from the liftoff point. At the moment the camera's elevation angle is $\pi/6$, the balloon is rising at the rate of 80 ft/min . At what rate is the angle of elevation changing at that moment?

(A) 0.12 radian per minute
(B) 0.16 radian per minute
(C) 0.2 radian per minute
(D) 0.4 radian per minute

4. A car is approaching a right-angled intersection from the north at 70 mph and a truck is traveling to the east at 60 mph. When the car is 1.5 miles north of the intersection and the truck is 2 miles to the east, at what rate, in miles per hour, is the distance between the car and truck is changing?

   (A) Decreasing 15 miles per hour

   (B) Decreasing 9 miles per hour

   (C) Increasing 6 miles per hour

   (D) Increasing 12 miles per hour

---

5. The radius r of a sphere is increasing at a constant rate. At the time when the surface area and the radius of sphere are increasing at the same numerical rate, what is the radius of the sphere? (The surface area of a sphere is $S = 4\pi r^2$.)

   (A) $\frac{1}{8\pi}$ (B) $\frac{1}{4\pi}$ (C) $\frac{1}{3\pi}$ (D) $\frac{\pi}{8}$

---

6. If the radius r of a cone is decreasing at a rate of 2 centimeters per minute while its height h is increasing at a rate of 4 centimeters per minute, which of the following must be true about the volume V of the cone? ($V = \frac{1}{3}\pi r^2 h$)

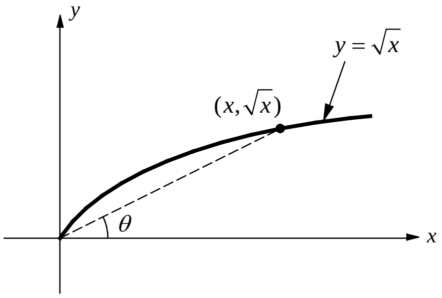
   (A) V is always decreasing.

   (B) V is always increasing.

   (C) V is increasing only when $r > h$.

   (D) V is increasing only when $r < h$.

Free Response Questions

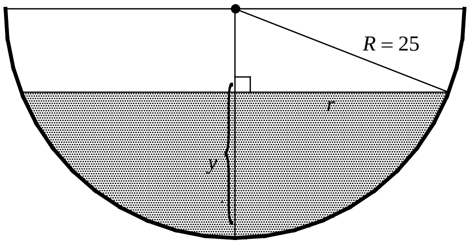


7. A particle moves along the curve $y = \sqrt{x}$. When $y = 2$ the x-component of its position is increasing at the rate of 4 units per second.

   (a) What is the value of $\frac{dy}{dt}$ when $y = 2$?

   (b) How fast is the distance from the particle to the origin changing when $y = 2$?

   (c) What is the value of $\frac{d\theta}{dt}$ when $y = 2$?

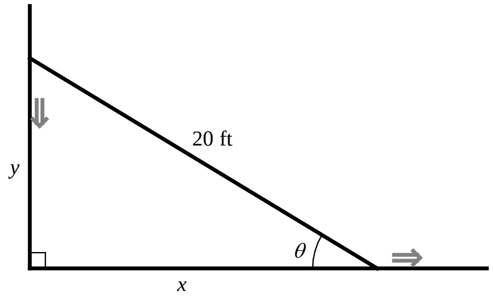


8. As shown in the figure above, water is draining at the rate of 12 ft^3/min from a hemispherical bowl of radius 25 feet. The volume of water in a hemispherical bowl of radius R when the depth of the water is y meters is given as $V = \frac{\pi}{3} y^2 (3R - y)$.

   (a) Find the rate at which the depth of water is decreasing when the water is 18 meters deep. Indicate units of measure.

   (b) Find the radius r of the water's surface when the water is y feet deep.

   (c) At what rate is the radius r changing when the water is 18 meters deep. Indicate units of measure.



9. In the figure shown above, the top of a 20-foot ladder is sliding down a vertical wall at a constant rate of 2 feet per second.

(a) When the top of the ladder is 12 feet from the ground, how fast is the bottom of the ladder moving away from the wall?

(b) The triangle is formed by the wall, the ladder and the ground. At what rate is the area of the triangle is changing when the top of the ladder is 12 feet from the ground?

(c) At what rate is the angle θ between the ladder and the ground is changing when the top of the ladder is 12 feet from the ground?

---

10. Consider the curve given by $2y^2 + 3xy = 1$.

(a) Find $\frac{dy}{dx}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has a slope of $-\frac{3}{4}$.

(c) Let x and y be functions of time t that are related by the equation $2y^2 + 3xy = 1$. At time $t = 3$, the value of y is 2 and $\frac{dy}{dt} = -2$. Find the value of $\frac{dx}{dt}$ at time $t = 3$.

3.2 Position, Velocity , and Acceleration

Let $\boldsymbol{s(t)}$ be the **position function** for an object moving along a straight line.

Velocity is the derivative of position with respect to time.

$$\boldsymbol{v(t) = \lim_{h\to 0}\frac{s(t+h)-s(t)}{h} = s'(t)} .$$

S**peed** is the absolute value of velocity.

$$\textbf{speed} = |\boldsymbol{v(t)}|$$

Acceleration is the derivative of velocity with respect to time.

$$\boldsymbol{a(t) = v'(t) = s''(t)}$$

If $v > 0$, then the particle is **moving to the right**.
If $v < 0$, then the particle is **moving to the left**.

If $a > 0$, then $\boldsymbol{v}$ **is increasing**.
If $a < 0$, then $\boldsymbol{v}$ **is decreasing**.

If a and v have the same sign, the particle's **speed is increasing**.
If a and v have the opposite signs, the particle's **speed is decreasing**.

Note: For movement on a horizontal line, $x(t)$ is used to represent the position function; rightward movement is considered to be in the positive direction.
For movement on a vertical line, $y(t)$ is used to represent the position function; upward movement is considered to be in the positive direction.

Example 1 □ A particle starts moving at time $t = 0$ and moves along the x-axis so that its position at time $t \ge 0$ is given by $x(t) = t^3 - \frac{9}{2}t^2 + 7$.

(a) Find the velocity of the particle at any time $t \ge 0$.

(b) For what values of t is the particle moving to the left.

(c) Find the values of t for which the particle is moving but its acceleration is zero.

(d) For what values of t is the speed of the particle decreasing?

Solution □ (a) $v(t) = x'(t) = 3t^2 - 9t$

(b) The particle is moving to the left when $v(t) < 0$.

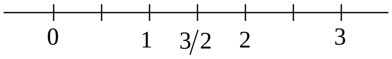
$v(t) = 3t^2 - 9t = 3t(t-3) < 0 \;\Rightarrow\; 0 < t < 3$.

(c) $a(t) = v'(t) = 6t - 9 = 3(2t-3)$

The acceleration is zero when $2t - 3 = 0$. So, $t = 3/2$.

(d) The speed of particle is decreasing when $\frac{3}{2} < t < 3$, since within this interval, $a(t)$ and $v(t)$ have the opposite signs.

Signs of $v(t)$	$-\;-\;-$	$-\;-\;-$	$+\;+$
Signs of $a(t)$	$-\;-\;-$	$+\;+\;+$	$+\;+$



Exercises - Position, Velocity, and Acceleration

Multiple Choice Questions

1. A particle moves along the x-axis so that at any time $t \geq 0$, its position is given by $x(t) = -\frac{1}{2}\cos t - 3t$.

   What is the acceleration of the particle when $t = \frac{\pi}{3}$?

   (A) $-\frac{\sqrt{3}}{4}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{4}$ (D) $\frac{\sqrt{3}}{4}$

---

2. A point moves along the x-axis so that at any time t, its position is given by $x(t) = \sqrt{x}\ln x$. For what values of t is the particle at rest?

   (A) No values (B) $\frac{1}{e^2}$ (C) $\frac{1}{e}$ (D) e

---

3. A particle moves along the x-axis so that at any time t, its position is given by $x(t) = 3\sin t + t^2 + 7$. What is velocity of the particle when its acceleration is zero?

   (A) 1.504 (B) 1.847 (C) 2.965 (D) 3.696

---

4. Two particles start at the origin and move along the x-axis. For $0 \leq t \leq 8$, their respective position functions are given by $x_1(t) = \sin^2 t$ and $x_2(t) = e^{-t}$. For how many values of t do the particles have the same velocity?

   (A) 3 (B) 4 (C) 5 (D) 6

5. A particle moves along a line so that at time t, where $0 \le t \le 5$, its velocity is given by $v(t) = -t^3 + 6t^2 - 15t + 10$. What is the minimum acceleration of the particle on the interval?

(A) -30 (B) -15 (C) -3 (D) 0

---

6. A particle moves along the x-axis so that at any time $t \ge 0$, its velocity is given by $v(t) = -t^3 e^{-t}$. At what value of t does v attain its minimum?

(A) $3\sqrt[3]{e}$ (B) 3 (C) 0 (D) $\sqrt[3]{e}$

---

7. The position of a particle moving along a line is given by $s(t) = t^3 - 12t^2 + 21t + 10$ for $t \ge 0$. For what value of t is the speed of the particle increasing?

(A) $1 < t < 7$ only
(B) $4 < t < 7$ only
(C) $0 < t < 1$ and $4 < t < 7$
(D) $1 < t < 4$ and $t > 7$

Free Response Questions

8. A particle moves along the x-axis so that its position at any time $t \ge 0$ is given by $x(t) = (t-2)^3(t-6)$.

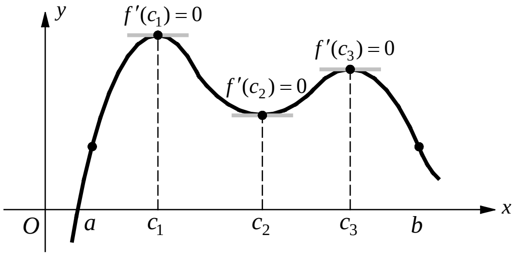
(a) Find the velocity and acceleration of the particle at any time $t \ge 0$.
(b) Find the value of t when the particle is moving and the acceleration is zero.
(c) When is the particle moving to the right?
(d) When is the velocity of the particle decreasing?
(e) When is the speed of the particle increasing?

3.3 Rolle's Theorem and The Mean Value Theorem

Rolle's Theorem

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$, then there is at least one number c in the open interval (a,b) such that

$$\boldsymbol{f'(c)} = 0 .$$

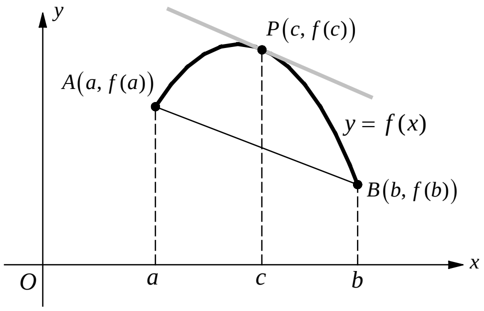


Rolle's Theorem says that
a differentiable curve has
at least one horizontal tangent
between a and b if $f(a)=f(b)$.

The Mean Value Theorem

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c in the open interval (a,b) such that

$$\boldsymbol{f'(c)} = \frac{\boldsymbol{f(b) - f(a)}}{\boldsymbol{b - a}} .$$



The Mean Value Theorem says that
there is at least one point $P(c, f(c))$,
on the curve where the tangent line
is parallel to the secant line AB.

Example 1 □ Let f be the function given by $f(x) = x^3 - 9x + 1$. Find all numbers c that satisfies the conclusion of Rolle's Theorem for f, such that $f'(c) = 0$ on the closed interval $\left[0,3\right]$.

Solution □

$$f(x) = x^3 - 9x + 1$$
$$f(0) = 1$$
$$f(3) = (3)^3 - 9(3) + 1 = 1$$

So, $f(0) = f(3)$, and from Rolle's Theorem there exists at least one number c in the open interval $\left(0,3\right)$ such that $f'(c) = 0$.

$$f'(x) = 3x^2 - 9 \qquad \text{Differentiate.}$$
$$f'(c) = 3c^2 - 9 = 0 \qquad \text{Set } f'(c) \text{ equal to 0.}$$
$$c^2 = 3 \;\Rightarrow\; c = \pm\sqrt{3}$$

But $-\sqrt{3}$ is not in $\left(0,3\right)$, so $c = \sqrt{3}$.

Example 2 □ Let f be the function given by $f(x) = x^3 - 2x^2 + x - 5$. Find all numbers c that satisfy the conclusion of the Mean Value Theorem for f on the closed interval $\left[-1,2\right]$.

Solution □

$$f(x) = x^3 - 2x^2 + x - 5$$
$$f'(x) = 3x^2 - 4x + 1 \qquad \text{Differentiate.}$$
$$f'(c) = 3c^2 - 4c + 1$$
$$f(2) = (2)^3 - 2(2)^2 + 2 - 5 = -3$$
$$f(-1) = (-1)^3 - 2(-1)^2 - 1 - 5 = -9$$
$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{Mean Value Theorem}$$
$$3c^2 - 4c + 1 = \frac{f(2) - f(-1)}{2 - (-1)} \qquad b = 2 \text{ and } a = -1$$
$$3c^2 - 4c + 1 = \frac{-3 + 9}{2 + 1} \qquad f(2) = -3 \text{ and } f(-1) = -9$$
$$3c^2 - 4c - 1 = 0 \qquad \text{Simplify.}$$
$$c = \frac{2 \pm \sqrt{7}}{3}$$
$$c = \frac{2 + \sqrt{7}}{3} \approx 1.549 \text{ and } c = \frac{2 - \sqrt{7}}{3} \approx -.215$$

Both numbers lie in the open interval $\left(-1,2\right)$, so $\frac{2 \pm \sqrt{7}}{3}$ satisfy the conclusion of the Mean Value Theorem.

Exercises - Roll's Theorem and the Mean Value Theorem

Multiple Choice Questions

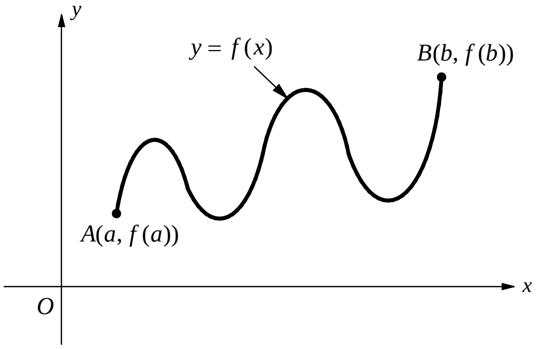
1. Let f be the function given by $f(x) = \sin(\pi x)$. What are the values of c that satisfy Roll's Theorem on the closed interval $[0, 2]$?

(A) $\frac{1}{4}$ only (B) $\frac{1}{2}$ only (C) $\frac{1}{4}$ and $\frac{1}{2}$ (D) $\frac{1}{2}$ and $\frac{3}{2}$

---

2. Let f be the function given by $f(x) = -x^3 + 3x + 2$. What are the values of c that satisfy Mean Value Theorem on the closed interval $[0, 3]$?

(A) $-\sqrt{3}$ only (B) $-\sqrt{3}$ and $\sqrt{3}$ (C) $\sqrt{3}$ only (D) 1.5 and $\sqrt{3}$

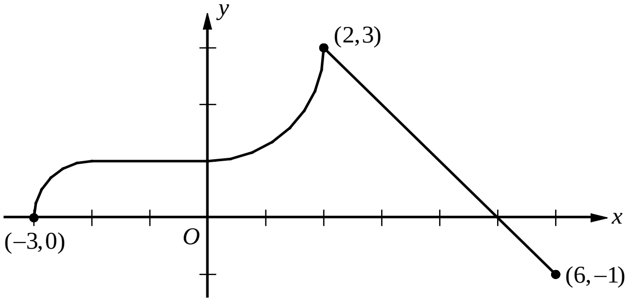


3. The figure above shows the graph of f. On the closed interval $[a, b]$, how many values of c satisfy the conclusion of the Mean Value Theorem?

(A) 2 (B) 3 (C) 4 (D) 5

4. Let f be the function given by $f(x) = \dfrac{x}{x+2}$. What are the values of c that satisfy the Mean Value Theorem on the closed interval $[-1, 2]$?

(A) -4 only (B) 0 only (C) 0 and $\dfrac{3}{2}$ (D) -4 and 0



Graph of f

5. The continuous function f is defined on the interval $-3 \le x \le 6$. The graph of f consists of two quarter circles and two line segments, as shown in the figure above. Which of the following statements must be true?

I. The average rate of change of f on the interval $-3 \le x \le 6$ is $-\dfrac{1}{9}$.

II. There is a point c on the interval $-3 < x < 6$, for which $f'(c)$ is equal to the average rate of change of f on the interval $-3 \le x \le 6$.

III. If h is the function given by $h(x) = f(\dfrac{1}{2}x)$, then $h'(6) = -\dfrac{1}{2}$.

(A) I and II only

(B) I and III only

(C) II and III only

(D) I, II, and III

Free Response Questions

t (min)	0	5	10	15	20	25	30	35	40	45	50
$v(t)$ (km/min)	1.5	1.8	2.3	2.4	1.8	1.3	0.8	0.3	0	–0.4	–1.2

6. A car drives on a straight road with positive velocity $v(t)$, in kilometers per minute at time t minutes. The table above gives selected values of $v(t)$ for $0 \le t \le 50$. The function $v(t)$ is a twice-differentiable function of t.

(a) For $0 < t < 50$, must there be a time t when $v(t) = -1$? Justify your answer.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the car could equal zero in the open interval $0 < t < 50$? Justify your answer.

3.4 The First Derivative Test and the Extreme Values of Functions

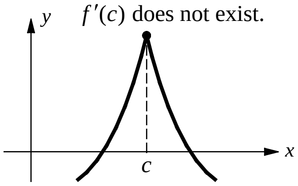
Definition of Absolute and Relative Extrema

A function f has an **absolute minimum** at c if $f(c) \le f(x)$ for all x in the domain of f.
A function f has an **absolute maximum** at c if $f(c) \ge f(x)$ for all x in the domain of f.

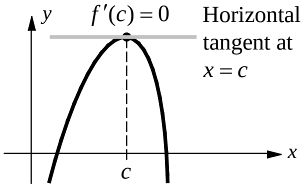
A function f has a **relative minimum** at c if $f(c) \le f(x)$ for all x in the vicinity of c.
A function f has a **relative maximum** at c if $f(c) \ge f(x)$ for all x in the vicinity of c.

Definition of a Critical Number

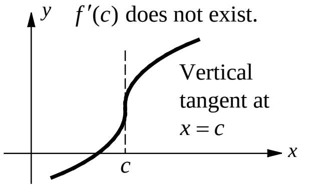
If $\boldsymbol{f'(c)} = 0$ or if $\boldsymbol{f'(c)}$ does not exist, then c is called a **critical number** of f.



c is a critical number of f.



c is a critical number of f.



c is a critical number of f.

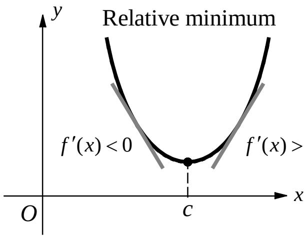
Test for Increasing and Decreasing Functions

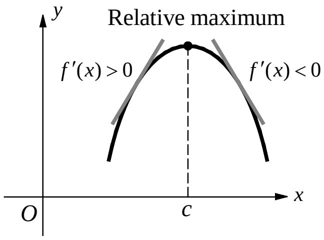
1. A function f is **increasing** on an interval if $f'(x) > 0$ on that interval.
2. A function f is **decreasing** on an interval if $f'(x) < 0$ on that interval.

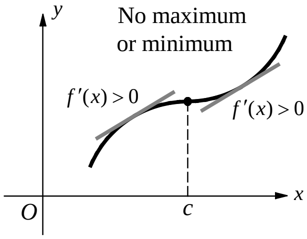
First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval.

1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative minimum of f.
2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum of f.
3. If $f'(x)$ does not change signs at c, then $f(c)$ is neither a relative minimum nor a relative maximum.



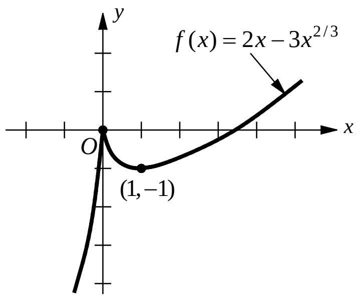




Relative Extrema Occur Only at Critical Numbers

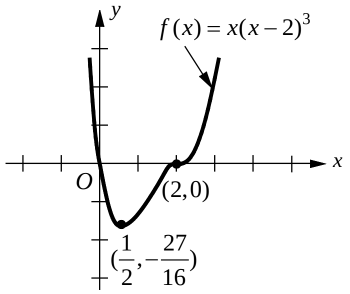
If f has a relative minimum or a relative maximum at $x = c$, then c is a critical number of f. But the converse of this theorem is not necessarily true. The critical numbers of a function need not produce relative extrema.

(a)



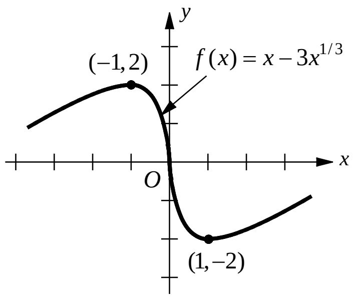
The number 0 is a critical number because $f'(0)$ does not exist, and the number 1 is a critical number because $f'(1) = 0$. The relative maximum is $f(0) = 0$ and the relative minimum is $f(1) = -1$.
f is increasing on $(-\infty, 0)$ and $(1, \infty)$.
f is decreasing on $(0,1)$.

(b)



The numbers 1/2 and 2 are critical numbers because $f'(1/2) = 0$ and $f'(2) = 0$. The absolute minimum is $f(1/2) = -27/16$. Note that the critical number $x = 2$ does not yield a relative minimum or a relative maximum. Not every critical number of a function produces a relative extrema.
f is decreasing on $(-\infty, 1/2)$ and f is increasing on $(1/2, \infty)$.

(c)



The number 0 is a critical number because $f'(0)$ does not exist. The numbers -1 and 1 are critical numbers because $f'(-1) = 0$ and $f'(1) = 0$.
The relative maximum is $f(-1) = 2$ and the relative minimum is $f(1) = -2$.
f is increasing on $(-\infty, -1)$ and $(1, \infty)$.
f is decreasing on $(-1, 1)$.

Example 1 □ Find the critical numbers of $f(x) = x^{1/3}(x-1)^2$.

Solution □ $f'(x) = x^{1/3} \cdot 2(x-1) + \frac{1}{3}x^{-2/3}(x-1)^2 = 2x^{1/3}(x-1) + \frac{1}{3x^{2/3}}(x-1)^2$

$= \frac{6x(x-1)+(x-1)^2}{3x^{2/3}} = \frac{(7x-1)(x-1)}{3x^{2/3}}$

$f'(x)$ does not exist when $x = 0$.

$f'(x) = 0$ when $x = \frac{1}{7}$ and $x = 1$.

So, the critical numbers are 0, $\frac{1}{7}$, and 1.

The Extreme Value Theorem

If f is continuous on the closed interval $[a,b]$, then f has both a minimum and maximum on the interval.

Finding Extrema on a Closed Interval

To find the extrema of a continuous function f on a closed interval $[a,b]$, use the following steps.

1. Find the values of f at the critical numbers of f in (a,b).
2. Find the values of f at the **endpoints** of the interval.
3. The least of these values is the minimum. The greatest is the maximum.

Example 2 □ Find the absolute maximum and minimum values of $f(x) = x^4 - 2x^3$ on the interval $[-1,3]$.

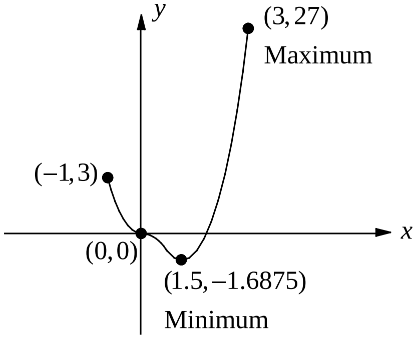
Solution □ $f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$

Since $f'(x)$ exists for all x, the only critical numbers of f occurs when $x = 0$ or $x = 3/2$.

$f(0) = 0$, $f(3/2) = -1.6875$

The values of f at the endpoints of the interval are $f(-1) = 3$ and $f(3) = 27$.

So, the absolute maximum value is $f(3) = 27$ and the absolute minimum value is $f(3/2) = -1.6875$.



Example 3 □ Let f be the function given by $f(x) = x - 2\sin x$ for $0 \le x \le 2\pi$.

(a) Find the intervals on which f is increasing and decreasing.

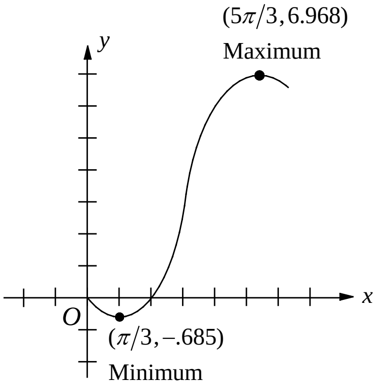
(b) Find the absolute minimum and maximum value of f on the closed interval $[0, 2\pi]$.

Solution □ $f'(x) = 1 - 2\cos x = 0$

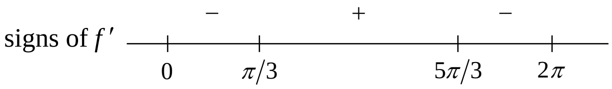
$$\Rightarrow \quad \cos x = \frac{1}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{3}, \ \frac{5\pi}{3}$$

Since there are no points for which f' does not exists, we can conclude that $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$ are the only critical numbers.



Make a diagram which shows the signs of f' .



(a) f is increasing on the interval $(\frac{\pi}{3}, \frac{5\pi}{3})$ and decreasing on the interval $(0, \frac{\pi}{3})$ and $(\frac{5\pi}{3}, 2\pi)$.

(b) Absolute maximum value is 6.968 and absolute minimum value is -0.685 .

Exercises - The First Derivative Test and the Extreme Values of Functions

Multiple Choice Questions

1. At what values of x does $f(x)=(x-1)^3(3-x)$ have the absolute maximum?

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) $\frac{5}{2}$

---

2. At what values of x does $f(x)=x-2x^{2/3}$ have a relative minimum?

(A) $\frac{64}{27}$ (B) $\frac{16}{9}$ (C) $\frac{4}{3}$ (D) 2

---

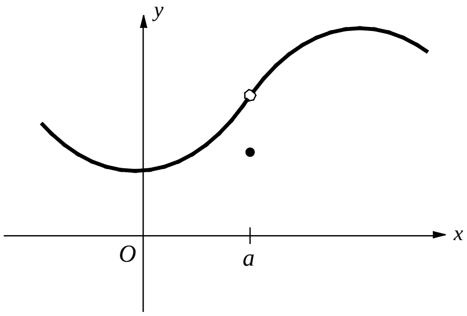
3. What is the minimum value of $f(x)=x^2\ln x$?

(A) $-e$

(B) $-\frac{1}{2e}$

(C) $-\frac{1}{e}$

(D) $-\frac{1}{\sqrt{e}}$



4. The graph of a function f is shown above. Which of the following statements about f are true?

I. $\lim_{x \to a} f(x)$ exists.
II. $x = a$ is the domain of f .
III. f has a relative minimum at $x = a$.

(A) I only
(B) I and II only
(C) I and III only
(D) I, II, and III

---

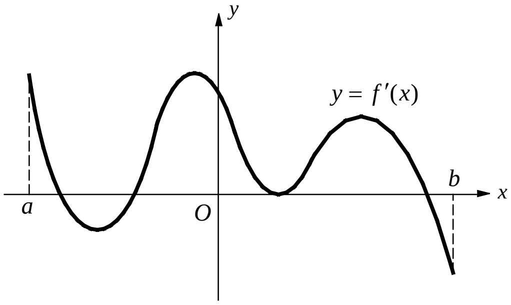
5. A polynomial $f(x)$ has a relative minimum at $(-4,2)$, a relative maximum at $(-1,5)$, a relative minimum at $(3,-3)$ and no other critical points. How many zeros does $f(x)$ have?

(A) one (B) two (C) three (D) four

---

6. At $x = 2$, which of the following is true of the function f defined by $f(x) = x^2 e^{-x}$?

(A) f has a relative maximum.
(B) f has a relative minimum.
(C) f is increasing.
(D) f is decreasing.



7. The graph of f', the derivative of f, is shown in the figure above. Which of the following describes all relative extrema of f on the open interval (a,b)?

(A) One relative maximum and two relative minima

(B) Two relative maxima and one relative minimum

(C) Two relative maxima and two relative minima

(D) Three relative maxima and two relative minim

---

8. The first derivative of a function f is given by $f'(x) = \dfrac{3\sin(2x)}{x^2}$. How many critical values does f have on the open interval $(0,10)$?

(A) four (B) five (C) six (D) seven

---

9. The function f is continuous on the closed interval $[-1,5]$ and differentiable on the open interval $(-1,5)$. If $f(-1)=4$ and $f(5)=-2$, which of the following statements could be false?

(A) There exist c, on $[-1,5]$, such that $f(c) \le f(x)$ for all x on the closed interval $[-1,5]$.

(B) There exist c, on $(-1,5)$, such that $f(c)=0$.

(C) There exist c, on $(-1,5)$, such that $f'(c)=0$.

(D) There exist c, on $(-1,5)$, such that $f(c)=2$.

x	-4	-3	-2	-1	0	1	2	3	4	5
$f'(x)$	-1	-2	0	1	2	1	0	-2	-3	-1

10. The derivative, f', of a function f is continuous and has exactly two zeros on $[-4,5]$. Selected values of $f'(x)$ are given in the table above. On which of the following intervals is f increasing?

(A) $-3 \le x \le 0$ or $4 \le x \le 5$

(B) $-2 \le x \le 0$ or $4 \le x \le 5$

(C) $-3 \le x \le 2$ only

(D) $-2 \le x \le 2$ only

---

11. The height h, in meters, of an object at time t is given by $h(t) = t^3 - 6t^2 + 20t$. What is the height of the object, in meters, at the instant it reaches its maximum upward velocity?

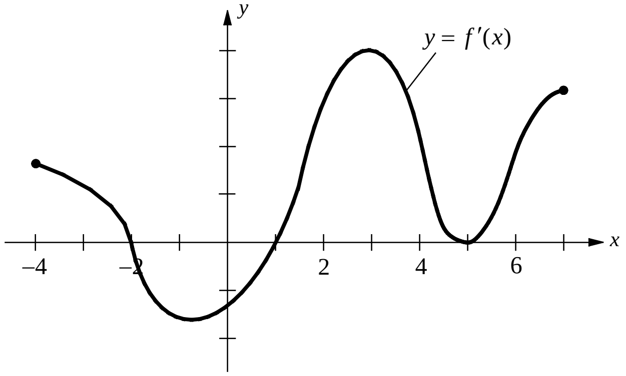
(A) 24 (B) 28 (C) 33 (D) 42

---

12. Which of the following is an equation of a curve that intersects at right angles every curve of the family $y = x^2 + c$, where c is a constant?

(A) $y = -\frac{1}{x}$ (B) $y = -x^2$ (C) $y = \frac{1}{\ln x}$ (D) $y = -\frac{1}{2}\ln x$

Free Response Questions



13. The figure above shows the graph of f', the derivative of the function f, for $-4 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -1$, $x = 3$, and $x = 5$.

(a) Find all values of x, for $-4 \le x \le 7$, at which f attains a relative minimum. Justify your answer.

(b) Find all values of x, for $-4 \le x \le 7$, at which f attains a relative maximum. Justify your answer.

(c) At what value of x, for $-4 \le x \le 7$, does f attain its absolute maximum. Justify your answer.

3.5 The Second Derivative Test

Test for Concavity

1. If $f''(x) > 0$ for all x in an open interval I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in an open interval I, then the graph of f is concave downward on I.

Definition of Points of Inflection

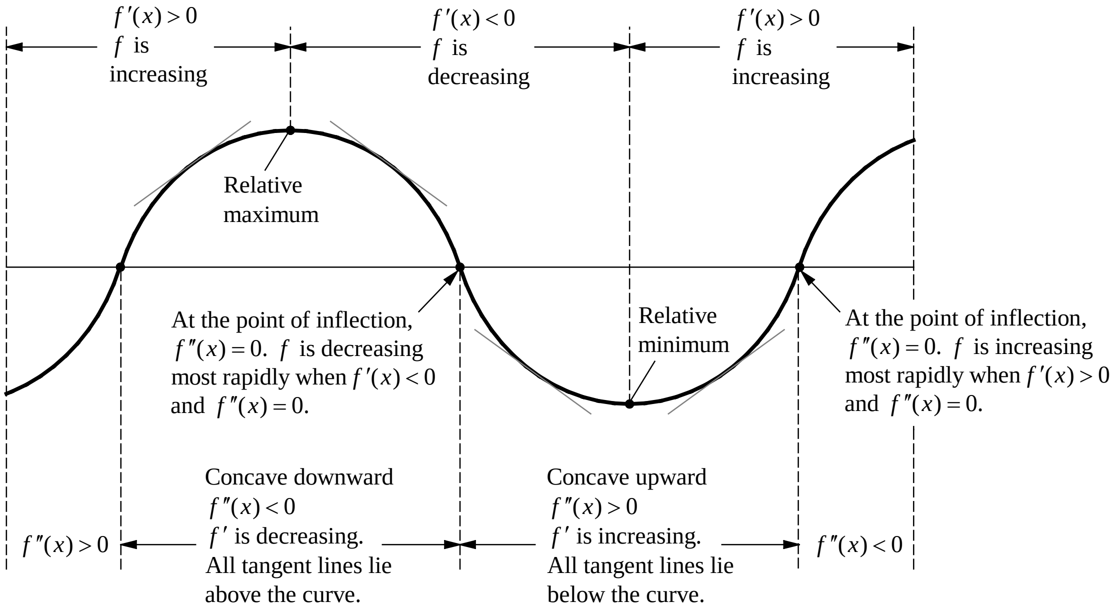
A point P on a curve $y = f(x)$ is called a point of inflection if f is continuous on P and the curve changes from concave upward to concave downward or vice versa.

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(x) = 0$ or f is not differentiable at $x = c$.

Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$, the test fails. Use the First Derivative Test.



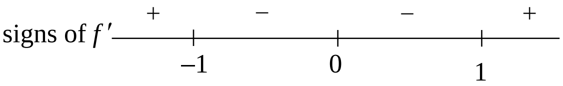
Example 1 □ Let f be the function given by $f(x) = 3x^5 - 5x^3 + 3$.

(a) Find the relative maximum and minimum value of f .
(b) Find the intervals on which f is increasing and decreasing.
(c) Find the x-coordinate of each inflection points on the graph of f .
(d) Find the intervals on which f is concave upwards and concave downwards.

Solution □ (a) Use the first derivative test to find the relative extreme values.

$f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$

$f'(x) = 0 \Rightarrow x = -1,\ 0,\ 1$



Since f' changes from positive to negative at $x = -1$,
$f(-1) = 5$ is a relative maximum.

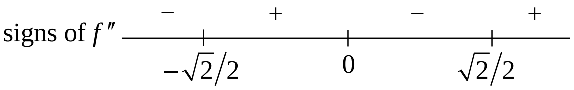
The sign of f' does not change at $x = 0$, so there is no maximum or minimum.

Since f' changes from negative to positive at $x = 1$,
$f(1) = 1$ is a relative minimum.

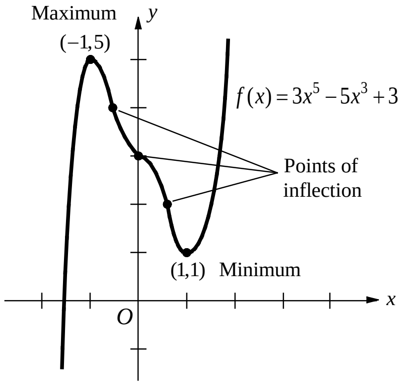
(b) f is increasing on $(\infty, -1)$ and $(1, \infty)$.
f is decreasing on $(-1, 1)$.

(c) $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

$f''(x) = 0 \Rightarrow x = -\dfrac{\sqrt{2}}{2},\ 0,\ \dfrac{\sqrt{2}}{2}$



f'' changes from negative to positive at $x = -\sqrt{2}/2$, from positive to negative at $x = 0$, and from negative to positive at $x = \sqrt{2}/2$, so, $-\sqrt{2}/2$, 0, and $\sqrt{2}/2$ are the x-coordinate of the points of inflection.



(d) The graph of f is concave downward on $(-\infty, -\sqrt{2}/2)$ and $(0, \sqrt{2}/2)$.

The graph of f is concave upward on $(-\sqrt{2}/2, 0)$ and $(\sqrt{2}/2, \infty)$.

Exercises - The Second Derivative Test

Multiple Choice Questions

1. The graph of $y = x^4 - 2x^3$ has a point of inflection at

(A) $(0,0)$ only

(B) $(0,0)$ and $(1,-1)$

(C) $(1,-1)$ only

(D) $(0,0)$ and $(\frac{3}{2},-\frac{27}{16})$

---

2. If the graph of $y = ax^3 - 6x^2 + bx - 4$ has a point of inflection at $(2,-2)$, what is the value of $a+b$?

(A) -2 (B) 3 (C) 6 (D) 10

---

3. At what value of x does the graph of $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ have a point of inflection?

(A) $\frac{1}{2}$ (B) 1 (C) 3 (D) $\frac{7}{2}$

---

4. The graph of $y = 3x^5 - 40x^3 - 21x$ is concave up for

(A) $x < 0$

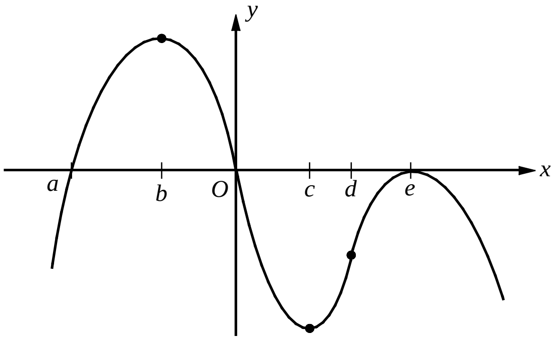
(B) $x > 2$

(C) $x < 0$ or $0 < x < 2$

(D) $-2 < x < 0$ or $x > 2$

5. Let f be a twice differentiable function such that $f(1) = 7$ and $f(3) = 12$. If $f'(x) > 0$ and $f''(x) < 0$ for all real numbers x, which of the following is a possible value for $f(5)$?

(A) 16 (B) 17 (C) 18 (D) 19



Graph of f''

6. The second derivative of the function f is given by $f''(x) = x(x+a)(x-e)^2$ and the graph of f'' is shown above. For what values of x does the graph of f have a point of inflection?

(A) b and c (B) b, c and e (C) b, c and d (D) a and 0

---

7. The first derivative of the function f is given by $f'(x) = (x^3 + 2)\, e^x$. What is the x-coordinate of the inflection point of the graph of f?

(A) –3.196 (B) –1.260 (C) –1 (D) 0

---

8. Let f be a twice differentiable function with $f'(x) > 0$ and $f''(x) > 0$ for all x, in the closed interval $[2,8]$. Which of the following could be a table of values for f?

(A)

x	$f(x)$
2	–1
4	3
6	6
8	8

(B)

x	$f(x)$
2	–1
4	2
6	5
8	8

(C)

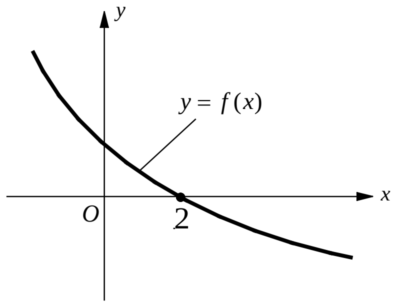
x	$f(x)$
2	–1
4	1
6	4
8	8

(D)

x	$f(x)$
2	8
4	4
6	1
8	–1

9. Let f be the function given by $f(x) = 3\sin(\frac{2x}{3}) - 4\cos(\frac{3x}{4})$. For $0 \le x \le 7$, f is increasing most rapidly when $x =$

(A) 0.823 (B) 1.424 (C) 1.571 (D) 3.206



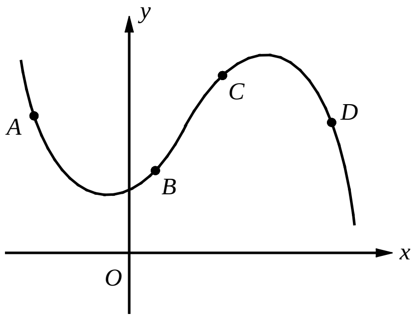
10. The graph of a twice differentiable function f is shown in the figure above. Which of the following is true?

(A) $f''(2) < f(2) < f'(2)$

(B) $f'(2) < f''(2) < f(2)$

(C) $f'(2) < f(2) < f''(2)$

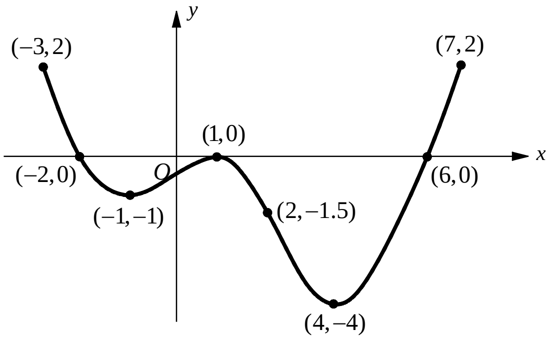
(D) $f(2) < f'(2) < f''(2)$



11. At which of the five points on the graph in the figure above is $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$?

(A) A (B) B (C) C (D) D

Free Response Questions



Graph of f'

12. The figure above shows the graph of f', the derivative of the function f, on the closed interval $[-3, 7]$. The graph of f' has horizontal tangent lines at $x = -1$, $x = 1$, and $x = 4$. The function f is twice differentiable and $f(-2) = \frac{1}{2}$.

(a) Find the x-coordinates of each of the points of inflection of the graph of f. Justify your answer.

(b) At what value of x does f attain its absolute minimum value on the closed interval $[-3, 7]$.

(c) Let h be the function defined by $h(x) = x^2 f(x)$. Find an equation for the line tangent to the graph of h at $x = -2$.

13. Let f be a twice differentiable function with $f(1) = -1$, $f'(1) = 2$, and $f''(1) = 0$. Let g be a function whose derivative is given by $g'(x) = x^2\left[2f(x) + f'(x)\right]$ for all x.

(a) Write an equation for the line tangent to the graph of f at $x = 1$.

(b) Does the graph of f have a point of inflection when $x = 1$? Explain.

(c) Given that $g(1) = 3$, write an equation for the line tangent to the graph of g at $x = 1$.

(d) Show that $g''(x) = 4x\,f(x) + 2x(x+1)\,f'(x) + x^2 f''(x)$. Does g have a local maximum or minimum at $x = 1$? Explain your reasoning.

3.6 Curves of f, f', f'' and Curve Sketching

The following guidelines provide the information you need to sketch a curve $y = f(x)$.
Not every item is relevant to every function.

Guidelines for Sketching a Curve

1. **Domain**
   Identify the domain of f, that is, the set of values of x for which $f(x)$ is defined.

2. **Intercepts**
   To find the *x*-intercepts, let $y = 0$ and solve for x.
   To find the *y*-intercepts, let $x = 0$ and solve for y.

3. **Symmetry**
   If $f(-x) = f(x)$, then f is an **even function** and the curve is **symmetric about the *y*-axis**.
   If $f(-x) = -f(x)$, then f is an **odd function** and the curve is **symmetric about the origin**.
   When replacing y by $-y$ yields an equivalent equation, the curve is **symmetric about the *x*-axis**.

4. **Asymptote**
   If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then the line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$.
   If one of the following is true: $\lim\limits_{x\to a^+} f(x) = \infty$, $\lim\limits_{x\to a^-} f(x) = \infty$, $\lim\limits_{x\to a^+} f(x) = -\infty$ or $\lim\limits_{x\to a^-} f(x) = -\infty$, then the line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$.

5. **Intervals of Increasing or Decreasing**
   Compute $f'(x)$. If $f'(x) > 0$ f is increasing and if $f'(x) < 0$ f is decreasing.

6. **Relative Maximum and Relative Minimum Values**
   Find the critical numbers then use the first derivative test.

7. **Points of Inflection and Concavity**
   Compute $f''(x)$. Inflection point occur where the direction of concavity changes. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$.

Example 1 □ Sketch the graph of $f(x) = 2xe^{-x^2}$.

Solution □ 1. Domain of f is $(-\infty, \infty)$.

2. x- intercepts: $(0,0)$ y- intercepts: $(0,0)$

3. $f(-x) = 2(-x)e^{-(-x)^2} = -2xe^{-x^2} = -f(x)$, so the curve is symmetric about the origin.

4. $\lim\limits_{x\to-\infty} 2xe^{-x^2} = \lim\limits_{x\to-\infty} \dfrac{2x}{e^{x^2}} = 0$, $\lim\limits_{x\to\infty} 2xe^{-x^2} = \lim\limits_{x\to\infty} \dfrac{2x}{e^{x^2}} = 0$, so
   $y = 0$ is a horizontal asymptote.

5. $f'(x) = 2(-2x^2e^{-x^2} + e^{-x^2}) = 2e^{-x^2}(1-2x^2) = \dfrac{2(1-2x^2)}{e^{x^2}}$

$f'(x) = 0 \;\Rightarrow\; x = \pm\dfrac{1}{\sqrt{2}}$

signs of f'

$-$		$+$		$-$
	$-1/\sqrt{2}$		$1/\sqrt{2}$	

f is increasing on $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$.

f is decreasing on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, \infty\right)$.

6. Relative maximum value $f(1/\sqrt{2}) = \sqrt{2/e}$

Relative minimum value $f(-1/\sqrt{2}) = -\sqrt{2/e}$

7. $f''(x) = 4xe^{-x^2}(2x^2-3)$

$f''(x) = 0 \;\Rightarrow\; x = 0,\; x = \pm\sqrt{3/2}$

signs of f''

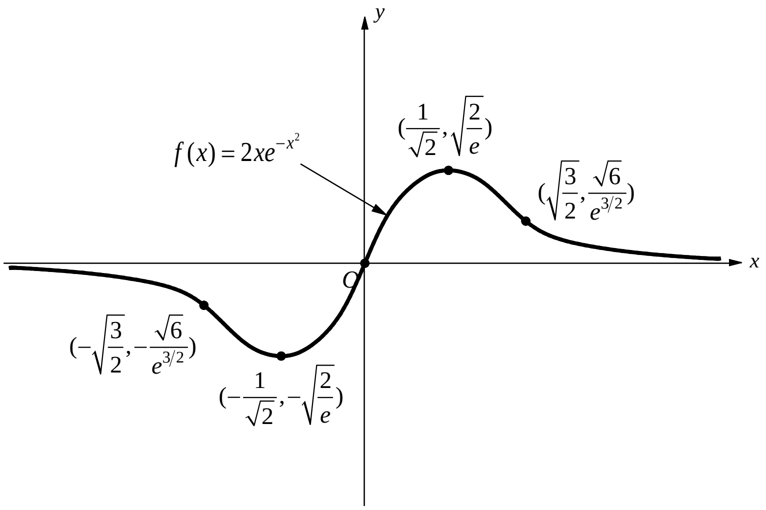
$-$		$+$		$-$		$+$
	$-\sqrt{3/2}$		0		$\sqrt{3/2}$	

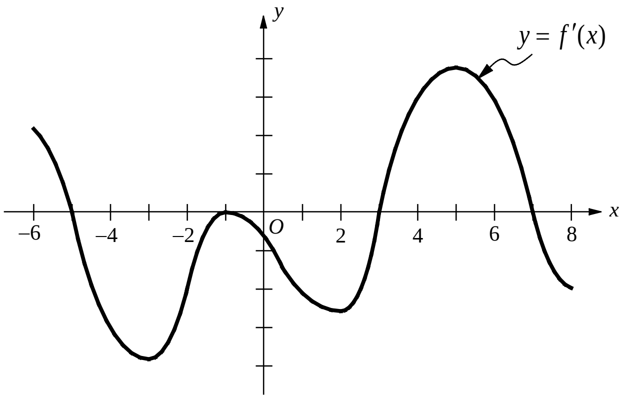
Points of inflection are $(0,0)$, $\left(-\sqrt{3/2}, -\sqrt{6}/e^{3/2}\right)$ and $\left(\sqrt{3/2}, \sqrt{6}/e^{3/2}\right)$.

f is concave upward on $\left(-\sqrt{3/2}, 0\right)$ and $\left(\sqrt{3/2}, \infty\right)$.

f is concave downward on $\left(-\infty, -\sqrt{3/2}\right)$ and $\left(0, \sqrt{3/2}\right)$.

Use the above information to sketch the graph of $f(x) = 2xe^{-x^2}$.





Note: This is the graph of f', not the graph of f.

Example 2 □ The figure above shows the graph of f'. The domain of f is the set of all real numbers x such that $-6 \le x \le 8$.

(a) For what values of x does f have a relative maximum?

(b) For what values of x does f have a relative minimum?

(c) For what values of x does the graph of f have a horizontal tangent?

(d) For what values of x is the graph of f concave upward?

(e) For what values of x is the graph of f concave downward?

(f) Suppose that $f(0) = 1$. Sketch a possible graph of f.

Solution □ (a) f has relative maximum at $x = -5$ and 7 because f' changes from positive to negative at $x = -5$ and 7.

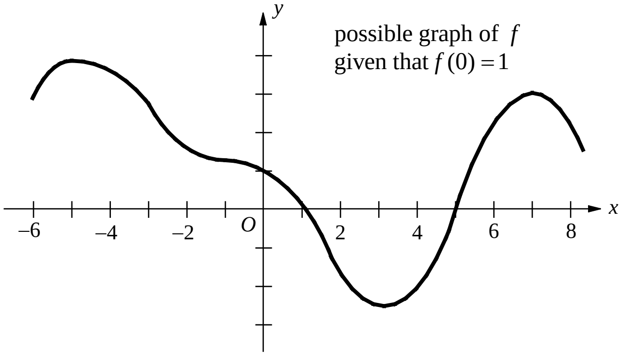
(b) f has relative minimum at $x = 3$ because f' changes from negative to positive at $x = 3$.

(c) The graph of f has horizontal tangent at $x = -5$, -1, 3, and 7 because $f'(x) = 0$ at these points.

(d) The graph of f is concave upward on $(-3, -1)$ and $(2, 5)$ because f' is increasing on these intervals.

(e) The graph of f is concave downward on $(-6, -3)$, $(-1, 2)$ and $(5, 8)$ because f' is decreasing on these intervals.

(f)

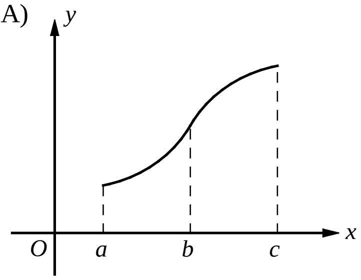


Exercises - Curves of f , f' , f'' and Curve Sketching

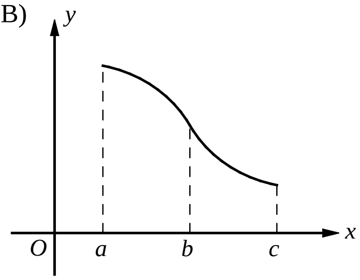
Multiple Choice Questions

1. If f is a function such that $f' > 0$ for $a < x < c$, $f'' < 0$ for $a < x < b$, and $f'' > 0$ for $b < x < c$ which of the following could be the graph of f ?

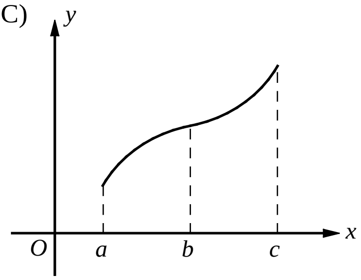
(A)



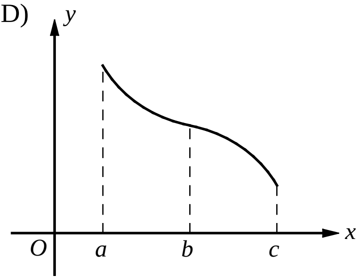
(B)



(C)



(D)



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2. The graph of $f(x) = xe^{-x^2}$ is symmetric about which of the following

I. The x-axis
II. The y-axis
III. The origin

(A) I only (B) II only (C) III only (D) II and III only

3. Let f be the function given by $f(x) = \dfrac{-3x^2}{\sqrt{3x^4+1}}$. Which of the following is the equation of horizontal asymptote of the graph of f ?

(A) $y = -3$ (B) $y = -\sqrt{3}$ (C) $y = \sqrt{3}$ (D) $y = 3$

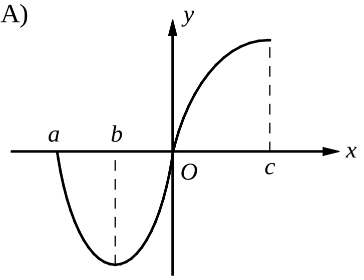
---

4. Let f be a function that is continuous on the closed interval $[a, c]$, such that the derivative of function f has the properties indicated on the table below.

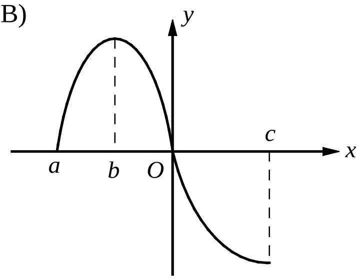
x	$a < x < b$	b	$b < x < 0$	0	$0 < x < c$
$f'(x)$	$-$	0	$+$	3	$+$
$f''(x)$	$+$	$+$	$+$	0	$-$

Which of the following could be the graph of f ?

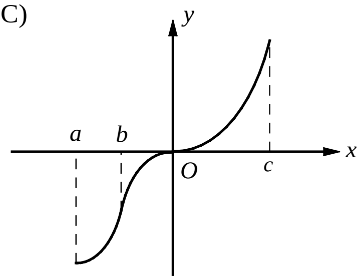
(A)



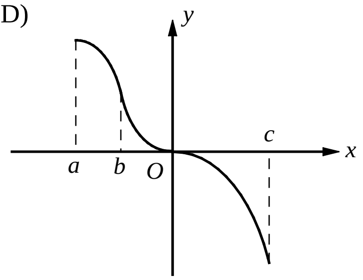
(B)

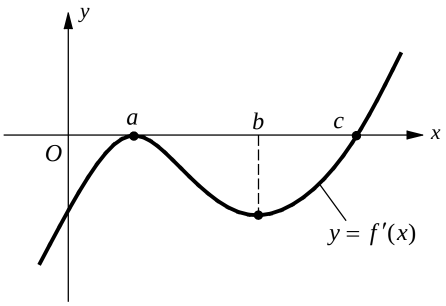


(C)



(D)





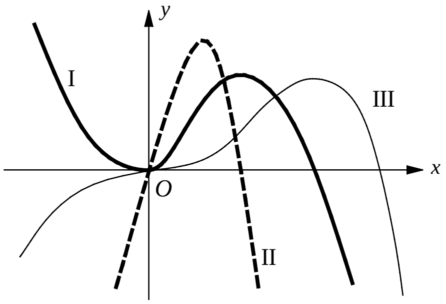
5. The graph of f', the derivative of function f, is shown above. If f is a twice differentiable function, which of the following statements must be true?

   I. $f(c) > f(a)$

   II. The graph of f is concave up on the interval $b < x < c$.

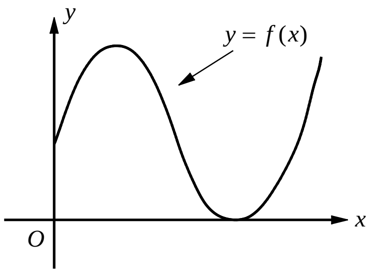
   III. f has a relative minimum at $x = c$.

   (A) I only (B) II only (C) III only (D) II and III only



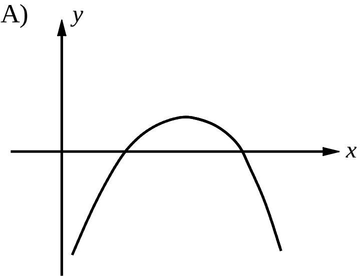
6. Three graphs labeled I, II, and III are shown above. They are the graphs of f, f', and f''. Which of the following correctly identifies each of the three graphs?

	f	f'	f''
(A)	I	II	III
(B)	II	I	III
(C)	III	I	II
(D)	I	III	II

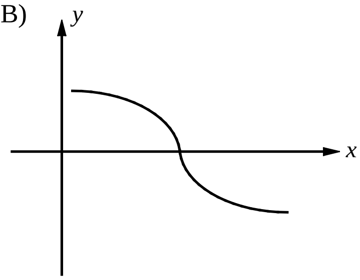


7. The graph of f is shown in the figure above. Which of the following could be the graph of f'?

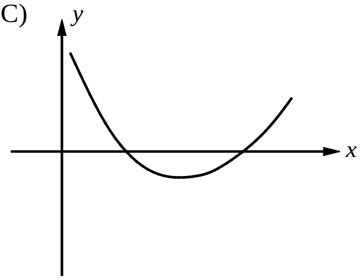
(A)



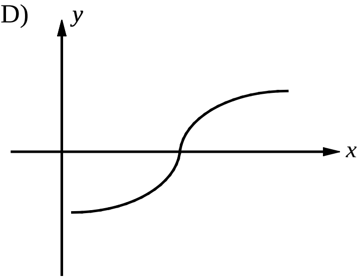
(B)



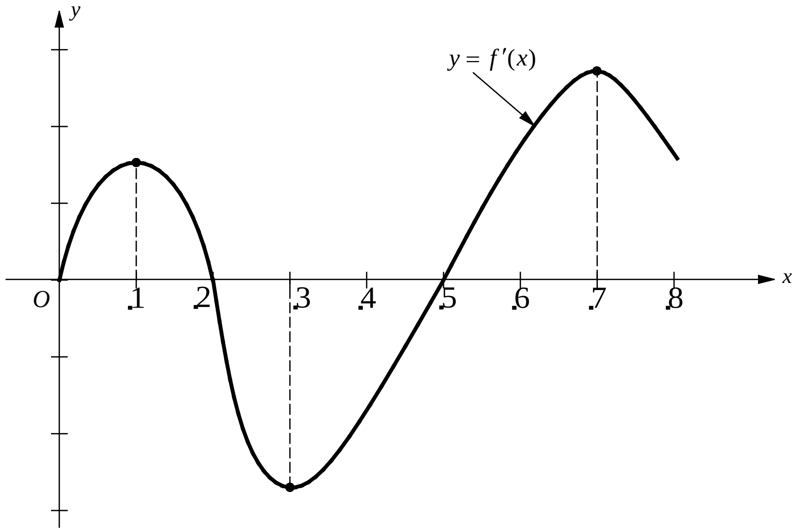
(C)



(D)



Free Response Questions



8. The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $0 \le x \le 8$.

(a) For what values of x does the graph of f have a horizontal tangent?

(b) On what intervals is f increasing?

(c) On what intervals is f concave upward?

(d) For what values of x does the graph of f have a relative maximum?

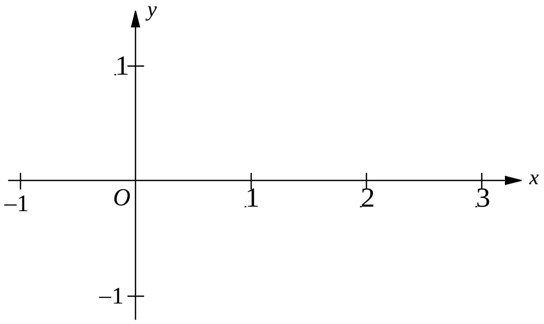
(e) Find the x-coordinate of each inflection point on the graph of f.

x	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	1	$+$	0	$-$	-1	$-$	0	$+$
$f'(x)$	-4	$-$	0	$-$	DNE	$+$	1	$+$
$f''(x)$	2	$+$	0	$-$	DNE	$-$	0	$+$

9. Let f be a function that is continuous on the interval $-1 \leq x < 3$. The function is twice differentiable except at $x = 1$. The function f and its derivatives have the properties indicated in the table above.

(a) For $-1 < x < 3$, find all values of x at which f has a relative extrema. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

(b) On the axis provided, sketch the graph of a function that has all the given characteristics of f.



(c) Let h be the function defined by $h'(x) = f(x)$ on the open interval $-1 < x < 3$. For $-1 < x < 3$, find all values of x at which h has a relative extremum. Determine whether h has a relative maximum or a relative minimum at each of these values. Justify your answer.

(d) For the function h, find all values of x, for $-1 < x < 3$, at which h has a point of inflection. Justify your answer.

3.7 Optimization Problems

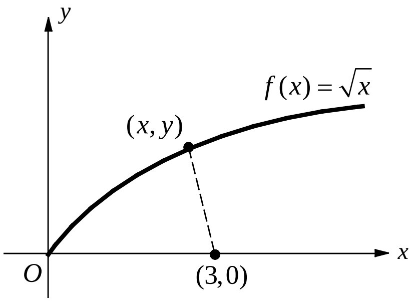
When you are given a word problem that asks for the maximum or minimum value of a certain quantity, you have to translate the word problem into a mathematical equation for the quantity that is to be maximized or minimized. The differential calculus is a powerful tool for solving these kind of problems.

Guidelines for Solving Optimization Problems

1. Read the problem carefully until you understand it.
2. In most problems it is useful to draw a picture. Label it with the quantities given in the problem.
3. Assign a variable to the unknown quantity and write an equation for the quantity that is to be maximized (or minimized), since this equation will usually involve two or more variables.
4. Use the given information to find relationships between these variables. Use these equations to eliminate all but one variable in the equation.
5. Use the first and second derivatives tests to find the critical points.

Example 1 □ Find the points on the curve $f(x) = \sqrt{x}$ that is nearest to the point $(3, 0)$.

Solution □ Sketch the graph of $f(x) = \sqrt{x}$.

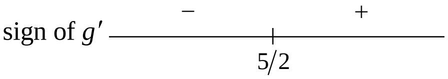


The distance between the point $(3, 0)$ and a point (x, y) on the graph of $f(x) = \sqrt{x}$ is given by

$$\begin{aligned} d &= \sqrt{(x-3)^2 + (y-0)^2} \\ &= \sqrt{(x-3)^2 + (\sqrt{x}-0)^2} \\ &= \sqrt{x^2 - 6x + 9 + x} \\ &= \sqrt{x^2 - 5x + 9} \ . \end{aligned}$$

We need only find the critical numbers of $g(x) = x^2 - 5x + 9$, because d is smallest when the expression inside the radical is smallest.

$$g'(x) = 2x - 5 = 0 \quad \Rightarrow \quad x = 5/2$$



The first derivative test verifies that $x = 5/2$ yields a relative minimum.

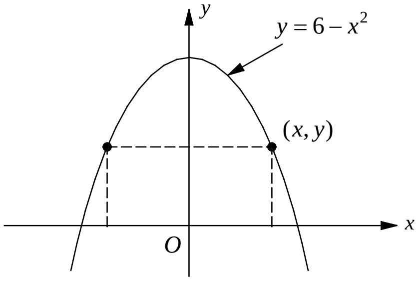
The point on the curve closest to $(3, 0)$ is $\left(5/2, \sqrt{5/2}\right)$.

Exercises - Optimization Problems

Multiple Choice Questions

1. The point on the curve $y = 2 - x^2$ nearest to $(3,2)$ is

(A) $(0,2)$ (B) $(\frac{1}{2},\frac{7}{4})$ (C) $(\frac{3}{4},\frac{23}{16})$ (D) $(1,1)$



2. What is the area of the largest rectangle that has its base on the x-axis and its other two vertices on the parabola $y = 6 - x^2$?

(A) $8\sqrt{2}$ (B) $6\sqrt{2}$ (C) $4\sqrt{3}$ (D) $3\sqrt{2}$

---

3. If $y = \frac{1}{\sqrt{x}} - \sqrt{x}$, what is the maximum value of the product of xy ?

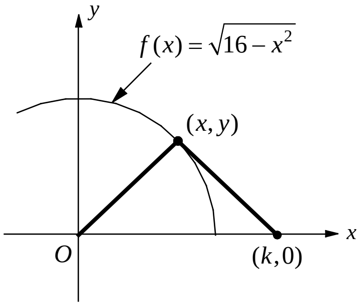
(A) $\frac{1}{9}$ (B) $\frac{\sqrt{3}}{9}$ (C) $\frac{2\sqrt{3}}{9}$ (D) $\frac{2}{3}$

---

4. If the maximum value of the function $y = \frac{\cos x - m}{\sin x}$ is at $x = \frac{\pi}{4}$, what the value of m ?

(A) $-\sqrt{2}$ (B) $\sqrt{2}$ (C) -1 (D) 1

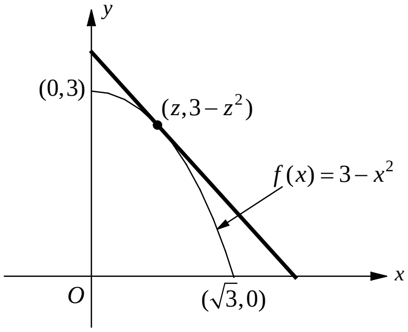
Free Response Questions



5. Let $f(x)=\sqrt{16-x^2}$. An isosceles triangle, whose base is the line segment from $(0,0)$ to $(k,0)$, where $k>0$, has its vertex on the graph of f as shown in the figure above.

   (a) Find the area of the triangle in terms of k .

   (b) For what values of k does the triangle have a maximum area?



6. The figure above shows the graph of the function $f(x)=3-x^2$. For $0<z<\sqrt{3}$, let $A(z)$ be the area of the triangle formed by the coordinate axes and the line tangent to the graph of f at the point $(z,3-z^2)$.

   (a) Find the equation of the line tangent to the graph of f at the point $(z,3-z^2)$.

   (b) For what values of z does the triangle bounded by the coordinate axis and tangent line have a minimum area?

3.8 Tangent Line Approximation and Differentials

The line tangent to a curve at a point is the line that best approximates the curve near that point. An equation for the tangent line at the point $\left(c, f(c)\right)$ is given by

$y - f(c) = f'(c)(x-c)$ or $y = f(c) + f'(c)(x-c)$

and the approximation

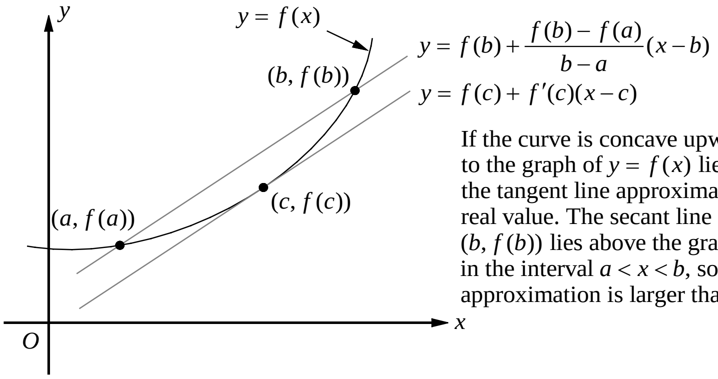
$$\boldsymbol{f(x) \approx f(c) + f'(c)(x-c)}$$

is called the **tangent line approximation** of f at c.

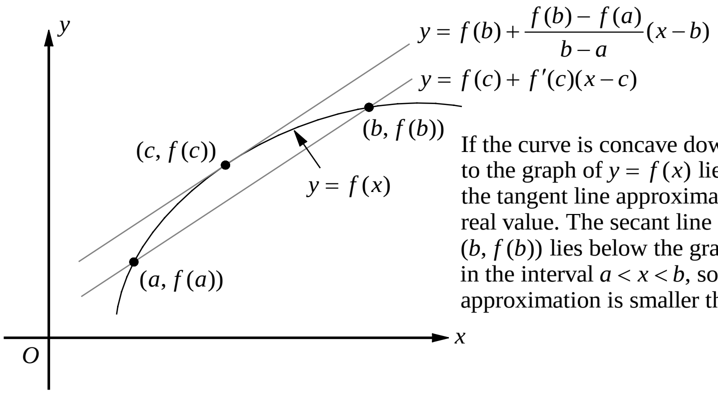
The linear function

$$\boldsymbol{L(x) = f(c) + f'(c)(x-c)}$$

is called the **linearization** of f at c.



If the curve is concave upward the line tangent to the graph of $y = f(x)$ lies below the graph, so the tangent line approximation is smaller than the real value. The secant line connecting $(a, f(a))$ and $(b, f(b))$ lies above the graph of $y = f(x)$ for all x in the interval $a < x < b$, so the secant line approximation is larger than the real value.



If the curve is concave downward the line tangent to the graph of $y = f(x)$ lies above the graph, so the tangent line approximation is larger than the real value. The secant line connecting $(a, f(a))$ and $(b, f(b))$ lies below the graph of $y = f(x)$ for all x in the interval $a < x < b$, so the secant line approximation is smaller than the real value.

Definition of Differentials

Let $y = f(x)$ be a differentiable function. The **differential** $\boldsymbol{dx}$ is an independent variable. The **differential** $\boldsymbol{dy}$ is

$$\boldsymbol{dy = f'(x)dx}\ .$$

Estimating with Differentials

Differentials can be used to approximate function values.

$$\boldsymbol{f(x + \Delta x) \approx f(x) + dy \approx f(x) + f'(x)dx}$$

Example 1 □ (a) Find the tangent line approximation of $f(x)=\sqrt{x-1}$ at $c=5$ and approximate the number $\sqrt{3.95}$.

(b) Find the tangent line approximation of $f(x)=\tan x$ at $c=\pi/4$ and approximate the number $\tan 47^\circ$.

Solution □ (a) $f(x)=\sqrt{x-1} \Rightarrow \quad f(5)=\sqrt{5-1}=2$

$$f'(x)=\frac{1}{2\sqrt{x-1}} \quad \Rightarrow \quad f'(5)=\frac{1}{2\sqrt{5-1}}=\frac{1}{4}$$

$$y=2+\frac{1}{4}(x-5) \qquad \text{Tangent line approximation}$$

$$y=\frac{1}{4}x+\frac{3}{4} \qquad \text{Simplify.}$$

$$\sqrt{x-1}\approx\frac{1}{4}x+\frac{3}{4} \qquad f(x)\approx f(c)+f'(c)(x-c)$$

$$\sqrt{3.95}=\sqrt{4.95-1} \qquad x=4.95$$

$$\approx\frac{1}{4}(4.95)+\frac{3}{4}=1.9875$$

(b) $f(x)=\tan x \quad \Rightarrow \quad f(\frac{\pi}{4})=\tan(\frac{\pi}{4})=1$

$$f'(x)=\sec^2 x \quad \Rightarrow \quad f'(\frac{\pi}{4})=\sec^2(\frac{\pi}{4})=2$$

$$y=1+2(x-\frac{\pi}{4}) \qquad \text{Tangent line approximation}$$

$$y=2x-\frac{\pi}{2}+1 \qquad \text{Simplify.}$$

$$\tan x\approx 2x-\frac{\pi}{2}+1 \qquad f(x)\approx f(c)+f'(c)(x-c)$$

$$\tan 47^\circ \approx 2(47^\circ\cdot\frac{\pi}{180})-\frac{\pi}{2}+1 \qquad x=47^\circ \text{ and } 47^\circ=47\cdot\frac{\pi}{180} \text{ radian}$$

$$\approx 1.0698$$

Exercises - Tangent Line Approximation and Differentials

Multiple Choice Questions

1. For small values of h, the function $h(x) = \sqrt[3]{8+h}$ is best approximated by which of the following?

(A) $\frac{h}{12}$ (B) $2-\frac{h}{12}$ (C) $2+\frac{h}{12}$ (D) $3+\frac{h}{12}$

---

2. The approximate value of $y = \sqrt{1-\sin x}$ at $x = -0.1$, obtained from the line tangent to the graph at $x = 0$, is

(A) 0.9 (B) 0.95 (C) 1.01 (D) 1.05

---

3. Let $y = x^2 \ln x$. When $x = e$ and $dx = 0.1$, the value of dy is

(A) $\frac{e}{10}$ (B) $\frac{e}{5}$ (C) $\frac{3e}{10}$ (D) $\frac{2e}{5}$

---

4. Let f be a differentiable function such that $f(2) = \frac{5}{2}$ and $f'(2) = \frac{1}{2}$. If the line tangent to the graph of f at $x = 2$ is used to find an approximation of a zero of f, that approximation is

(A) -3 (B) -2.4 (C) -1.8 (D) -1.2

5. The approximate value of $y = \frac{1}{\sqrt{x}}$ at $x = 4.1$, obtained from the line tangent to the graph at $x = 4$ is

(A) $\frac{39}{80}$ (B) $\frac{79}{160}$ (C) $\frac{1}{2}$ (D) $\frac{81}{160}$

---

6. Let f be the function given by $f(x) = x^2 - 4x + 5$. If the line tangent to the graph of f at $x = 1$ is used to find an approximate value of f, which of the following is the greatest value of x for which the error resulting from this tangent line approximation is less than 0.5?

(A) 1.5 (B) 1.6 (C) 1.7 (D) 1.8

---

7. The linear approximation to the function f at $x = a$ is $y = \frac{1}{2}x - 3$. What is the value of $f(a) + f'(a)$ in terms of a?

(A) $a - 4$

(B) $a - \frac{5}{2}$

(C) $\frac{1}{2}a - 4$

(D) $\frac{1}{2}a - \frac{5}{2}$

Free Response Questions

8. Let f be the function given by $f(x) = \dfrac{2}{e^{\sin x} + 1}$.

(a) Write an equation for the line tangent to the graph of f at $x = 0$.

(b) Using the tangent line to the graph of f at $x = 0$, approximate $f(0.1)$.

(c) Find $f^{-1}(x)$.

x	-2	0	1	3	6
$f(x)$	-1	-4	-3	0	7

9. Let f be a twice differentiable function such that $f'(3) = \frac{9}{5}$. The table above gives values of f for selected points in the closed interval $-2 \le x \le 6$.

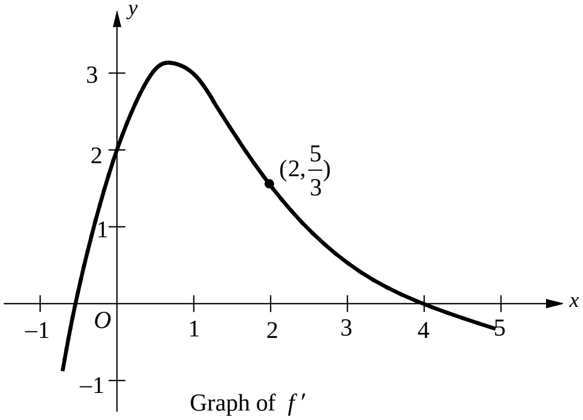
(a) Estimate $f'(0)$. Show the work that leads to your answer.

(b) Write an equation for the line tangent to the graph of f at $x = 3$.

(c) Write an equation of the secant line for the graph of f on $1 \le x \le 6$.

(d) Suppose $f''(x) > 0$ for all x in the closed interval $1 \le x \le 6$. Use the line tangent to the graph of f at $x = 3$ to show $f(5) \ge \frac{18}{5}$.

(e) Suppose $f''(x) > 0$ for all x in the closed interval $1 \le x \le 6$. Use the secant line for the graph of f on $1 \le x \le 6$ to show $f(5) \le 5$.



Graph of f'

10. Let f be twice differentiable function on the interval $-1 < x < 5$ with $f(1) = 0$ and $f(2) = 3$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x- axis at $x = -0.5$ and $x = 4$. Let h be the function given by $h(x) = f(\sqrt{x+1})$.

(a) Write an equation for the line tangent to the graph of h at $x = 3$.

(b) The second derivative of h is $h''(x) = \frac{1}{4}\left[\frac{\sqrt{x+1}f''(\sqrt{x+1}) - f'(\sqrt{x+1})}{(x+1)^{3/2}}\right]$. Is $h''(3)$ positive, negative, or zero? Justify your answer.

(c) Suppose $h''(x) < 0$ for all x in the closed interval $0 \le x \le 3$. Use the line tangent to the graph of h at $x = 3$ to show $h(2) \le \frac{31}{12}$. Use the secant line for the graph of h on $0 \le x \le 3$ to show $h(2) \ge 2$.

Chapter 4
Integration

4.1 Antiderivatives and Indefinite Integrals

Definition of an Antiderivative

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x on I .

Representation Antiderivatives

If F is an antiderivative of f on an interval I , then $F(x)+C$ represents the most general antiderivative of f on I , where C is a constant.

Example 1 □ Find an antiderivative for each of the following functions.

a. $f(x)=3x^2$
b. $g(x)=\cos x+3$

Solution □ a. $F(x)=x^3+C$ Derivative of x^3 is $3x^2$.

b. $G(x)=\sin x+3x+C$ Derivative of $\sin x$ is $\cos x$, and derivative of $3x$ is 3.

Definition of Indefinite Integral

The set of all antiderivatives of f is the indefinite integral of f with respect to x denoted by

$$\int f(x)dx \, .$$

Thus $\int f(x)dx = F(x)+C$ means $F'(x) = f(x)$.

Example 2 □ Find the general solution of $F'(x)=\sec^2 x$.

Solution □ $F(x)=\int \sec^2 x = \tan x + C$ Derivative of $\tan x$ is $\sec^2 x$.

Table of Indefinite Integrals

$\int k\ dx = kx + C$ $\int k\,f(x)\ dx = k\int f(x)\ dx$

$\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

$\int x^n\ dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$ Power Rule

$\int e^x\ dx = e^x + C$

$\int \sin x\ dx = -\cos x + C$ $\int \cos x\ dx = \sin x + C$

$\int \sec^2 x\ dx = \tan x + C$ $\int \csc^2 x\ dx = -\cot x + C$

$\int \sec x \tan x\ dx = \sec x + C$ $\int \csc x \cot x\ dx = -\csc x + C$

Example 2 □ Find the antiderivative of $x^3 - 3x + 2$.

Solution □ $\int (x^3 - 3x + 2)\ dx = \int x^3\ dx - \int 3x\ dx + \int 2\ dx$

$= \dfrac{x^{3+1}}{3+1} - 3\cdot\dfrac{x^{1+1}}{1+1} + 2x + C$ Power Rule

$= \dfrac{x^4}{4} - \dfrac{3x^2}{2} + 2x + C$ Answer

Check: $\dfrac{d}{dx}(\dfrac{x^4}{4} - \dfrac{3x^2}{2} + 2x + C) = \dfrac{1}{4}\cdot 4x^3 - \dfrac{3}{2}\cdot 2x + 2\cdot 1 + 0$

$= x^3 - 3x + 2 \quad \surd$

Example 3 □ Find the general indefinite integral

$\int (\sqrt{x} - \sec x \tan x)\ dx$.

Solution □ $\int (\sqrt{x} - \sec x \tan x)\ dx = \int \sqrt{x}\ dx - \int \sec x \tan x\ dx$

$= \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \sec x + C = \dfrac{2}{3}x^{\frac{3}{2}} - \sec x + C$

Check: $\dfrac{d}{dx}(\dfrac{2}{3}x^{\frac{3}{2}} - \sec x + C) = \dfrac{2}{3}\cdot\dfrac{3}{2}x^{\frac{3}{2}-1} - \sec x \tan x + 0$

$= x^{\frac{1}{2}} - \sec x \tan x = \sqrt{x} - \sec x \tan x \quad \surd$

Exercises - Antiderivatives and Indefinite Integrals

Multiple Choice Questions

1. If $\frac{dy}{dx} = 3x^2 - 1$, and if $y = -1$ when $x = 1$, then $y =$

   (A) $x^3 - x + 1$

   (B) $x^3 - x - 1$

   (C) $-x^3 + x - 1$

   (D) $-x^3 + 1$

---

2. Which of the following is the antiderivative of $f(x) = \tan x$?

   (A) $\sec x + \tan x + C$

   (B) $\csc x + \cot x + C$

   (C) $\ln\left|\csc x\right| + C$

   (D) $-\ln\left|\cos x\right| + C$

---

3. A curve has a slope of $-x + 2$ at each point (x, y) on the curve. Which of the following is an equation for this curve if it passes through the point $(2,1)$?

   (A) $\frac{1}{2}x^2 - 2x - 4$

   (B) $2x^2 + x - 8$

   (C) $-\frac{1}{2}x^2 + 2x - 1$

   (D) $x^2 - 2x + 1$

4. $\int (x^2 - 2)\sqrt{x}\ dx =$

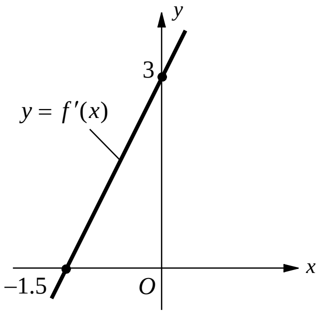
(A) $\frac{2}{5}x^2\sqrt{x} - \frac{2}{3}x\sqrt{x} + C$

(B) $\frac{2}{5}x^2\sqrt{x} - \frac{4}{3}x\sqrt{x} + C$

(C) $\frac{2}{7}x^3\sqrt{x} - \frac{4}{3}x\sqrt{x} + C$

(D) $\frac{2}{7}x^3\sqrt{x} - \frac{2}{3}x^2\sqrt{x} + C$

Free Response Questions



5. The graph of f', the derivative of f, is the line shown in the figure above. If $f(3) = 11$, then $f(-3) =$

4.2 Riemann Sum and Area Approximation

The **area of a region S** that lies under the curve $y = f(x)$ from a to b can be approximated by summing the areas of a collection of rectangles.

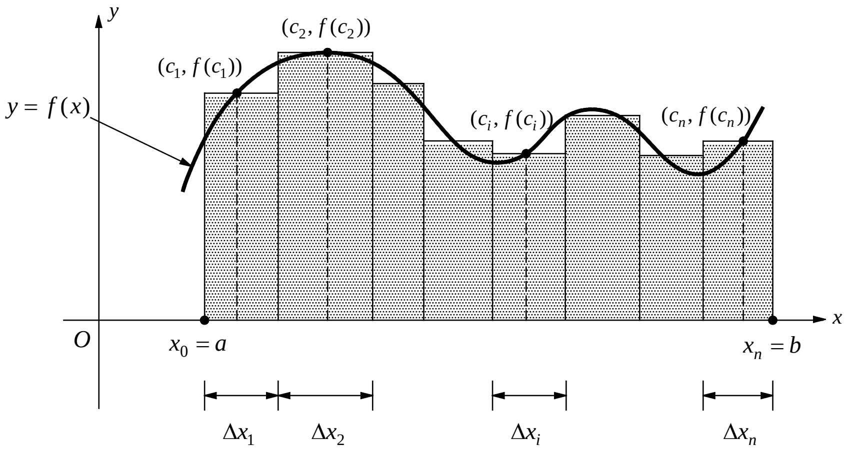


Figure 4.1 The rectangles approximate the area between the graph of the function $y = f(x)$ and the x-axis.

Definition of Riemann Sum

Let f be a continuous function defined on the closed interval $[a,b]$, and let Δ be a partition of $[a,b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the i th interval. If c_i is any point in the i th interval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_i)\Delta x_i + \cdots + f(c_n)\Delta x_n$$

is called a **Riemann sum** for f on the interval $[a,b]$.

If every subinterval is of equal width, the partition is **regular** and $\Delta x = \dfrac{b-a}{n}$.

Then the Riemann sum can be written

$$\sum_{i=1}^{n} f(c_i)\Delta x = \Delta x\left[f(c_1) + f(c_2) + \cdots + f(c_i) + \cdots + f(c_n)\right]$$

where $c_i = a + i(\Delta x)$.

Left, Right, and Midpoint Riemann Sum Approximation

If c_i is the left endpoint of each subinterval then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a **left Riemann sum**.

If c_i is the right endpoint of each subinterval then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a **right Riemann sum**.

If c_i is the midpoint of each subinterval then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a **midpoint Riemann sum**.

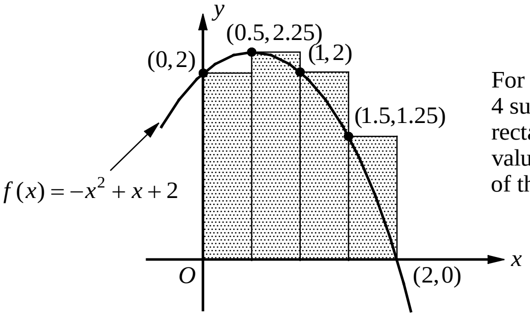
Example 1 □ Approximate the area of the region bounded by the graph of $f(x) = -x^2 + x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$,

(a) by using a left Riemann sum with four subintervals,

(b) by using a right Riemann sum with four subintervals, and

(c) by using a midpoint Riemann sum with four subintervals.

Solution □ (a)



For a left Riemann sum with 4 subintervals, we use the four rectangles whose heights are the values of f at the left endpoints of their bases.

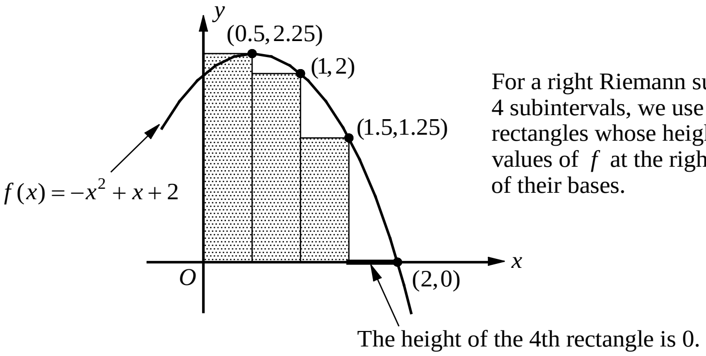
The left endpoints of each subinterval are 0, 0.5, 1, and 1.5 and $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{4} = \dfrac{1}{2}$.

The left Riemann sum is

$$\sum_{i=1}^{4} f(c_i)\Delta x_i = f(0)\cdot(\frac{1}{2}) + f(0.5)\cdot(\frac{1}{2}) + f(1)\cdot(\frac{1}{2}) + f(1.5)\cdot(\frac{1}{2})$$

$$= \frac{1}{2}\left[2 + 2.25 + 2 + 1.25\right] = 3.75$$

(b)



For a right Riemann sum with 4 subintervals, we use the three rectangles whose heights are the values of f at the right endpoints of their bases.

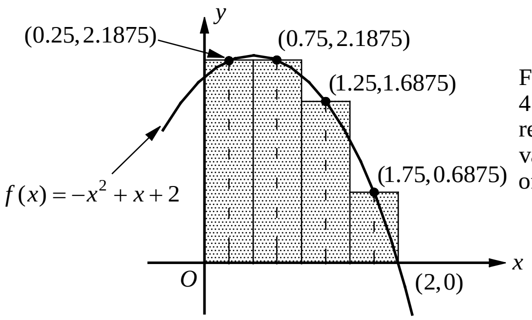
The right endpoints of each subinterval are 0.5, 1, 1.5, and 2, and $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{4} = \dfrac{1}{2}$.

The right Riemann sum is

$$\sum_{i=1}^{4} f(c_i)\Delta x_i = f(0.5)\cdot(\frac{1}{2}) + f(1)\cdot(\frac{1}{2}) + f(1.5)\cdot(\frac{1}{2}) + f(2)\cdot(\frac{1}{2})$$

$$= \frac{1}{2}\left[2.25 + 2 + 1.25 + 0\right] = 2.75$$

(c)



For a midpoint Riemann sum with 4 subintervals, we use the four rectangles whose heights are the values of f at the midpoints of their bases.

The midpoints of each subinterval are 0.25, 0.75, 1.25, and 1.75, and $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{4} = \dfrac{1}{2}$.

The midpoint Riemann sum is

$$\sum_{i=1}^{4} f(c_i)\Delta x_i = f(.25)\cdot(\frac{1}{2}) + f(.75)\cdot(\frac{1}{2}) + f(1.25)\cdot(\frac{1}{2}) + f(1.75)\cdot(\frac{1}{2})$$

$$= \frac{1}{2}\left[2.1875 + 2.1875 + 1.6875 + 0.6875\right] = 3.375$$

Example 2 ☐

x	0	1.5	3	4.5	6	7.5	9	10.5	12
$f(x)$	1	1.45	2.8	5.05	8.2	12.25	17.2	23.05	29.8

The function f is continuous on the closed interval $[0,12]$ and has values as shown in the table above. Use a midpoint Riemann sum with 4 subintervals of equal length to approximate the area that lies under f and above the x-axis from $x = 0$ to $x = 12$.

Solution ☐ The four intervals are $[0,3]$, $[3,6]$, $[6,9]$, and $[9,12]$.

1.5, 4.5, 7.5, and 10.5 are the midpoints of each interval.

Midpoint Riemann sum is

$$f(1.5)\cdot 3 + f(4.5)\cdot 3 + f(7.5)\cdot 3 + f(10.5)\cdot 3$$
$$= 3\cdot[1.45 + 5.05 + 12.25 + 23.05]$$
$$= 125.4$$

Exercises - Riemann Sums

Multiple Choice Questions

1. Using a left Riemann sum with three subintervals $[0,1]$, $[1,2]$, and $[2,3]$, what is the approximation of $\int_0^3 (3-x)(x+1)\, dx$?

   (A) 7.5 (B) 9 (C) 10 (D) 11.5

---

x	1	3	5	8	10
$f(x)$	7	12	16	23	17

2. The function f is continuous on the closed interval $[1,10]$ and has values as shown in the table above. Using a right Riemann sum with four subintervals $[1,3]$, $[3,5]$, $[5,8]$, $[8,10]$,what is the approximation of $\int_1^{10} f(x)\, dx$?

   (A) 96 (B) 116 (C) 132 (D) 159

---

3. The expression $\frac{1}{20}\left[\left(\frac{1}{20}\right)^2+\left(\frac{2}{20}\right)^2+\left(\frac{3}{20}\right)^2+\ldots+\left(\frac{20}{20}\right)^2\right]$ is a Riemann sum approximation for

   (A) $\frac{1}{20}\int_0^{20} x^2\, dx$

   (B) $\frac{1}{20}\int_0^{1} x^2\, dx$

   (C) $\int_0^{1} x^2\, dx$

   (D) $\int_0^{1} \frac{1}{x^2}\, dx$

4. Using a midpoint Riemann sum with three subintervals $[0,1]$, $[1,2]$, and $[2,3]$, what is the approximation of $\int_0^3 \sqrt{1+x^2}\,dx$?

(A) 5.613 (B) 6.213 (C) 6.812 (D) 7.195

---

5. The expression $\frac{1}{30}\left[\sqrt{\frac{1}{30}}+\sqrt{\frac{2}{30}}+\sqrt{\frac{3}{30}}+\ldots+\sqrt{\frac{30}{30}}\right]$ is a Riemann sum approximation for

(A) $\int_0^1 \sqrt{x}\,dx$

(B) $\frac{1}{30}\int_0^1 \sqrt{x}\,dx$

(C) $\frac{1}{30}\int_0^{30} \sqrt{x}\,dx$

(D) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

---

6. The expression $\frac{1}{10}\left[\frac{1}{10}+\frac{2}{10}+\frac{3}{10}+\ldots+\frac{20}{10}\right]$ is a Riemann sum approximation for

(A) $\int_0^2 2\,dx$

(B) $\int_0^2 x\,dx$

(C) $\int_0^2 \frac{x}{10}\,dx$

(D) $\frac{1}{10}\int_0^1 x\,dx$

4.3 Definite Integral, Area Under a Curve, and Application

Definition of a Definite Integral

If f is a continuous function defined for $a \le x \le b$, then the definite integral of f from a to b is

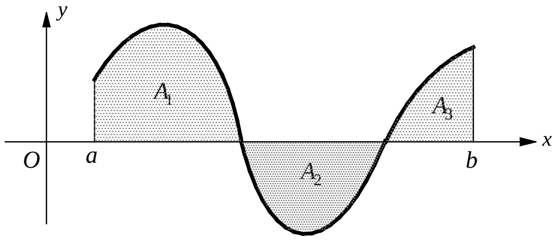
$$\int_a^b \boldsymbol{f(x)\,dx} = \lim_{n\to\infty}\sum_{i=1}^{n} \boldsymbol{f(c_i)\Delta x},$$

where $c_i = a + i\Delta x$ and $\Delta x = (b-a)/n$.

If $y = f(x)$ is continuous and nonnegative over a closed interval $[a,b]$ then the **area** of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\textbf{Area} = \int_a^b \boldsymbol{f(x)\,dx}.$$

If $f(x)$ takes on both positive and negative values over a closed interval $[a,b]$, then the area of the region bounded by the graph of f and the x-axis is obtained by adding the absolute value of the definite integral over each subinterval where $f(x)$ does not change sign.



The figure above shows how both the total area and the value of definite integral can be interpreted in terms of areas between the graph of $f(x)$ and the x-axis.

The definite integral of $f(x)$ over $[a,b] = \int_a^b f(x)\,dx = A_1 - A_2 + A_3$.

The total area between the curve and the x-axis over $[a,b] = \int_a^b |f(x)|\,dx = A_1 + A_2 + A_3$.

Application of Definite Integral

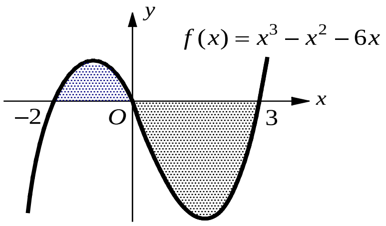
If the population density of a region at a distance x from a straight road is $D(x)$, then the **total population** of the region between $x = a$ and $x = b$ is given by

$$\textbf{Total Population} = \int_a^b \boldsymbol{f(x)\cdot D(x)\,dx}.$$

If the population density of a circular region at a distance r from the center is $D(r)$, then the total population of the circular region between $r = a$ and $r = b$ is given by

$$\textbf{Total Population} = 2\pi\int_a^b \boldsymbol{r\,D(r)\,dr}.$$

Example 1 □ The figure on the right shows the graph of $f(x) = x^3 - x^2 - 6x$.



(a) Find the definite integral of $f(x)$ on $[-2,3]$.

(b) Find the area between the graph of $f(x)$ and the x-axis on $[-2,3]$.

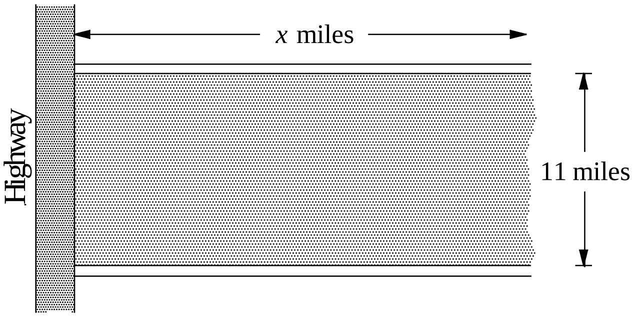
Solution □ (a) $\int_{-2}^{3} (x^3 - x^2 - 6x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{3}$

$$= (\frac{81}{4} - \frac{27}{3} - 27) - (\frac{16}{4} + \frac{8}{3} - 12) = -\frac{125}{12}$$

(b) Area $= \int_{-2}^{3} \left|x^3 - x^2 - 6x\right| dx = \left|\left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{0}\right| + \left|\left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{0}^{3}\right|$

$$= \left|\frac{16}{3}\right| + \left|-\frac{63}{4}\right| = \frac{253}{12}$$

Example 2 □



A rectangular region located beside a highway and between two straight roads 11 miles apart are shown in the figure above. The population density of the region at a distance x miles from the highway is given by $D(x) = 15x\sqrt{x} - 3x^2$, where $0 \le x \le 25$.
How many people live between 16 to 25 miles from the highway?

. Population $= \int_a^b f(x) \cdot D(x) dx$

$= \int_{16}^{25} 11(15x\sqrt{x} - 3x^2)\, dx$ $f(x) = 11$ and $D(x) = 15x\sqrt{x} - 3x^2$

$= \int_{16}^{25} 11(15x^{3/2} - 3x^2)\, dx$

$= 11\left[15(\frac{2}{5})x^{5/2} - x^3\right]_{16}^{25} = 11\left[6x^{5/2} - x^3\right]_{16}^{25}$

$= 11\left[\left\{6(25)^{5/2} - (25)^3\right\} - \left\{6(16)^{5/2} - (16)^3\right\}\right]$

$= 11[3{,}125 - 2{,}048]$

$= 11{,}847$

Exercises - Definite Integral and Area Under a Curve

Multiple Choice Questions

1. $\int_0^3 \frac{dx}{\sqrt{1+x}} =$

(A) 2 (B) 2.5 (C) 3 (D) 4

---

2. The area of the region in the first quadrant enclosed by the graph of $f(x) = 4x - x^3$ and the x-axis is

(A) $\frac{11}{4}$ (B) $\frac{7}{2}$ (C) 4 (D) $\frac{11}{2}$

---

3. $\int_0^5 \sqrt{25 - x^2}\, dx =$

(A) $\frac{25\pi}{8}$ (B) $\frac{25\pi}{4}$ (C) $\frac{25\pi}{2}$ (D) 25π

---

4. The population density of a circular region is given by $f(r) = 10 - 3\sqrt{r}$ people per square mile, where r is the distance from the center of the city, in miles. Which of the following expressions gives the number of people who live within a 3 mile radius from the center of the city?

(A) $\pi\int_0^3 r^2(10 - 3\sqrt{r})\, dr$

(B) $\pi\int_0^3 (r+3)^2(10 - 3\sqrt{r})\, dr$

(C) $2\pi\int_0^3 (r+3)(10 - 3\sqrt{r})\, dr$

(D) $2\pi\int_0^3 r(10 - 3\sqrt{r})\, dr$

5. Which of the following limits is equal to $\int_1^3 x^3 \; dx$?

(A) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{i}{n})^3\frac{1}{n}$

(B) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{i}{n})^3\frac{2}{n}$

(C) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{2i}{n})^3\frac{1}{n}$

(D) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{2i}{n})^3\frac{2}{n}$

---

6. Which of the following integrals is equal to $\lim_{n\to\infty}\sum_{i=1}^{n}(-1+\frac{3i}{n})^2\frac{3}{n}$?

(A) $\int_{-1}^{2} x^2 \; dx$

(B) $\int_{-1}^{0} x^2 \; dx$

(C) $\int_{-1}^{2} (-1+x)^2 \; dx$

(D) $\int_{-1}^{0} (-1+\frac{x}{3})^2 \; dx$

---

7. The closed interval $[a,b]$ is partitioned into n equal subintervals, each of width Δx, by the numbers $x_0, x_1, ..., x_n$ where $0 < a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. What is $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{x_i}}\;\Delta x$?

(A) $\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}$

(B) $\frac{(\sqrt{b}-\sqrt{a})}{2}$

(C) $2(\sqrt{b}-\sqrt{a})$

(D) $\sqrt{b}-\sqrt{a}$

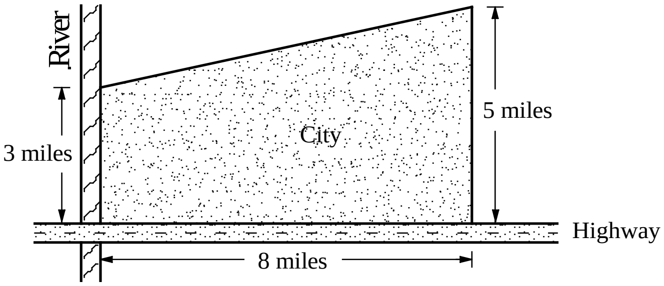
8. If n is a positive integer, then $\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right]$ can be expressed as

(A) $\int_0^1 \frac{1}{x}\,dx$ (B) $\int_0^1 \frac{1}{x^2}\,dx$ (C) $\int_0^1 x^2\,dx$ (D) $\frac{1}{2}\int_0^1 x^2\,dx$

---

9. If n is a positive integer, then $\lim_{n\to\infty}\frac{2}{n}\left[\sqrt{\frac{2}{n}}+\sqrt{\frac{4}{n}}+\cdots+\sqrt{\frac{2n}{n}}\right]$ can be expressed as

(A) $\int_0^1 \sqrt{x}\,dx$ (B) $\int_0^2 \sqrt{x}\,dx$ (C) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (D) $\int_0^2 \frac{1}{\sqrt{x}}\,dx$



10. The region shown in the figure above represents the boundary of a city that is bordered by a river and a highway. The population density of the city at a distance of x miles from the river is modeled by $D(x)=\frac{6}{\sqrt{x+16}}$, where $D(x)$ is measured in thousands of people per square mile. According to the model, which of the following expressions gives the total population, in thousands of the city?

(A) $\int_0^8 (4)(\frac{6}{\sqrt{x+16}})\,dx$

(B) $\int_0^8 (4x)(\frac{6}{\sqrt{x+16}})\,dx$

(C) $\int_0^8 (\frac{1}{4}x)(\frac{6}{\sqrt{x+16}})\,dx$

(D) $\int_0^8 (\frac{1}{4}x+3)(\frac{6}{\sqrt{x+16}})\,dx$
]

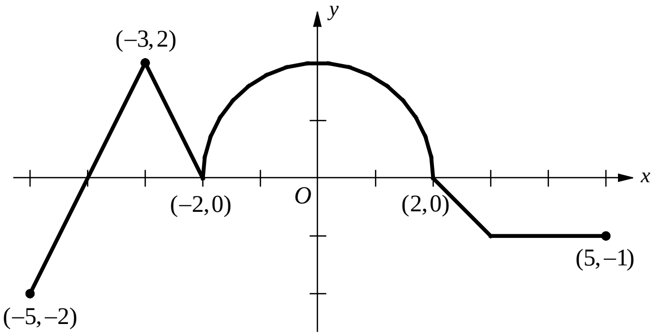
Free Response Questions

11. Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = ky^2 + 1$ ($k > 0$), and the line $x = 2$.

(a) Write an integral expression for the area of the region R.

(b) If the area of the region R is 2, what is the value of k ?

(c) Show that for all $k > 0$, the line tangent to the graph of $x = ky^2 + 1$ at the point $(2, \sqrt{\frac{1}{k}})$ passes through the origin.



12. The graph of $y = f(x)$ consists of four line segments and a semicircle as shown in the figure above. Evaluate each definite integral by using geometric formulas.

(a) $\int_{-5}^{-2} f(x)\,dx$ (b) $\int_{-2}^{2} f(x)\,dx$ (c) $\int_{2}^{5} f(x)\,dx$ (d) $\int_{-5}^{5} |f(x)|\,dx$

4.4 Properties of Definite Integral

Definition

$\int_a^a f(x)\,dx = 0$ $\qquad$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Constant Multiple

$\int_a^b c\,dx = c(b-a)$ $\qquad$ $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

Sum and Difference

$$\int_a^b \left[f(x) \pm g(x)\right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Additivity

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Integrals of Symmetric Functions

If f is even $f(-x) = f(x)$, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

If f is odd $f(-x) = -f(x)$, then $\int_{-a}^a f(x)\,dx = 0$

Comparison Property

If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge 0$

If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Example 1 □ Suppose that $\int_{-3}^4 f(x)\,dx = 5$, $\int_{-3}^4 g(x)\,dx = -4$, and $\int_{-3}^1 f(x)\,dx = 2$.

Find (a) $\int_{-3}^4 \left[2f(x) - 3g(x)\right] dx$ (b) $\int_1^4 f(x)\,dx$ (c) $\int_{-3}^4 \left[g(x) + 2\right] dx$.

Solution □ (a) $\int_{-3}^4 \left[2f(x) - 3g(x)\right] dx = \int_{-3}^4 2f(x)\,dx - \int_{-3}^4 3g(x)\,dx$

$$= 2\int_{-3}^4 f(x)\,dx - 3\int_{-3}^4 g(x)\,dx = 2\cdot 5 - 3\cdot(-4) = 22$$

(b) $\int_{-3}^4 f(x)\,dx = \int_{-3}^1 f(x)\,dx + \int_1^4 f(x)\,dx$

$$\Rightarrow\ 5 = 2 + \int_1^4 f(x)\,dx \quad \Rightarrow\ \int_1^4 f(x)\,dx = 3$$

(c) $\int_{-3}^4 \left[g(x) + 2\right] dx = \int_{-3}^4 g(x)\,dx + \int_{-3}^4 2\,dx = -4 + 2(4+3) = 10$

Exercises - Properties of Definite Integral

Multiple Choice Questions

1. If $\int_a^b f(x)\,dx = 2a-5b$, then $\int_a^b [\,f(x)-2\,]dx =$

(A) $-7b$ (B) $-3b$ (C) $4a-7b$ (D) $4a-3b$

---

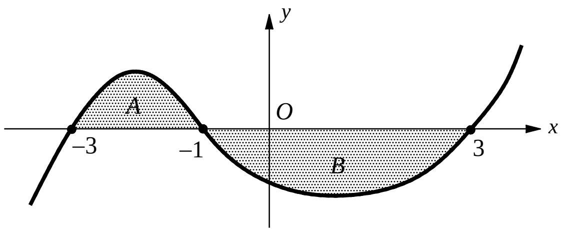
2. If $\int_1^6 f(x)\,dx = \frac{15}{2}$ and $\int_6^4 f(x)\,dx = 5$, then $\int_1^4 f(x)\,dx =$

(A) $\frac{5}{2}$ (B) $\frac{9}{2}$ (C) $\frac{19}{2}$ (D) $\frac{25}{2}$

---

3. If $\int_{-2}^6 f(x)\,dx = 10$ and $\int_2^6 f(x)\,dx = 3$, then $\int_2^6 f(4-x)\,dx =$

(A) 3 (B) 6 (C) 7 (D) 10



4. The graph of $y = f(x)$ is shown in the figure above. If A and B are positive numbers that represent the areas of the shaded regions, what is the value of $\int_{-3}^{3} f(x)\,dx - 2\int_{-1}^{3} f(x)\,dx$, in terms of A and B?

(A) $-A-B$ (B) $A+B$ (C) $A-2B$ (D) $A-B$

Free Response Questions

5. Let f and g be a continuous function on the interval $[1,5]$. Given $\int_1^3 f(x)\,dx = -3$, $\int_1^5 f(x)\,dx = 7$, and $\int_1^5 \mathrm{g}(x)\,dx = 9$, find the following definite integrals.

(a) $\int_3^5 f(x)\,dx$

(b) $\int_1^3 [f(x)+3]\,dx$

(c) $\int_5^1 2g(x)\,dx$

(d) $\int_5^5 g(x)\,dx + \int_5^3 f(x)\,dx$

(e) $\int_{-1}^3 f(x+2)\,dx$

---

6. Let f and g be continuous functions with the following properties.

(1) $g(x) = f(x) - n$ where n is a constant.

(2) $\int_0^4 f(x)\,dx - \int_4^6 g(x)\,dx = 1$

(3) $\int_4^6 f(x)\,dx = 5n - 1$

(a) Find $\int_0^4 f(x)\,dx$ in terms of n.

(b) Find $\int_0^6 g(x)\,dx$ in terms of n.

(c) Find the value of k if $\int_0^2 f(2x)\,dx = kn$.

4.5 Trapezoidal Rule

Let f be continuous on $[a,b]$. The **trapezoidal rule** for approximating $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\ \cdots\ +2f(x_{n-1})+f(x_n)\right],$$

where $x_0 = a,\ x_1 = a+\Delta x,\ x_2 = a+2\Delta x,\ \ldots,\ x_{n-1} = a+(n-1)\Delta x,\ x_n = b$, and $\Delta x = \dfrac{b-a}{n}$.

Example 1 □ Use the trapezoidal rule to approximate the integral $\int_1^3 \sqrt{1+x^2}\,dx$ with four subintervals.

Solution □ $\Delta x = \dfrac{b-a}{n} = \dfrac{3-1}{4} = \dfrac{1}{2}$

$x_0 = 1$, $x_1 = 1+0.5 = 1.5$, $x_2 = 1+2(0.5) = 2$, $x_3 = 1+3(0.5) = 2.5$, and $x_4 = 1+4(0.5) = 3$.

$$\int_1^3 \sqrt{1+x^2}\,dx \approx \frac{\Delta x}{2}\left[f(1)+2f(1.5)++2f(2)++2f(2.5)+f(3)\right]$$

$$= \frac{1}{4}\left[\sqrt{2}+2\sqrt{3.25}+2\sqrt{5}+2\sqrt{7.25}+\sqrt{10}\right]$$

$$\approx 4.509$$

Example 2 □

x	-1	1	4	6	8
$f(x)$	5	7	11	8	7

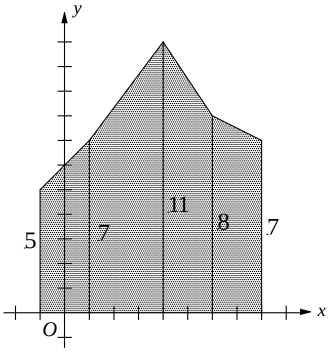
The function f is continuous on the closed interval $[-1,8]$ and has values that are given in the table above. What is the trapezoidal approximation of $\int_{-1}^{8} f(x)\,dx$?

Solution □ Make a sketch and use the formula for the area of trapezoid.

$$\text{Area} = \frac{1}{2}(5+7)(2)+\frac{1}{2}(7+11)(3)$$

$$+\frac{1}{2}(11+8)(2)+\frac{1}{2}(8+7)(2)$$

$$= 12+27+19+15 = 73$$



Exercises - Trapezoidal Rule

Multiple Choice Questions

1. If four equal subdivisions on $[0,2]$ are used, what is the trapezoidal approximation of $\int_0^2 e^x \, dx$?

   (A) $\frac{1}{4}\left[1+2\sqrt{e}+2e+2e\sqrt{e}+e^2\right]$

   (B) $\frac{1}{2}\left[1+2\sqrt{e}+2e+2e\sqrt{e}+e^2\right]$

   (C) $\frac{1}{4}\left[1+\sqrt{e}+e+e\sqrt{e}+e^2\right]$

   (D) $\frac{1}{2}\left[1+\sqrt{e}+e+e\sqrt{e}+e^2\right]$

---

2. If three equal subdivisions on $\left[\frac{\pi}{2},\pi\right]$ are used, what is the trapezoidal approximation of $\int_{\pi/2}^{\pi} \sin x \, dx$?

   (A) $\frac{\pi}{12}(\sin\frac{2\pi}{3}+\sin\frac{5\pi}{6}+\sin\pi)$

   (B) $\frac{\pi}{12}(\sin\frac{\pi}{2}+\sin\frac{2\pi}{3}+\sin\frac{5\pi}{6}+\sin\pi)$

   (C) $\frac{\pi}{12}(\sin\frac{\pi}{2}+2\sin\frac{2\pi}{3}+2\sin\frac{5\pi}{6}+\sin\pi)$

   (D) $\frac{\pi}{6}(\sin\frac{\pi}{2}+\sin\frac{2\pi}{3}+\sin\frac{5\pi}{6}+\sin\pi)$

3. If three equal subdivisions on $[0,6]$ are used, what is the trapezoidal approximation of $\int_0^6 \ln(x+1)\, dx$?

(A) $\frac{1}{3}(\ln 1 + \ln 9 + \ln 25 + \ln 7)$

(B) $\frac{1}{2}(\ln 1 + \ln 9 + \ln 25 + \ln 7)$

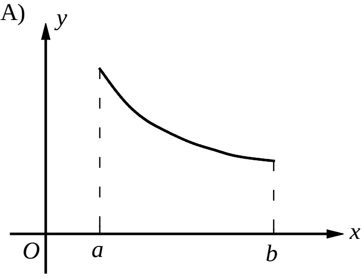
(C) $\ln 1 + \ln 3 + \ln 5 + \ln 7$

(D) $\ln 1 + \ln 9 + \ln 25 + \ln 7$

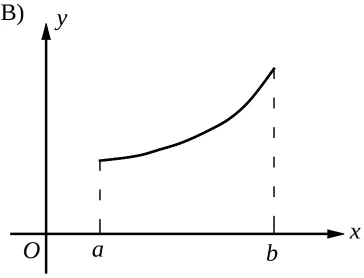
---

4. If a trapezoidal sum underapproximates $\int_a^b f(x)\, dx$, and a right Riemann sum overapproximates $\int_a^b f(x)\, dx$, which of the following could be the graph of $y = f(x)$?

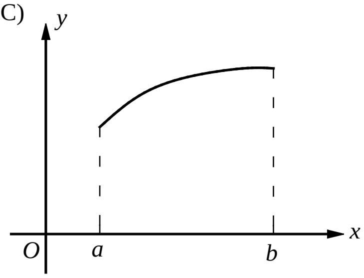
(A)



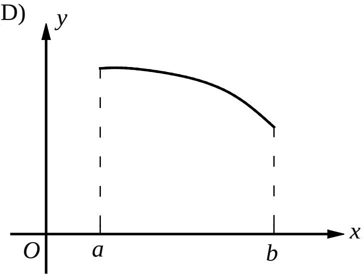
(B)



(C)



(D)



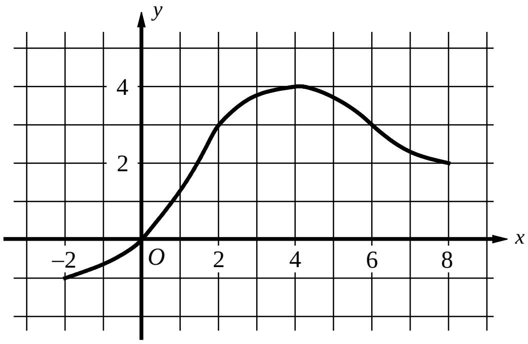
x	1	3	5	9	12
$f(x)$	4	10	14	11	7

5. A function f is continuous on the closed interval $[1,12]$ and has values that are given in the table above. Using subintervals $[1,3]$, $[3,5]$, $[5,9]$, and $[9,12]$, what is the trapezoidal approximation of $\int_1^{12} f(x)\,dx$?

(A) 97 (B) 115 (C) 128 (D) 136

Free Response Questions

6. Find a trapezoidal approximation of $\int_0^2 \cos(x^2)\,dx$ using four subdivisions of length $\Delta x = 0.5$.



7. The graph of a differentiable function f on the closed interval $[-2,8]$ is shown in the figure above. Find a trapezoidal approximation of $\int_{-2}^{8} f(x)\,dx$ using five subdivisions of length $\Delta x = 2$.

4.6 Fundamental Theorem of Calculus Part 1

If f is continuous on $[a,b]$ then $F(x)=\int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b), so

$$\boldsymbol{F'(x)=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)}.$$

If $F(x)=\int_a^{u(x)} f(t)\,dt$, then by the chain rule

$$\boldsymbol{F'(x)=\frac{d}{dx}\int_a^{u(x)} f(t)\,dt=f(u)\cdot\frac{du}{dx}}.$$

Example 1 □ If $F(x)=\int_1^x \frac{1}{1+t^3}\,dt$, then $F'(x)=$

Solution □ $F'(x)=\frac{d}{dx}\int_1^x \frac{1}{1+t^3}\,dt$

$=\frac{1}{1+x^3}$ Fundamental Theorem of Calculus

Example 2 □ If $F(x)=\int_1^{x^2+1} \sqrt{t}\,dt$, then $F'(x)=$

Solution □ $F'(x)=\frac{d}{dx}\int_1^{x^2+1} \sqrt{t}\,dt$

$=\sqrt{x^2+1}\cdot\frac{d}{dx}(x^2+1)$ Fundamental Theorem of Calculus

$=2x\sqrt{x^2+1}$

Exercises - Fundamental Theorem of Calculus Part 1

Multiple Choice Questions

1. $\dfrac{d}{dx}\displaystyle\int_1^{x^2} \sqrt{3+t^2}\, dt =$

(A) $\sqrt{3+x^2}$ (B) $\sqrt{3+x^4}$ (C) $2x\sqrt{3+x^4}$ (D) $2\sqrt{3+x^2}$

---

2. For $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, if $F(x) = \displaystyle\int_0^{\sin x} \dfrac{dt}{\sqrt{1-t^2}}$, then $F'(x) =$

(A) $\dfrac{\sin x}{\sqrt{1-x^2}}$ (B) $\dfrac{\cos x}{\sqrt{1-x^2}}$ (C) 1 (D) $\csc x$

---

3. If $F(x) = \displaystyle\int_0^{\sqrt{x}} \cos(t^2)\, dt$, then $F'(4) =$

(A) $\cos 2$ (B) $\dfrac{\cos 4}{4}$ (C) $\dfrac{\cos 4}{\sqrt{2}}$ (D) $\sqrt{2}\cos 4$

---

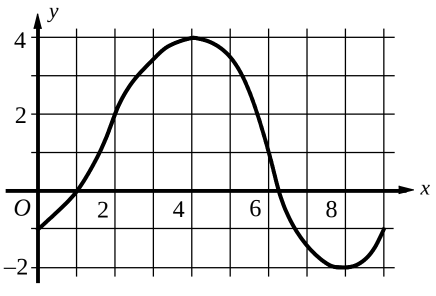
4. Let f be the function given by $f(x) = \displaystyle\int_0^{x} \cos(t^2+2)\, dt$ for $0 \le x \le \pi$. On which of the following intervals is f increasing?

(A) $0 \le x \le \dfrac{\pi}{2}$

(B) $0 \le x \le 1.647$

(C) $1.647 \le x \le 2.419$

(D) $\dfrac{\pi}{2} \le x \le \pi$



graph of g

5. The graph of the function g, shown in the figure above, has horizontal tangents at $x = 4$ and $x = 8$. If $f(x) = \int_0^{\sqrt{x}} g(t)\, dt$, what is the value of $f'(4)$?

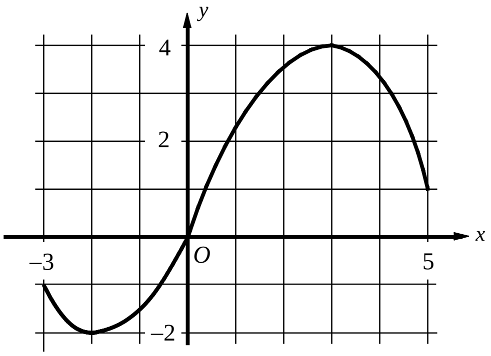
(A) 0 (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{3}{2}$

---

6. If $F(x) = \int_0^{x^2} \frac{\sqrt{t^2+3}}{2t}\, dt$, then $F''(1) =$

(A) −1 (B) 0 (C) 1 (D) $\frac{3}{2}$ (E) $\frac{8}{5}$

Free Response Questions



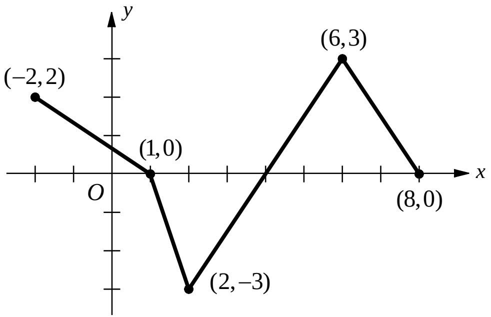
graph of f

7. The graph of a function f, whose domain is the closed interval $[-3, 5]$, is shown above. Let g be the function given by $g(x) = \int_{-3}^{2x-1} f(t)\, dt$.

(a) Find the domain of g.

(b) Find $g'(3)$.

(c) At what value of x is $g(x)$ a maximum? Justify your answer.



graph of f

8. The graph of f, consisting of four line segments, is shown in the figure above. Let g be the function given by $g(x) = \int_{-2}^{x} f(t)\,dt$.

(a) Find $g'(1)$.

(b) Find the x-coordinate for each point of inflection of the graph of g on the interval $-2 < x < 8$.

(c) Find the average rate of change of g on the interval $2 \le x \le 8$.

(d) For how many values of c, where $2 < c < 8$, is $g'(c)$ equal to the average rate found in part (c)? Explain your reasoning.

4.7 Fundamental Theorem of Calculus Part 2

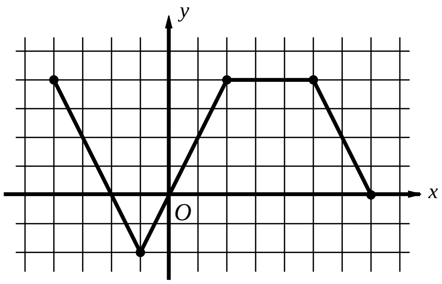
If f is continuous on $[a,b]$ and F is an antiderivative of f on $[a,b]$, then

$$\int_a^b \boldsymbol{f(x)\,dx = F(b) - F(a)}\,.$$

Example 1 □ $\int_{\pi/2}^{x} \cos t\; dt =$

Solution □ $\int_{\pi/2}^{x} \cos t\; dt = [\sin t]_{\pi/2}^{x} = \left[\sin x - \sin\frac{\pi}{2}\right] = \sin x - 1$

Example 2 □ The graph of the function f shown below consists of four line segments. If g is the function defined by $g(x) = \int_{-4}^{x} f(t)\; dt$, find the value of $g(6)$, $g'(6)$, and $g''(6)$.



Graph of f

Solution □

$g(6) = \int_{-4}^{6} f(t)\; dt$	Substitute 6 for *x*.
$= \frac{1}{2}(2)(4) - \frac{1}{2}(2)(2) + \frac{1}{2}(2)(4)$ $+(3)(4) + \frac{1}{2}(2+4)\cdot 1$ $= 21$	$\int_{-4}^{6} f(t)\; dt =$ sum of the area above the *x*-axis minus sum of the area below the *x*-axis, between $x = -4$ and $x = 6$.
$g'(x) = f(x)$	Fundamental Theorem of Calculus
$g'(6) = f(6) = 2$	Substitute 6 for *x*, and $f(6) = 2$.
$g''(6) = f'(6) = -2$	Slope of f at $x = 6$ is -2.

Exercises - Fundamental Theorem of Calculus Part 2

Multiple Choice Questions

1. If f is the antiderivative of $\frac{\sqrt{x}}{1+x^3}$ such that $f(1)=2$, then $f(3)=$

(A) 1.845 (B) 2.397 (C) 2.906 (D) 3.234

---

2. If $f'(x)=\cos(x^2-1)$ and $f(-1)=1.5$, then $f(5)=$

(A) 1.554 (B) 2.841 (C) 3.024 (D) 3.456

---

3. If $f(x)=\sqrt{x^4-3x+4}$ and g is the antiderivative of f, such that $g(3)=7$, then $g(0)=$

(A) –2.966 (B) –1.472 (C) –0.745 (D) 1.086

---

4. If f is a continuous function and $F'(x)=f(x)$ for all real numbers x, then $\int_2^{10} f(\frac{1}{2}x)\,dx=$

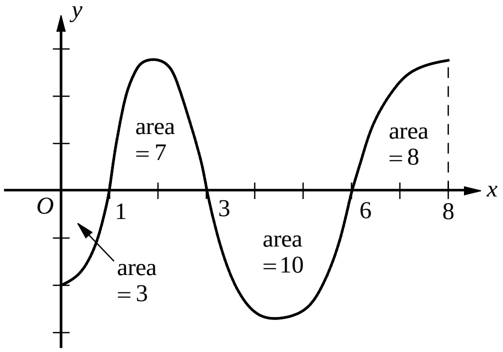
(A) $\frac{1}{2}\left[F(5)-F(1)\right]$

(B) $\frac{1}{2}\left[F(10)-F(2)\right]$

(C) $2\left[F(5)-F(1)\right]$

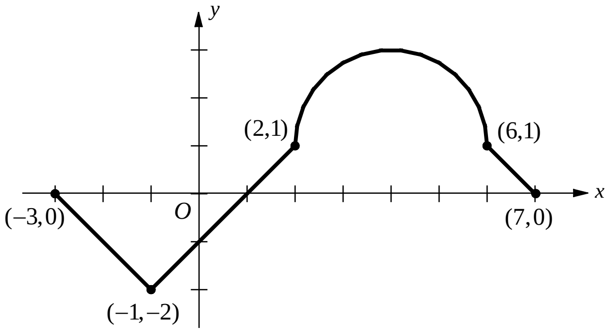
(D) $2\left[F(10)-F(2)\right]$

Free Response Questions



5. The figure above shows the graph of f', the derivative of a differentiable function f, on the closed interval $0 \le x \le 8$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. Given $f(6) = 9$, find each of the following.

(a) $f(0)$ (b) $f(1)$ (c) $f(3)$ (d) $f(8)$



graph of f'

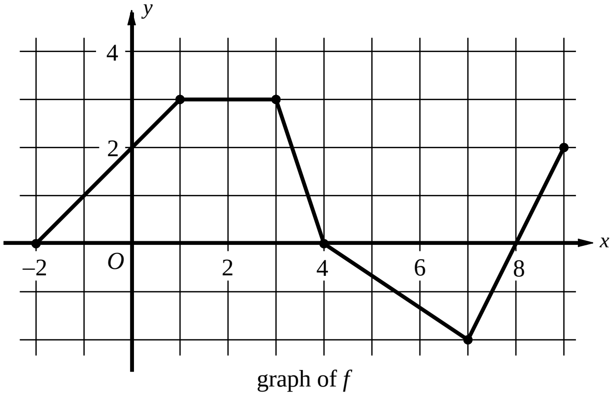
6. Let f be a function defined on the closed interval $[-3,7]$ with $f(2) = 3$. The graph of f' consists of three line segments and a semicircle, as shown above.

(a) Find $f(-3)$ and $f(7)$.

(b) Find an equation for the line tangent to the graph of f at $(2,3)$.

(c) On what interval is f increasing? Justify your answer.

(d) On what interval is f concave up? Justify your answer.



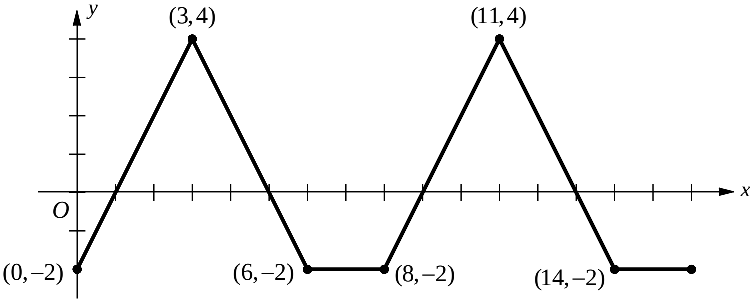
graph of f

7. Let g be the function given by $g(x)=\int_{-2}^{x} f(t)\,dt$. The graph of the function f, shown above, consists of five line segments.

   (a) Find $g(0)$, $g'(0)$ and $g''(0)$.

   (b) For what values of x, in the open interval $(-2,9)$, is the graph of g concave up?

   (c) For what values of x, in the open interval $(-2,9)$, is g increasing?



graph of f

8. The graph above shows two periods of f. The function f is defined for all real numbers x and is periodic with a period of 8. Let h be the function given by $h(x)=\int_{0}^{x} f(t)\,dt$.

   (a) Find $h(8)$, $h'(6)$, and $h''(4)$.

   (b) Find the values of x at which h has its minimum and maximum on the closed interval $[0,8]$. Justify your answer.

   (c) Write an equation for the line tangent to the graph of h at $x=35$.

4.8 Integration by Substitution

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int \boldsymbol{f(g(x))g'(x)\,dx = \int f(u)\,du}$$

Evaluating the integral $\int f(g(x))g'(x)\,dx$, when f and g' are continuous functions

1. Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

The Substitution Rule for Definite Integrals

If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b \boldsymbol{f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$

Example 1 □ Find $\int \cos(5\theta-3)\,d\theta$.

Solution □ $\int \cos(5\theta-3)\,d\theta = \int \cos u\cdot\frac{1}{5}du$ Let $u = 5\theta-3$, $du = 5\,d\theta$, $\frac{1}{5}du = d\theta$.

$=\frac{1}{5}\int \cos u\,du$ The integral is now in standard form.

$=\frac{1}{5}\sin u + C$ Integrate with respect to *u*.

$=\frac{1}{5}\sin(5\theta-3)+C$ Replace *u* by $5\theta-3$.

Example 2 □ Evaluate $\int \frac{x}{\sqrt{1-x^2}}\,dx$.

Solution □ $\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{-1/2\ du}{\sqrt{u}}$ Let $u = 1-x^2$, $du = -2x\,dx$, $-\frac{1}{2}du = x\,dx$.

$=-\frac{1}{2}\int u^{-1/2}\,du$ In the form $\int u^n\,du$

$=-\frac{1}{2}(2\sqrt{u})+C$ Integrate with respect to *u*.

$=-\sqrt{u}+C$

$=-\sqrt{1-x^2}+C$ Replace u with $1-x^2$.

Example 3 □ Evaluate $\int_0^{\pi/2} \frac{3\cos x}{\sqrt{1+3\sin x}}\,dx$.

Solution □ We have two choices.

Method 1: Transform the integral and evaluate the integral with the transformed limits.

$$\int_0^{\pi/2} \frac{3\cos x}{\sqrt{1+3\sin x}}\,dx$$

Let $u = 1+3\sin x$, $du = 3\cos x\,dx$.
When $x = 0$, $u = 1+3\sin 0 = 1$.
When $x = \pi/2$,

$$= \int_1^4 \frac{du}{\sqrt{u}}$$

$$= \left[2u^{1/2}\right]_1^4$$

Evaluate the new integral.

$$= 2\left[4^{1/2} - 1\right]$$

$$= 2\left[2-1\right] = 2$$

Method 2: Transform the integral to an indefinite integral, integrate, change back to *x*, and use the original *x*-limits.

$$\int \frac{3\cos x}{\sqrt{1+3\sin x}}\,dx = \int \frac{du}{\sqrt{u}}$$

Let $u = 1+3\sin x$, $du = 3\cos x\,dx$.

$$= 2u^{\frac{1}{2}} + C$$

Integrate with respect to *u*.

$$= 2(1+3\sin x)^{\frac{1}{2}} + C$$

Replace u with $1+3\sin x$.

$$\int_0^{\pi/2} \frac{3\cos x}{\sqrt{1+3\sin x}}\,dx$$

$$= 2\left[(1+3\sin x)^{\frac{1}{2}}\right]_0^{\pi/2}$$

Use the integral found, with limits of integration for *x*.

$$= 2\left[(1+3\sin\frac{\pi}{2})^{\frac{1}{2}} - (1+3\sin 0)^{\frac{1}{2}}\right]$$

$$= 2\left[4^{\frac{1}{2}} - 1^{\frac{1}{2}}\right] = 2$$

Exercises - Integration by Substitution

Multiple Choice Questions

1. $\int \sqrt{x}\sin(x^{3/2})\;dx =$

(A) $\frac{2}{3}\cos(x^{3/2})+C$

(B) $\frac{2}{3}\sqrt{x}\cos(x^{3/2})+C$

(C) $-\frac{2}{3}x^{3/2}\cos(x^{3/2})+C$

(D) $-\frac{2}{3}\cos(x^{3/2})+C$

---

2. If the substitution $u = 2-x$ is made, $\int_1^3 x\sqrt{2-x}\;dx =$

(A) $\int_{-1}^{1} u\sqrt{u}\;du$

(B) $-\int_1^3 u\sqrt{u}\;du$

(C) $\int_1^3 (2-u)\sqrt{u}\;du$

(D) $\int_{-1}^{1} (u-2)\sqrt{u}\;du$

---

3. If $\int_{-1}^{3} f(x+k)\;dx = 8$, where k is a constant, then $\int_{k-1}^{k+3} f(x)\;dx =$

(A) $8-k$ (B) $8+k$ (C) 8 (D) $k-8$

4. If $\int_0^6 f(x)\,dx = 12$, what is the value of $\int_0^6 f(6-x)\,dx$?

(A) 12 (B) 6 (C) 0 (D) –6

---

5. If the substitution $u = 1+\sqrt{x}$ is made, $\int \dfrac{(1+\sqrt{x})^{3/2}}{\sqrt{x}}\,dx =$

(A) $\frac{1}{2}\int u^{3/2}\,du$ (B) $2\int u^{3/2}\,du$ (C) $\frac{1}{2}\int \sqrt{u}\,du$ (D) $2\int \sqrt{u}\,du$

---

6. Using the substitution $u = \sec\theta$, $\int_0^{\pi/4} \dfrac{\tan\theta}{\sqrt{\sec\theta}}\,d\theta$ is equivalent to

(A) $\int_1^{\sqrt{2}} \dfrac{du}{\sqrt{u}}$

(B) $\int_1^{\sqrt{2}} \dfrac{du}{u\sqrt{u}}$

(C) $\int_1^{\sqrt{2}} \sqrt{u}\,du$

(D) $\int_1^{\sqrt{2}} u\sqrt{u}\,du$

7. If the substitution $u = \ln x$ is made, $\int_e^{e^2} \frac{1-(\ln x)^2}{x}\, dx =$

(A) $\int_e^{e^2} (\frac{1}{u} - u^2)\, du$

(B) $\int_e^{e^2} (\frac{1}{u} - u)\, du$

(C) $\int_1^2 (1-u^2)\, du$

(D) $\int_1^2 (1-u)\, du$

Free Response Questions

8. If f is continuous and $\int_1^8 f(x)\, dx = 15$, find the value of $\int_1^2 x^2 f(x^3)\, dx$.

4.9 Integration of Exponential and Logarithmic Function

Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\mathbf{\ln x = \int_1^x \frac{1}{t}\, dt}\,, \qquad x > 0$$

If u is a differentiable function such that $u \neq 0$,

$$\int \frac{1}{\boldsymbol{u}}\, \boldsymbol{du} = \ln|\boldsymbol{u}| + \boldsymbol{C}\,.$$

Whether u is positive or negative, the integral of $(1/u)\ du$ is $\ln|u| + C$.

Integration of Exponential Function

$$\int \boldsymbol{e^u\ du = e^u + C}$$

Example 1 □ Evaluate $\int_1^e \frac{x^2+3}{x}\, dx$.

Solution □ $\int_1^e \frac{x^2+3}{x}\, dx = \int_1^e \left(\frac{x^2}{x} + \frac{3}{x}\right) dx = \int_1^e x\, dx + \int_1^e \frac{3}{x}\, dx = \left[\frac{x^2}{2}\right]_1^e + \left[3\ln x\right]_1^e$

$= \left[\frac{e^2}{2} - \frac{1}{2}\right] + 3\left[\ln e - \ln 1\right] = \frac{e^2}{2} - \frac{1}{2} + 3 = \frac{e^2}{2} + \frac{5}{2}$

Example 2 □ Evaluate $\int_0^{\pi/4} (e^{\tan x} + 2)\sec^2 x\, dx$.

Solution □ $\int_0^{\pi/4} (e^{\tan x} + 2)\sec^2 x\, dx$

$= \int_0^1 (e^u + 2)\, du$

$= \left[e^u + 2u\right]_0^1$

$= \left[(e^1 + 2) - (e^0 + 0)\right] = e + 1$

Let $u = \tan x$, $du = \sec^2 x\ dx$.
When $x = 0$, $u = \tan 0 = 0$.
When $x = \pi/4$, $u = \tan \pi/4 = 1$.

Example 3 □ Evaluate $\int_e^{e^2} \frac{(\ln x)^2}{x}\, dx$.

Solution □ $\int_e^{e^2} \frac{(\ln x)^2}{x}\, dx$

$= \int_1^2 u^2\, du$

$= \left[\frac{1}{3}u^3\right]_1^2 = \frac{1}{3}\left[2^3 - 1^3\right] = \frac{7}{3}$

Let $u = \ln x$, $du = dx/x$.
When $x = e$, $u = \ln e = 1$.
When $x = e^2$, $u = \ln e^2 = 2$.

Exercises - Integration of Exponential and Logarithmic Functions

Multiple Choice Questions

1. $\int_1^3 \frac{x+3}{x^2+6x}\,dx =$

(A) $\ln\frac{3}{2}$ (B) $\frac{\ln 27 - \ln 7}{2}$ (C) $\ln 3$ (D) $\frac{\ln 20 - \ln 5}{2}$

---

2. $\int_0^1 \frac{x}{e^{x^2}}\,dx =$

(A) $e-1$ (B) $(1-\frac{1}{e})$ (C) $\frac{1}{2}(1-\frac{1}{e})$ (D) $\frac{1}{2}(1-\frac{1}{e^2})$

---

3. $\int_0^{\pi/2} \cos x\, e^{\sin x}\,dx =$

(A) $-e$ (B) $1-e$ (C) $\frac{e}{2}$ (D) $e-1$

---

4. What is the area of the region in the first quadrant bounded by the curve $y = \frac{\cos x}{2+\sin x}$ and the vertical line $x = \frac{\pi}{2}$?

(A) $\ln\frac{1}{2}$ (B) $\ln\frac{3}{2}$ (C) $\ln 3$ (D) $\frac{\ln 3}{\ln 2}$

5. Let $F(x)$ be an antiderivetive of $\ln(\sin^2 x)+3$. If $F(1)=2$, then $F(3)=$

(A) 6.595 (B) 7.635 (C) 10.036 (D) 12.446

---

6. $\int_0^2 \frac{x^2}{x+1}\,dx =$

(A) $\ln 3$ (B) $\ln 3+2$ (C) $\ln 6$ (D) $\ln 6+4$

---

7. $\int_1^e \frac{\cos(\ln x)}{x}\,dx =$

(A) $\frac{1}{\sin 1}$ (B) $\frac{1}{\cos 1}$ (C) $\sin(e)$ (D) $\sin 1$

Free Response Questions

8. The sales of a new product, after it has been on the market for t years, is given by $S(t)=Ce^{k/t}$.

(a) Find C and k if 7000 units have been sold after one year and $\lim_{t\to\infty} S(t)=45,000$.

(b) Find the total number of units sold during the year $t=5$ and $t=10$.

Chapter 5
Applications of Integration

5.1 Area of a Region Between Two Curves

Area of a Region Between Two Curves

If f and g are continuous on $[a,b]$ and $g(x) \le f(x)$ for all x on $[a,b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$\boldsymbol{A = \int_a^b \left[f(x) - g(x) \right] dx} .$$

In general, to determine the area between two curves, we can use

$$\boldsymbol{A = \int_{x_1}^{x_2} \left[(\textbf{top curve}) - (\textbf{bottom curve}) \right] dx} ,$$

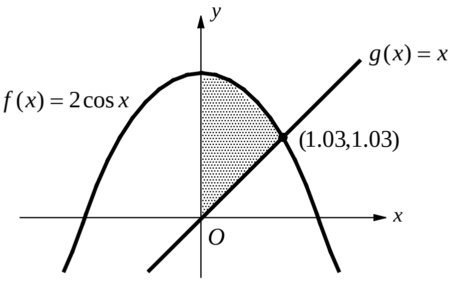
if the curves are defined by functions of x, and

$$\boldsymbol{A = \int_{y_1}^{y_2} \left[(\textbf{right curve}) - (\textbf{left curve}) \right] dy} ,$$

if the curves are defined by functions of y.

Example 1 □ Find the area of the region in the first quadrant enclosed by the graphs of $f(x) = 2\cos x$, $g(x) = x$ and the *y*-axis.

Solution □ Use a graphing calculator to find the point of intersection and sketch the two curves. The point of intersection in the first quadrant is $(1.03, 1.03)$.



The area between the curves is

$$\begin{aligned} A &= \int_a^b \left[f(x) - g(x) \right] dx \\ &= \int_0^{1.03} \left[2\cos x - x \right] dx \\ &= \left[2\sin x - \frac{x^2}{2} \right]_0^{1.03} \\ &= (2\sin 1.03 - \frac{1.03^2}{2}) - (2\sin 0 - 0) \\ &= 1.184 \end{aligned}$$

The limits of integration are $a = 0$ and $b = 1.03$.

Exercises - Area of a Region Between Two Curves

Multiple Choice Questions

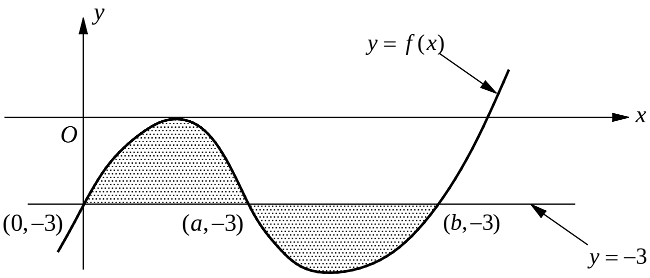
1. What is the area of the region enclosed by the graphs of $f(x) = x + 2$ and $g(x) = x^3 - 4x^2 + 6$?

(A) $\frac{193}{12}$ (B) $\frac{218}{12}$ (C) $\frac{253}{12}$ (D) $\frac{305}{12}$

---

2. What is the area of the region in the first quadrant, bounded by the curve $y = \sqrt[3]{x}$ and $y = x$?

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$



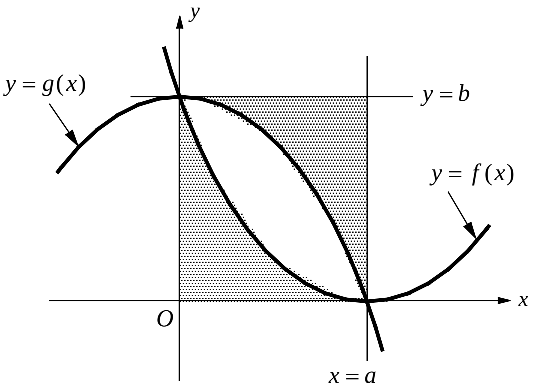
3. The curve $y = f(x)$ and the line $y = -3$, shown in the figure above, intersect at the points $(0,-3)$, $(a,-3)$, and $(b,-3)$. The sum of area of the shaded region enclosed by the curve and the line is given by

(A) $\int_0^a [3 - f(x)]\, dx + \int_a^b [-3 + f(x)]\, dx$

(B) $\int_0^a [-3 + f(x)]\, dx + \int_a^b [3 - f(x)]\, dx$

(C) $\int_0^a [f(x) + 3]\, dx + \int_a^b [-3 - f(x)]\, dx$

(D) $\int_0^a [f(x) - 3]\, dx + \int_a^b [3 - f(x)]\, dx$



4. Which of the following is the area of the shaded region in the figure above?

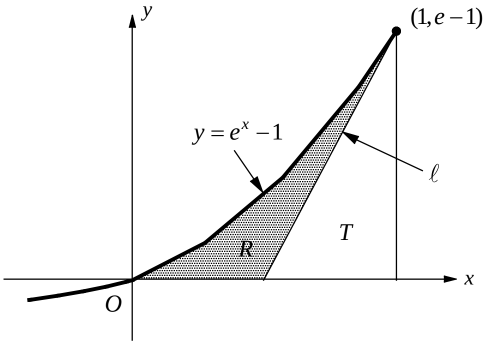
(A) $\int_0^a [g(x)-f(x)]\,dx$

(B) $\int_0^a [b+g(x)-f(x)]\,dx$

(C) $\int_0^a [b-g(x)-f(x)]\,dx$

(D) $\int_0^a [b-g(x)+f(x)]\,dx$

Free Response Questions



5. The figure above shows the graph of $y=e^x-1$ and the line ℓ tangent to the graph at $(1, e-1)$.

(a) Find the area of the triangular region T, which is bounded by the line $x=1$, x-axis and ℓ.

(b) Find the area of region R, which is bounded by the graph of $y=e^x-1$, x-axis and ℓ.

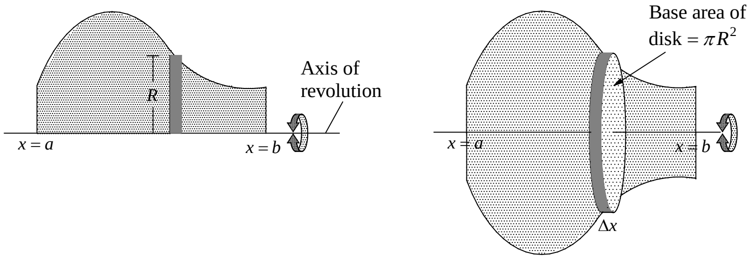
5.2 Volumes by Disk and Washers

The solid generated by rotating a plane region about an axis is called a **solid of revolution**. The simplest such solid is a right circular cylinder or **disk**.

Volume of disk = (base area of disk)(width of disk)

$\Delta V = A(x)\,\Delta x$.

So the definition of volume gives us $\boldsymbol{V = \int_a^b A(x)\,dx}$.



The Disk Method

Horizontal Axis of Revolution

$$\boldsymbol{V = \pi\int_a^b [R(x)]^2\,dx}$$

Vertical Axis of Revolution

$$\boldsymbol{V = \pi\int_{\mathrm{c}}^{\mathrm{d}} [R(y)]^2\,dy}$$

Horizontal axis of revolution

Vertical axis of revolution

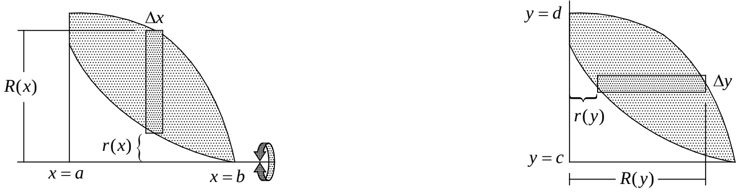
The Washer Method

Horizontal Axis of Revolution

$$\boldsymbol{V = \pi\int_a^b ([R(x)]^2 - [r(x)]^2)\,dx}$$

Vertical Axis of Revolution

$$\boldsymbol{V = \pi\int_c^d ([R(y)]^2 - [r(y)]^2)\,dy}$$



Horizontal axis of revolution

Vertical axis of revolution

Example 1 ☐ Find the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x}$, $x = 1$, and $x = 4$ about the x-axis.

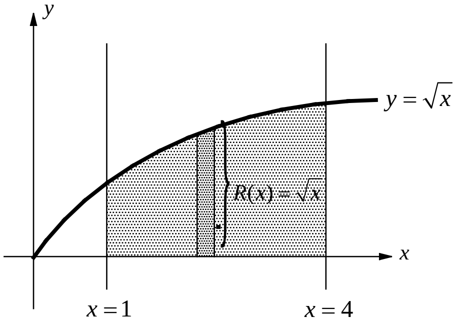
Solution ☐ Sketch the graphs.
The volume is

$$V = \pi \int_a^b \left[R(x)\right]^2 dx$$

$$= \pi \int_1^4 \left[\sqrt{x}\right]^2 dx = \pi \int_1^4 x \, dx$$

$$= \pi \left[\frac{x^2}{2}\right]_1^4 = \pi(\frac{16}{2} - \frac{1}{2})$$

$$= \frac{15\pi}{2}$$



Example 2 ☐ Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^{1/3}$, $y = x^2$,

(a) about the x-axis (b) about the y-axis.

Solution ☐ Sketch the graphs.

(a) The region runs from $x = 0$ to $x = 1$.

The volume is

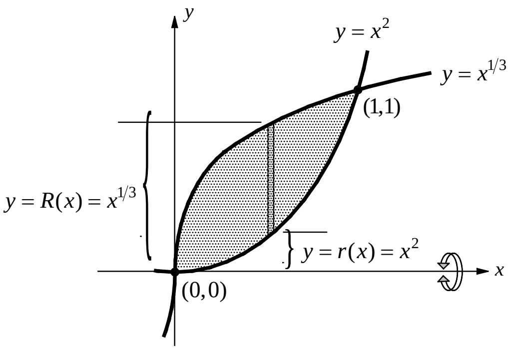
$$V = \pi \int_a^b (\left[R(x)\right]^2 - \left[r(x)\right]^2) \, dx$$

$$= \pi \int_0^1 (\left[x^{1/3}\right]^2 - \left[x^2\right]^2) \, dx$$

$$= \pi \int_0^1 (x^{2/3} - x^4) \, dx$$

$$= \pi \left[\frac{3}{5} x^{5/3} - \frac{1}{5} x^5\right]_0^1$$

$$= \frac{2\pi}{5}$$



(b) The region runs from $y = 0$ to $y = 1$.

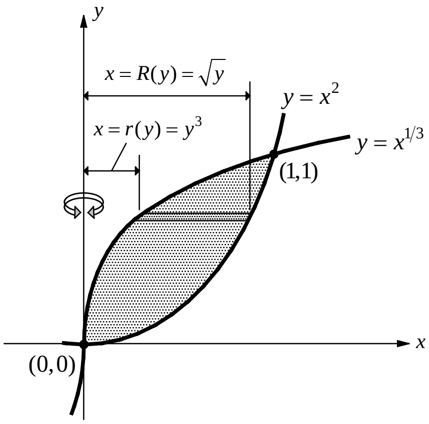
$$V = \pi \int_c^d (\left[R(y)\right]^2 - \left[r(y)\right]^2) \, dy$$

$$= \pi \int_0^1 (\left[\sqrt{y}\right]^2 - \left[y^3\right]^2) \, dy$$

$$= \pi \int_0^1 (y - y^6) \, dx$$

$$= \pi \left[\frac{1}{2} y^2 - \frac{1}{7} y^7\right]_0^1$$

$$= \frac{5\pi}{14}$$



Exercises - Volumes by Disk and Washers

Multiple Choice Questions

1. The region in the first quadrant bounded by the graph of $y = \sec x$, $x = \frac{\pi}{3}$, and the coordinate axes is rotated about the *x*-axis. What is the volume of the solid generated?

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\sqrt{3}\pi$ (D) 3π

---

2. The region enclosed by the graphs of $y = e^{(x/2)}$ and $y = (x-1)^2$ from $x = 0$ to $x = 1$ is rotated about the *x*-axis. What is the volume of the solid generated?

(A) $\frac{11}{4}\pi$ (B) $2(e-1)\pi$ (C) $(e-\frac{3}{2})\pi$ (D) $(e-\frac{6}{5})\pi$

---

3. Let R be the region between the graphs of $y = 1+\sin(\pi x)$ and $y = x^3$ from $x = 0$ to $x = 1$. The volume of the solid obtained by revolving R about the *x*-axis is given by

(A) $\pi\int_0^1 \left[1+\sin(\pi x) - x^3\right] dx$

(B) $\pi\int_0^1 \left[\left(1+\sin(\pi x)\right)^2 - x^6\right] dx$

(C) $\pi\int_0^1 \left[1+\sin(\pi x) - x^3\right]^2 dx$

(D) $2\pi\int_0^1 \left[1+\sin(\pi x) - x^3\right] dx$

4. The region R is enclosed by the graph of $y=\sqrt{x+1}$, the line $y=x-1$, and the y-axis. The volume of the solid generated when is R rotated about the line $y=2$ is

(A) $\frac{13}{2}\pi$ (B) $\frac{20}{3}\pi$ (C) $\frac{49}{6}\pi$ (D) 9π

---

5. The region R is enclosed by the graph of $y=3x-x^2$ and the line $y=x$. If the region R is rotated about the line $y=-1$, the volume of the solid that is generated is represented by which of the following integrals?

(A) $\pi\int_0^2 \left[3x-x^2-x+1\right]^2 dx$

(B) $\pi\int_0^2 \left[\left(3x-x^2+1\right)^2-(x+1)^2\right] dx$

(C) $\pi\int_0^2 \left[\left(3x-x^2+1\right)-(x+1)\right]^2 dx$

(D) $\pi\int_0^2 \left[\left(3x-x^2-1\right)^2-(x-1)^2\right] dx$

---

6. The region R is enclosed by the graph of $y=x+\frac{3}{x}$ and the line $y=4$. The volume of the solid generated when is R rotated about the x-axis is

(A) $\frac{16}{3}\pi$ (B) 4π (C) 6π (D) $\frac{15\pi}{2}$

---

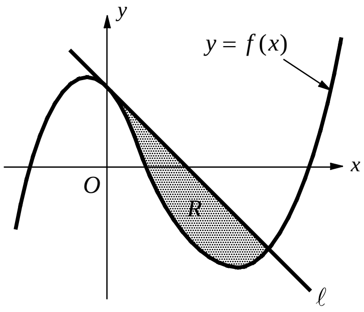
7. The volume of the solid generated by revolving the region enclosed by the ellipse $x^2+9y^2=36$ about the x-axis is

(A) 14π (B) 16π (C) 24π (D) 32π

8. The volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x}$, $y = 2$, and y-axis about the y-axis is

(A) $\frac{32}{5}\pi$ (B) $\frac{16}{3}\pi$ (C) $\frac{10}{3}\pi$ (D) $\frac{8}{3}\pi$

Free Response Questions



9. Let f be the function given by $f(x) = x^3 - 2x^2 - x + \cos x$. Let R be the shaded region bounded by the graph of f and the line ℓ, which is the line tangent to the graph of f at $x = 0$, as shown above.

(a) Find the equation of the line ℓ.

(b) Find the area of R.

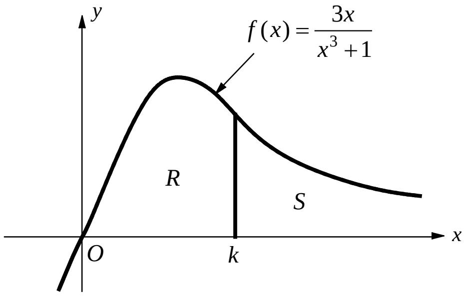
(c) Set up, but do not evaluate, an integral expression for the volume of the solid generated when R is revolved about the line $y = 2$.

10. Let R be the region between the graphs of $y = e^x$, $y = 2$ and $x = -1$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the line $x = -1$.

(c) Find the volume of the solid generated when R is revolved about the line $y = -1$.



11. Let f be the function given by $f(x) = \frac{3x}{x^3+1}$. Let R be the region bounded by the graph of f, the x- axis, and the vertical line $x = k$, where $k > 0$. $\boxed{BC}$

(a) Find the volume of the solid generated when R is revolved about the x-axis in terms of k.

(b) Let S be the unbounded region in the first quadrant to the right of the vertical line $x = k$ and below the graph of f, as shown in the figure above. Find the value of k such that the volume of the solid generated when S is revolved about the x-axis is equal to the volume of the solid found in part (a).

5.3 Volumes of Solids with Known Cross Sections

The volume of a solid of known cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b.

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis, the volume is

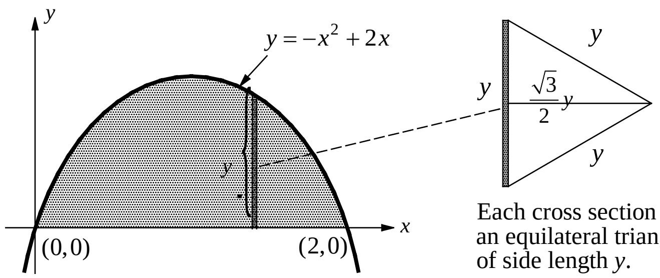
$$\boldsymbol{V = \int_a^b A(x)\,dx}.$$

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis, the volume is

$$\boldsymbol{V = \int_c^d A(y)\,dy}.$$

Example 1 □ The base of a solid is the region in the first quadrant enclosed by the graph of $y = -x^2 + 2x$ and the x-axis. If every cross section of the solid perpendicular to the x-axis is an equilateral triangle, what is the volume of the solid?

Solution □ Sketch the typical cross section.



Each cross section is an equilateral triangle of side length y.

The area of each cross section is

$$A(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(y)(\frac{\sqrt{3}}{2}y) = \frac{\sqrt{3}}{4}y^2$$

$$= \frac{\sqrt{3}}{4}(-x^2 + 2x)^2$$ Substitute $-x^2 + 2x$ for y.

$$= \frac{\sqrt{3}}{4}(x^4 - 4x^3 + 4x^2)$$ FOIL

The equilateral triangles lie on the plane from $x = 0$ to $x = 2$.
The volume is

$$V = \int_a^b A(x)\,dx = \int_0^2 \frac{\sqrt{3}}{4}(x^4 - 4x^3 + 4x^2)\,dx$$

$$= \frac{\sqrt{3}}{4}\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2 = \frac{4\sqrt{3}}{15}$$

Exercises - Volumes of Solids with Known Cross Sections

Multiple Choice Questions

1. The base of a solid is the region enclosed by the graph of $y = e^x$, the coordinate axes, and the line $x = 1$. If the cross sections of the solid perpendicular to the *x*-axis are squares, what is the volume of the solid?

   (A) $\frac{e^2}{4}$ (B) $\frac{e^2 - 1}{2}$ (C) $\frac{e^2 + 1}{2}$ (D) $e^2 - \frac{1}{2}$

---

2. The base of a solid is the region enclosed by the graph of $y = \sqrt{x}$, the *x*-axis, and the line $x = 2$. If each cross section perpendicular to the *x*-axis is an equilateral triangle , what is the volume of the solid?

   (A) $\frac{\sqrt{3}}{8}$ (B) $\frac{\sqrt{3}}{6}$ (C) $\frac{\sqrt{3}}{4}$ (D) $\frac{\sqrt{3}}{2}$

---

3. The base of a solid is the region in the first quadrant bounded by the coordinate axes, and the line $2x + 3y = 6$. If the cross sections of the solid perpendicular to the *x*-axis are semicircles, what is the volume of the solid?

   (A) $\frac{\pi}{2}$ (B) $\frac{3\pi}{4}$ (C) π (D) $\frac{3\pi}{2}$

---

4. The base of a solid S is the semicircular region enclosed by the graph of $y = \sqrt{9 - x^2}$ and the *x*-axis. If the cross sections of S perpendicular to the *x*-axis are semicircles, what is the volume of the solid?

   (A) $\frac{20\pi}{3}$ (B) 6π (C) $\frac{9\pi}{2}$ (D) $\frac{7\pi}{2}$

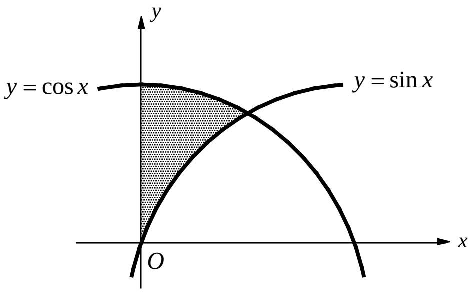
5. The base of a solid is the region bounded by the graph of $y = \sqrt{x}$, the x-axis and the line $x = 4$. If the cross sections of the solid perpendicular to the y-axis are squares, the volume of the solid is given by

(A) $\int_0^2 (4 - y^2)^2 \, dy$

(B) $\int_0^2 (4 - y)^2 \, dy$

(C) $\int_0^2 \left[(2 - y)^2\right]^2 dy$

(D) $\int_0^4 \left[(2 - y)^2\right]^2 dy$



6. The base of a solid is the region in the first quadrant bounded by the y-axis and the graphs of $y = \cos x$ and $y = \sin x$, as shown in the figure above. If the cross sections of the solid perpendicular to the x-axis are squares, what is the volume of the solid?

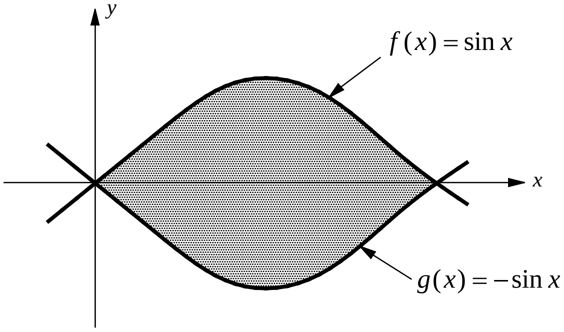
(A) $\pi - 1$ (B) $\pi + 1$ (C) $\frac{\pi - 2}{4}$ (D) $\frac{\pi + 2}{4}$

---

7. Let R be the region enclosed by the graph of $y = 3\sqrt{x} - x$ and the x-axis. The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $g(x) = \frac{1}{\sqrt{x}}$. What is the volume of the water in the pond?

(A) $2\sqrt{3}$ (B) 6 (C) $4\sqrt{3}$ (D) 9

Free Response Questions



8. Let $f(x) = \sin x$ and $g(x) = -\sin x$ for $0 \le x \le \pi$. The graphs of f and g are shown in the figure above.

   (a) Find the area of the shaded region enclosed by the graphs of f and g.

   (b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 3$.

   (c) Let h be the function given by $h(x) = k\sin x$ for $0 \le x \le \pi$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. If the volume of the solid is equal to 8π, what is the value of k?

---

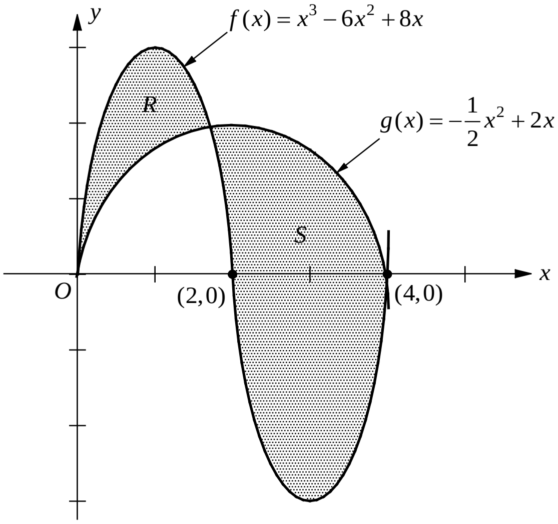
x (meters)	0	2	4	6	8	10	12
$D(x)$ (meters)	1.7	1.5	1.46	1.42	1.5	1.38	1.21

9. A 12 meter long tree trunk with circular cross sections of varying diameter is represented in the table above. The distance, x, of the tree trunk is measured from the ground and $D(x)$ represents the diameter at that point.

   (a) Write an integral expression in terms of $D(x)$ that represents the volume of the tree trunk between $x = 0$ and $x = 12$.

   (b) Approximate the volume of the tree trunk between $x = 0$ and $x = 12$ using the data from the table and a midpoint Riemann sum with three subintervals of equal length.

   (c) Explain why there must be a value x for $0 < x < 12$ such that $D'(x) = 0$?



10. Let R and S be the region bounded by the graphs of $f(x) = x^3 - 6x^2 + 8x$ and $g(x) = -\frac{1}{2}x^2 + 2x$ as shown in the figure above.

(a) Write, but do not evaluate, an integral expression that can be used to find the area of R.

(b) Write, but do not evaluate, an integral expression that can be used to find the area of S.

(c) The region R is the base of a solid. At all points in R at a distance x from the y-axis, the height of the solid is given by $g(x) = 4e^{-x}$. Find the volume of this solid.

(d) The region S models the surface of a small pond. At all points in S at a distance x from the y-axis, the depth of the water is given by $h(x) = 4 - \sqrt{x}$. Find the volume of water in the pond.

5.4 The Total Change Theorem (Application of FTC)

The **total change** (**accumulated change**) in a quantity over a time period is the integral of the rate of change of the quantity.

$$\textbf{Total Change} = \int_a^b F'(x)\,dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in natural and social sciences.
The following are a few examples of these idea:

1. If $F'(t)$ is the rate of growth of a population at time t, then
$$\int_a^b F'(t)\,dt = F(b) - F(a)$$
is the increase in population between times $t = a$ and $t = b$.

2. If $F'(t)$ is the rate of decomposition of a certain chemical, then
$$\int_a^b F'(t)\,dt = F(b) - F(a)$$
is the total amount that has decomposed between times $t = a$ and $t = b$.

3. If $F'(t)$ is the rate of consumption of a certain product, then
$$\int_a^b F'(t)\,dt = F(b) - F(a)$$
is the total amount that has been consumed between times $t = a$ and $t = b$.

4. If $F'(t)$ is the rate of change of temperature in a room, then
$$\int_a^b F'(t)\,dt = F(b) - F(a)$$
is the total change in temperature between times $t = a$ and $t = b$.

Example 1 □ For $0 \le t \le 60$, the rate of change of the number of mosquitoes at time t days is modeled by $f(t) = 6\sqrt{t}\sin(\frac{t}{6})$ mosquitoes per day. There are 1200 mosquitoes at time $t = 0$.

(a) At time $t = 20$, is the number of mosquitoes increasing or decreasing?

(b) According to the model, how many mosquitoes will be there at time $t = 60$.

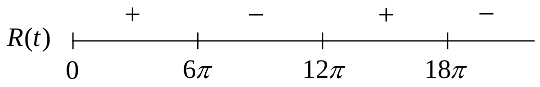
(c) To the nearest whole number, what is the maximum number of mosquitoes for $0 \le t \le 60$?

Solution □ (a) Since $f(20) = 6\sqrt{20}\sin(\frac{20}{6}) = -5.113 < 0$, the number of mosquitoes is decreasing at $t = 20$.

(b) The number of mosquitoes at time $t = 60$ is

$1200 + \int_a^b f(t)\,dt$	Initially there were 1200 mosquitoes.
$= 1200 + \int_0^{60} 6\sqrt{t}\,\sin(\frac{t}{6})\,dt$	$\int_0^{60} 6\sqrt{t}\,\sin(\frac{t}{6})\,dt$ represents the increase in mosquitoe population between times $t = 0$ and $t = 60$.
$= 1200 + 282.272 = 1482.272$	Use a graphing calculator to find the value of $\int_0^{60} 6\sqrt{t}\,\sin(\frac{t}{6})\,dt$.

(c) $f(t) = 0$ when $t = 0$, $t = 6\pi$, $t = 12\pi$, $t = 18\pi$



The absolute maximum number of mosquitoes occurs at $t = 6\pi$ or $t = 18\pi$.

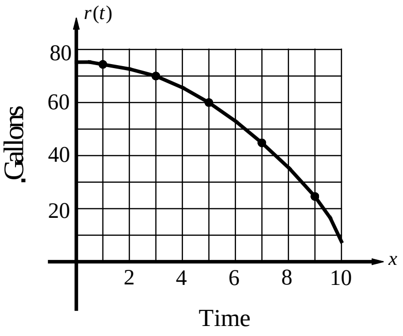
$$1200 + \int_0^{6\pi} 6\sqrt{t}\,\sin(\frac{t}{6})\,dt = 1200 + 214.751 = 1414.751$$

$$1200 + \int_0^{18\pi} 6\sqrt{t}\,\sin(\frac{t}{6})\,dt = 1200 + 326.710 = 1526.710$$

There are 1415 mosquitoes at $t = 6\pi$ and 1527 mosquitoes at $t = 18\pi$,

so the maximum number of mosquitoes is 1527 to the nearest whole number.

Example 2 □ Water leaks from a tank at a rate of $r(t)$ gallons per hour. The graph of r , for the time interval $0 \le t \le 10$ hours, is shown at the right. Use the midpoint Riemann sum with five intervals of equal length to estimate the total amount of water leaked out during the first 10 hours.



Solution □ The five intervals are $[0,2]$, $[2,4]$, $[4,6]$, $[6,8]$, and $[8,10]$.
75, 70, 60, 45, and 25 are the heights at each midpoint.

Total amount of water leaked out during the first 10 hours is
$75 \cdot 2 + 70 \cdot 2 + 60 \cdot 2 + 45 \cdot 2 + 25 \cdot 2$
$= 550$ gallons.

Exercises - The Total Change Theorem (Application of FTC)

Multiple Choice Questions

1. Oil is pumped out from a tank at the rate of $\dfrac{20e^{(-0.1t)}}{1+e^{-t}}$ gallons per minute, where t is measured in minutes. To the nearest gallon, how many gallons of oil are pumped out from a tank during the time interval $0 \le t \le 6$?

   (A) 62 (B) 78 (C) 85 (D) 93

---

2. Pollutant is released into a lake at the rate of $\dfrac{50e^{-t/2}}{\sqrt{t+1}}$ gallons per hour. To the nearest gallon, how many gallons of pollutant are released during the time interval $0 \le t \le 12$?

   (A) 53 (B) 58 (C) 66 (D) 75

---

3. Oil is pumped into an oil tank at the rate of $S(t)$ gallons per hour during the time interval $0 \le t \le 8$ hours. During the same time interval, oil is removed from the tank at the rate of $R(t)$ gallons per hour. If the oil tank contained 200 gallons of oil at time $t = 0$, which of the following expressions shows the amount of oil in the tank at time $t = 6$ hours?

   (A) $200 + S(6) - R(6)$

   (B) $200 + S'(6) - R'(6)$

   (C) $200 + \int_0^6 (S(t) - R(t))\, dt$

   (D) $200 + \int_0^6 (S'(t) - R'(t))\, dt$

4. The rate at which people enter a supermarket, measured in people per hour on a given day, is modeled by the function S defined by $S(t) = \dfrac{720}{t^2 - 28t + 205}$, for $6 \le t \le 22$. To the nearest whole number, how many people entered the supermarket from time $t = 6$ to $t = 22$?

(A) 426 (B) 475 (C) 524 (D) 582

---

5. The height of the water in a cylindrical storage tank is modeled by a differential function $h(t)$, where h is measured in meters and t is measured in hours. At time $t = 0$ the height of the water in the tank is 8 meters. During the time interval $0 \le t \le 20$ hours, the height is changing at the rate $h'(t) = 0.01t^3 - 0.3t^2 + 2.2t - 1.5$ meters per hour. What is the maximum height of the water in meters during the time period $0 \le t \le 20$?

(A) 28.156 (B) 30.108 (C) 32.654 (D) 33.975

Free Response Questions

6. Water is pumped into a tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\dfrac{1}{2}t^{2/3}$ gallons per minute, for $0 \le t \le 90$ minutes. At time $t = 0$, the tank contains 50 gallons of water.

(a) How many gallons of water leak out of the tank from time $t = 0$ to $t = 10$ minutes?

(b) How many gallons of water are in the tank at time $t = 10$ minutes?

(c) Write an expression for $f(t)$, the total number of gallons of water in the tank at time t.

(d) At what time t, for $0 \le t \le 90$, is the amount of water in the tank a maximum?

7. The rate at which the amount of granules of plastic at a toy factory is changing during a workday is modeled by $P(t) = 5 - 2\sqrt{x} - 4\sin(\frac{x^2}{12})$ tons per hour, where $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the factory has 6 tons of granules of plastic.

   (a) Find $P'(3)$. Using correct unit, interpret your answer in the context of the problem.

   (b) At what time during the 8 hours was the amount of granules of plastic decreasing most rapidly?

   (c) What was the maximum amount of granules of plastic at the factory during the 8 working hours?

5.5 Motion of a Particle, Distance, and Displacement

If a particle moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\boldsymbol{s(b) - s(a) = \int_a^b v(t)\,dt}$$

is the change of the position, or displacement, of the particle during the time period from $t = a$ to $t = b$.

The average velocity and the average acceleration over the time interval from $t = a$ to $t = b$ is

$$\textbf{Average velocity} = \frac{\textbf{displacement}}{\textbf{time}} = \frac{s(b)-s(a)}{b-a} = \frac{1}{b-a}\int_a^b v(t)\,dt$$

$$\textbf{Average acceleration} = \frac{v(b)-v(a)}{b-a} = \frac{1}{b-a}\int_a^b a(t)\,dt\,.$$

To find the total distance traveled we have to consider when the particle moves to the right, $v(t) \ge 0$, and when the particle moves to the left, $v(t) \le 0$.

In both cases, the distance is computed by integrating $|v(t)|$, the speed of the particle. Therefore

$$\textbf{Total distance traveled} = \int_a^b |v(t)|\,dt$$

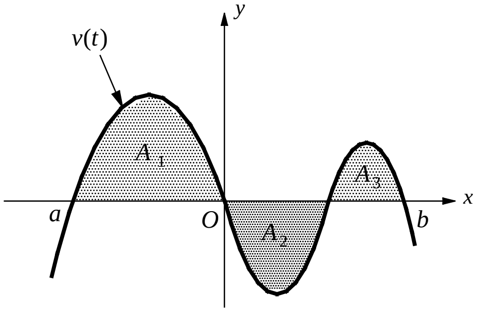
$$\textbf{Average speed} = \frac{\textbf{distance traveled}}{\textbf{time}} = \frac{1}{b-a}\int_a^b |v(t)|\,dt\,.$$

The acceleration of the object is $a(t) = v'(t)$, so

$$\boldsymbol{v(b) - v(a) = \int_a^b a(t)\,dt}$$

is the change in velocity from $t = a$ to $t = b$.

The figure at the right shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



displacement

$$= \int_a^b v(t)\,dt = A_1 - A_2 + A_3$$

total distance traveled

$$= \int_a^b |v(t)|\,dt = A_1 + A_2 + A_3$$

Example 1 □ A particle moves along the x-axis so that its velocity at any time $t \ge 0$ is given by $v(t) = t^2 - 3t - 4$.

(a) In which direction (left or right) is the particle moving at time $t = 5$?

(b) Find the acceleration of the particle at time $t = 5$.

(c) Given that $x(t)$ is the position of the particle at time t and that $x(0) = 12$, find $x(3)$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 6$.

(e) Find the average speed of the particle from $t = 0$ to $t = 6$.

(f) For what values of t, $0 \le t \le 6$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0,6]$?

Solution ☐ (a) $v(5) = (5)^2 - 3(5) - 4 > 0$

Since $v(5) > 0$, particle is moving to the right.

(b) $a(t) = v'(t) = 2t - 3$ So $a(5) = 2(5) - 3 = 7$.

(c) $x(3) - x(0) = \int_0^3 v(t)\,dt = \int_0^3 (t^2 - 3t - 4)\,dt$

$$= \left[\frac{1}{3}t^3 - \frac{3}{2}t^2 - 4t\right]_0^3 = -16.5$$

$x(3) = x(0) - 16.5 = 12 - 16.5 = -4.5$

(d) Total distance traveled $= \int_a^b |v(t)|\,dt = \int_0^6 |t^2 - 3t - 4|\,dt$

$$= -\int_0^4 (t^2 - 3t - 4)\,dt + \int_4^6 (t^2 - 3t - 4)\,dt = \frac{56}{3} + \frac{38}{3} = \frac{94}{3}$$

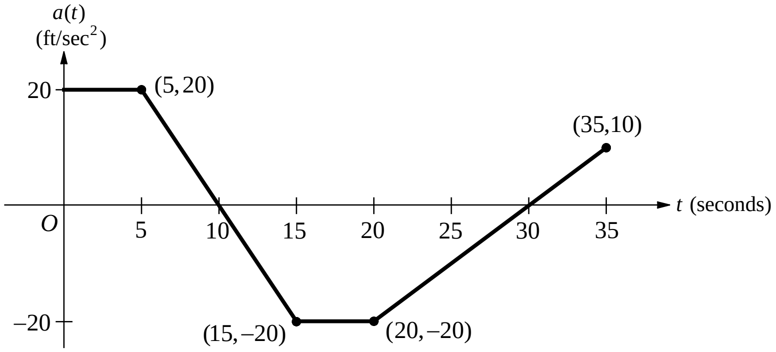
(e) Average speed $= \dfrac{\text{distance traveled}}{\text{time}} = \dfrac{94/3}{6} = \dfrac{47}{9}$

(f) Average velocity $= \dfrac{\text{displacement}}{\text{time}} = \dfrac{x(6) - x(0)}{6 - 0}$

$$= \frac{\int_0^6 (t^2 - 3t - 4)\,dt}{6} = \frac{-6}{6} = -1$$

$t^2 - 3t - 4 = -1 \Rightarrow t^2 - 3t - 3 = 0$ Since $t \ge 0$, $t = \dfrac{3 + \sqrt{21}}{2}$.

Example 2 ☐



A car is traveling on a straight road with velocity 80 ft / sec at time $t = 0$. For $0 \le t \le 35$ seconds, the cars acceleration $a(t)$, in ft / sec^2, is the piecewise linear function defined by the graph above.

(a) Find $a(15)$ and $v(15)$.

(b) At what time, other than $t = 0$, on the interval $0 \le t \le 35$, is the velocity of the car $80 \text{ ft}/\text{sec}$?

(c) On the time interval $0 \le t \le 35$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur?

(d) On the time interval $0 \le t \le 35$, what is the car's absolute minimum velocity, in ft/sec, and at what time does it occur?

Solution ☐ (a) $a(15) = -20$

$$v(15) = v(0) + \int_0^{15} a(t)\,dt$$

$$= v(0) + \int_0^{10} a(t)\,dt + \int_{10}^{15} a(t)\,dt$$

$$= 80 + \frac{1}{2}(10+5)\cdot 20 - \frac{1}{2}(5)(20) \qquad v(0) = 80$$

$$= 180$$

(b) At time $t = 20$. Because the area above the x-axis from $t = 0$ to $t = 10$ is same as the area below the x-axis from $t = 10$ to $t = 20$.

(c) Check endpoints and the points where $a(t)$ changes signs.

$$v(0) = 80$$

$$v(10) = v(0) + \int_0^{10} a(t)\,dt \qquad \text{When } t = 10,\ a(t) \text{ changes signs.}$$

$$= 80 + \frac{1}{2}(10+5)\cdot 20 = 230$$

$$v(30) = v(0) + \int_0^{30} a(t)\,dt \qquad \text{When } t = 30,\ a(t) \text{ changes signs.}$$

$$= 80 + \frac{1}{2}(10+5)\cdot 20 - \frac{1}{2}(20+5)\cdot 20$$

$$= -20$$

$$v(35) = v(0) + \int_0^{35} a(t)\,dt$$

$$= 80 + \frac{1}{2}(10+5)\cdot 20 - \frac{1}{2}(20+5)\cdot 20 + \frac{1}{2}(5)(10)$$

$$= 5$$

The car's absolute maximum velocity is 230 ft/sec and it occurs at $t = 10$.

(d) The car's absolute minimum velocity is -20 ft/sec and it occurs at $t = 30$.

Exercises - Motion of a Particle, Distance, and Displacement

Multiple Choice Questions

1. The acceleration of a particle moving along the x-axis at time t is given by $a(t)=2t-6$.
If at $t=1$, the velocity of the particle is 3 and its position is $\frac{1}{3}$, then the position $x(t)=$

(A) $\frac{t^3}{3}-6t^2+5t+\frac{1}{3}$
(B) $\frac{t^3}{3}-3t^2+8t-5$
(C) $\frac{t^3}{3}-6t+9$
(D) $\frac{t^3}{3}-3t^2+8t-\frac{7}{3}$

---

2. The velocity of a particle moving along the x-axis at any time t is given by $v(t)=3e^{-t}-t$.
What is the average speed of the particle over the time interval $0 \le t \le 3$?

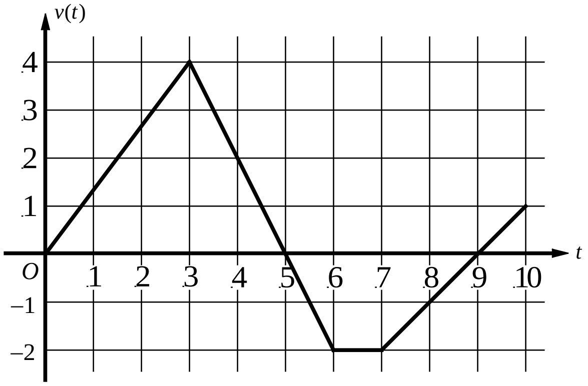
(A) 0.873 (B) 1.096 (C) 1.273 (D) 1.482

---

3. A particle travels along a straight line with a velocity of $v(t)=e^t(t^2-5t+6)$ meters per second.
What is the average velocity of the particle over the time interval $0 \le t \le 5$?

(A) 58.602 (B) 64.206 (C) 79.351 (D) 86.448

Questions 4-8 refer to the following situation.



A particle is moving along the x-axis. The velocity v of the particle at time t, $0 \le t \le 10$, is given by the function whose graph is shown above.

4. At what value(s) of t does the particle change direction?

(A) 3 only (B) 3 and 6 (C) 5 and 9 (D) 6 and 7

---

5. What is the total distance traveled by the particle over the time interval $0 \le t \le 10$?

(A) 15.5 (B) 12 (C) 9.5 (D) 8

---

6. At what time t during the time interval $0 \le t \le 10$ is the particle farthest to the right?

(A) 3 (B) 5 (C) 7 (D) 9

---

7. What is the velocity of the particle at time $t = 4$?

(A) –2 (B) 2 (C) 5 (D) 7

---

8. What is the acceleration of the particle at time $t = 4$?

(A) –2 (B) 2 (C) 5 (D) 7

9. A car is traveling on a straight road with position function given by $s(t) = (4t^2 - 3)e^{-0.5t}$, where s is measured in meters and t is measured in seconds. At time $t = 0$ seconds the brakes are applied to stop the car. To the nearest meters, how far does the car travel from time $t = 0$ to the moment the car stops?

(A) 9 (B) 10 (C) 11 (D) 12

Free Response Questions

10. A particle moves along the x-axis with a velocity given by $v(t) = t\cos(t^2 - 1)$ for $t \geq 0$.

(a) In which direction (left or right) is the particle moving at time $t = 2$?

(b) Find the acceleration of the particle at time $t = 2$. Is the velocity of the particle increasing at time $t = 2$?Justify your answer.

(c) Is the speed of the particle increasing at time $t = 2$? Justify your answer.

(d) Given that $x(t)$ is the position of the particle at time t and that $x(0) = 4$, find $x(2.5)$.

(e) During the time interval $0 \leq t \leq 2.5$, what is the greatest distance between the particle and the origin?

(f) Find the total distance traveled by the particle from $t = 0$ to $t = 2.5$.

11. A particle moves along the y-axis with a velocity given by $v(t) = 3t^2 - 14t + 8$ for $t \geq 0$. At time $t = 0$, the position of the particle is $y(0) = 2$.

(a) Find the minimum acceleration of the particle.

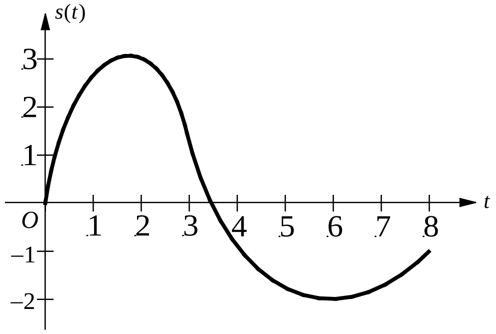
(b) For what values of t is the particle moving downward?

(c) What is the average velocity of the particle on the closed interval $[0,3]$?

(d) What is the average acceleration of the particle on the closed interval $[0,3]$?

(e) Find the position of the particle at time $t = 3$.

(f) Find the total distance traveled by the particle from $t = 0$ to $t = 3$.



12. A particle is moves along a horizontal line. The graph of the particle's position $s(t)$ at time t is shown above for $0 < t < 8$. The graph has horizontal tangents at $t = 2$ and $t = 6$ and has a point of inflection at $t = 3$.

(a) What is the velocity of the particle at time $t = 6$?

(b) The slope of tangent to the graph(not shown) at $t = 4$ is -1. What is the speed of the particle at time $t = 4$?

(c) For what values of t is the particle moving to the left?

(d) For what values of t is the velocity of the particle decreasing?

(e) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(f) During what time intervals, if any, is the acceleration of the particle positive? Justify your answer.

5.6 Average Value of a Function

Definition of the Average Value of a Function

If f is integrable on $\left[a,\ b\right]$, then the **average value** of f on the interval is

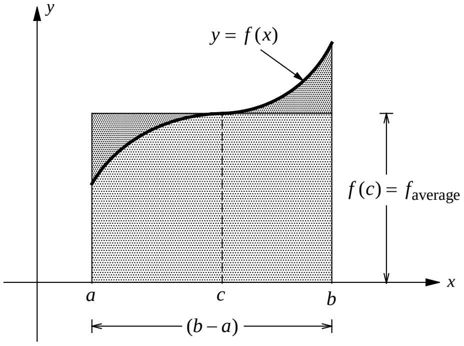
$$\frac{1}{\boldsymbol{b-a}}\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(x)\ dx}\ .$$

The Mean Value Theorem for Definite Integrals

If f is continuous on $\left[a,\ b\right]$, then there exists a number c in $\left[a,b\right]$, such that

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(x)\ dx = f(c)(b-a)}\ .$$

In the figure when $f \ge 0$, the area of the rectangle, which is $f(c)(b-a)$, is equal to the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.



Example 1 □ Find the average value of $f(x) = \dfrac{1}{2}x\cos(x^2) + x$ on the interval $\left[0, \sqrt{2\pi}\right]$.

Solution □ Average value

$$= \frac{1}{\sqrt{2\pi} - 0}\int_0^{\sqrt{2\pi}} (\frac{1}{2}x\cos x^2 + x)\ dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\sqrt{2\pi}} (\frac{1}{2}x\cos x^2)\ dx + \int_0^{\sqrt{2\pi}} x\ dx\right]$$

$$= \frac{1}{2\sqrt{2\pi}}\int_0^{\sqrt{2\pi}} x\cos x^2\ dx + \frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2\pi}} x\ dx$$

$$= \frac{1}{4\sqrt{2\pi}}\int_0^{2\pi} \cos u\ du + \frac{1}{2\sqrt{2\pi}}\int_0^{2\pi} du$$

Let $u = x^2$, then $du = 2xdx$. $\Rightarrow$ $\frac{1}{2}du = xdx$.

If $x = 0$, $u = 0$ and if $x = \sqrt{2\pi}$, $u = 2\pi$.

$$= \frac{1}{4\sqrt{2\pi}}\left[\sin u\right]_0^{2\pi} + \frac{1}{2\sqrt{2\pi}}\left[u\right]_0^{2\pi}$$

$$= \frac{1}{4\sqrt{2\pi}}\left[\sin 2\pi - \sin 0\right] + \frac{1}{2\sqrt{2\pi}}\left[2\pi - 0\right]$$

$$= \frac{\pi}{\sqrt{2\pi}} \approx 1.253$$

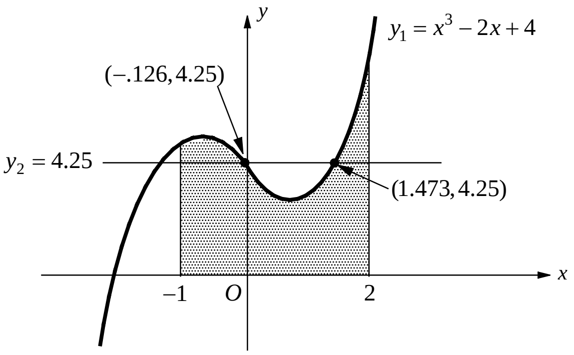
Example 2 □ Let f be the function given by $f(x) = x^3 - 2x + 4$ on the interval $[-1, 2]$. Find c such that average value of f on the interval is equal to $f(c)$.

Solution □ Average value

$$= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2} (x^3 - 2x + 4)\,dx$$

$$= \frac{1}{3}\left[\frac{x^4}{4} - x^2 + 4x\right]_{-1}^{2} = 4.25$$

Therefore, $f(c) = 4.25$.



Use a graphing calculator to graph $y_1 = x^3 - 2x + 4$ and $y_2 = 4.25$.
Find the points of intersection using a graphing calculator.
There are two points of intersection on the interval $[-1, 2]$, $(-.126, 4.25)$ and $(1.473, 4.25)$.

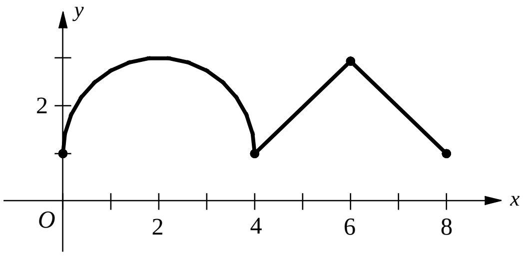
Therefore $c = -.126$ or $c = 1.473$.

Exercises -Average Value of a Function

Multiple Choice Questions

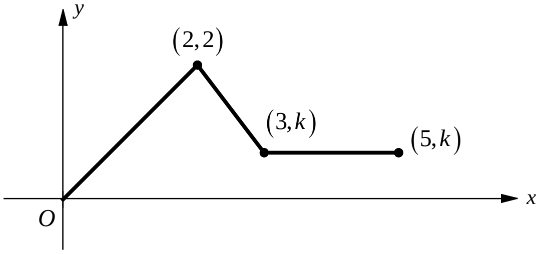
1. What is the average value of $f(x)=\sqrt{x}(4-x)$ on the closed interval $[0,4]$?

(A) $\frac{7}{3}$ (B) $\frac{21}{5}$ (C) $\frac{32}{15}$ (D) $\frac{35}{4}$



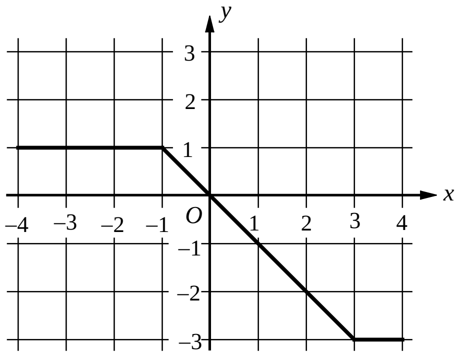
2. The graph of $y=f(x)$ consists of a semicircle and two line segments. What is the average value of f on the interval $[0,8]$?

(A) $\frac{\pi+2}{4}$ (B) $\frac{\pi+3}{4}$ (C) $\pi+1$ (D) $\frac{\pi+6}{4}$



3. The graph of $y=f(x)$ consists of three line segments as shown above. If the average value of f on the interval $[0,5]$ is 1 what is the value of k?

(A) $\frac{3}{5}$ (B) $\frac{7}{10}$ (C) $\frac{4}{5}$ (D) $\frac{9}{10}$



Graph of f

4. The function f is continuous for $-4 \le x \le 4$. The graph of f shown above consists of three line segments. What is the average value of f on the interval $-4 \le x \le 4$?

(A) -1

(B) $-\frac{1}{2}$

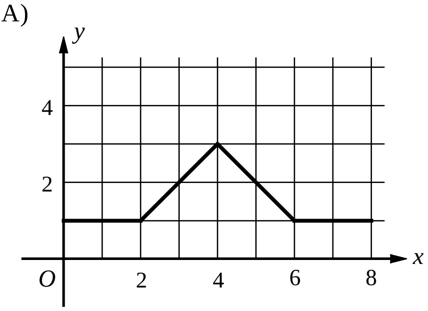
(C) $\frac{1}{2}$

(D) 1

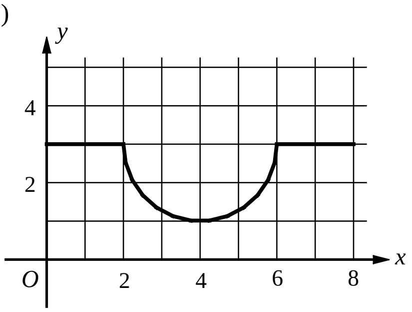
---

5. On the closed interval $[0,8]$, which of the following could be the graph of a function f with the property that $\frac{1}{8-0}\int_0^8 f(t)\,dt > 2$?

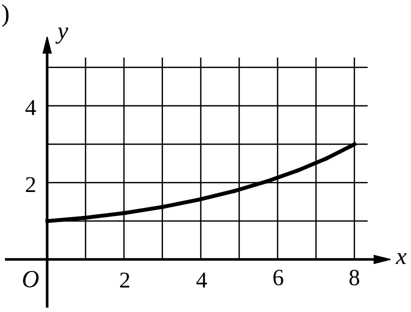
(A)



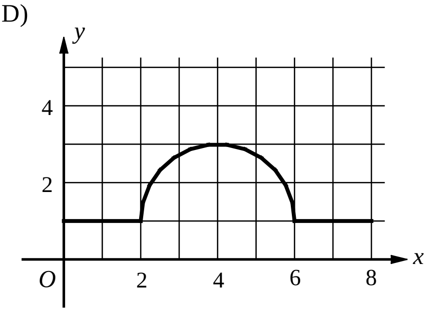
(B)



(C)



(D)



6. Let f be the function defined by

$$f(x)=\begin{cases}\frac{1}{16}x^2+1 & \text{for } 0\le x\le 4\\ 3\sqrt{x}-x & \text{for } 4<x\le 9.\end{cases}$$

What is the average value of f on the closed interval $0\le x\le 9$?

(A) $\frac{65}{54}$ (B) $\frac{35}{27}$ (C) $\frac{85}{27}$ (D) $\frac{55}{9}$

Free Response Questions

7. Let f be the function given by $f(x)=x\cos(x^2)$.

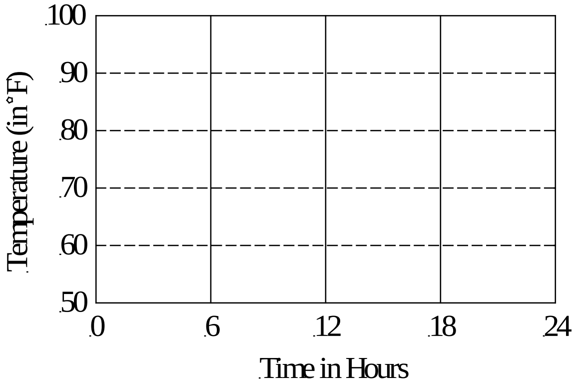
(a) Find the average rate of change of f on the closed interval $\left[0,\sqrt{\pi}\right]$.

(b) Find the average value of f on the closed interval $\left[0,\sqrt{\pi}\right]$.

(c) Find the average value of f' on the closed interval $\left[0,\sqrt{\pi}\right]$.

8. The temperature outside a house during a 24-hour period is given by $F(t) = 75 + 15\sin\left[\frac{\pi(t-6)}{12}\right]$, for $0 \le t \le 24$, where $F(t)$ is measured in degrees Fahrenheit and t is measured in hours.

(a) Sketch the graph of F on the grid below.



(b) Find the average temperature, to the nearest degree, between $t = 4$ and $t = 10$.

(c) An air conditioner cooled the house whenever the outside temperature was 80 degrees or above. For what values of t was the air conditioner cooling the house?

(d) The hourly cost of cooling the house is $0.12 for each degree the outside temperature exceeds 80 degrees. What is the total cost, to the nearest cent, to cool the house for the 24 hour period?

5.7 Length of a Curve (Distance Traveled Along a Curve) BC

If f' is continuous on the closed interval $[a,b]$, then the length of the curve $y=f(x)$ from $x=a$ to $x=b$ is

$$\boldsymbol{L=\int_a^b\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx=\int_a^b\sqrt{1+[f'(x)]^2}\,dx}$$

If g' is continuous on the closed interval $[c,d]$, then the length of the curve $x=g(y)$ from $y=c$ to $y=d$ is

$$\boldsymbol{L=\int_c^d\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy=\int_c^d\sqrt{1+[g'(y)]^2}\,dy}$$

Example 1 □ Find the length of the curve $y=2x^{3/2}+1$, from $x=1$ to $x=3$.

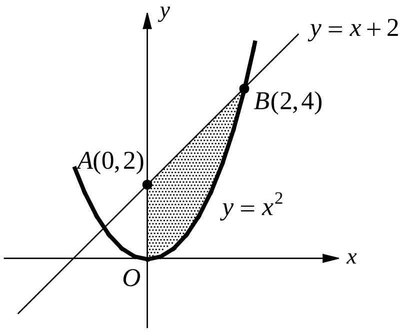
Solution □ $\frac{dy}{dx}=2\cdot\frac{3}{2}x^{1/2}=3x^{1/2}$

$$\left(\frac{dy}{dx}\right)^2=\left(3x^{1/2}\right)^2=9x$$

$$L=\int_1^3\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx=\int_1^3\sqrt{1+9x}\,dx$$

$$=\frac{2}{3}\cdot\frac{1}{9}\left[(1+9x)^{3/2}\right]_1^3\approx 8.633$$

Example 2 □ Let R be the region bounded by the y-axis and the graphs of $y=x^2$ and $y=x+2$. Find the perimeter of the region R.



Solution □ The y-intercept of the line is $A(0,2)$. We can find the point of intersection of the two graphs by solving $y=x^2$ and $y=x+2$ simultaneously for x. The point of intersection is $B(2,4)$.

The perimeter of the region R

$=OA+AB+$ length of the curve from O to B

$$=2+\sqrt{(4-2)^2+(2-0)^2}+\int_0^2\sqrt{1+(2x)^2}\,dx$$

$$=2+2\sqrt{2}+4.647$$

$$=9.475$$

$\frac{dy}{dx}=2x$

Use a graphing calculator to find the value of $\int_0^2\sqrt{1+(2x)^2}\,dx$.

Exercises - Length of a Curve (Distance Traveled Along a Curve) $\boxed{BC}$

Multiple Choice Questions

1. What is the length of the curve of $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 1$ to $x = 2$?

(A) $\frac{8}{3}$ (B) $\frac{10}{3}$ (C) 4 (D) $\frac{14}{3}$

---

2. Which of the following integrals gives the length of the graph of $y = \ln(\sin x)$ between $x = \frac{\pi}{3}$ to $x = \frac{2\pi}{3}$?

(A) $\int_{\pi/3}^{2\pi/3} \csc^2 x \, dx$

(B) $\int_{\pi/3}^{2\pi/3} \sqrt{1 + \cot x} \, dx$

(C) $\int_{\pi/3}^{2\pi/3} \csc x \, dx$

(D) $\int_{\pi/3}^{2\pi/3} \sqrt{1 + \csc^2 x} \, dx$

---

3. Which of the following integrals gives the length of the graph of $y = \frac{1}{3}x^{3/2} - x^{1/2}$ between $x = 1$ to $x = 4$?

(A) $\frac{1}{2}\int_{1}^{4} (\sqrt{x} + \frac{1}{\sqrt{x}}) \, dx$

(B) $\frac{1}{2}\int_{1}^{4} (\sqrt{x} - \frac{1}{\sqrt{x}}) \, dx$

(C) $\frac{1}{2}\int_{1}^{4} (1 + \sqrt{x} + \frac{1}{\sqrt{x}}) \, dx$

(D) $\frac{1}{2}\int_{1}^{4} (1 + \sqrt{x} - \frac{1}{\sqrt{x}}) \, dx$

4. What is the length of the curve of $y=\ln(x^2+1)-x$ from $x=0$ to $x=3$?

(A) 1.026 (B) 1.826 (C) 2.227 (D) 3.135

---

5. If the length of a curve from $(0,-3)$ to $(3,3)$ is given by $\int_0^3 \sqrt{1+(x^2-1)^2}\,dx$, which of the following could be an equation for this curve?

(A) $y=\frac{x^3}{3}-\frac{x}{3}-3$

(B) $y=\frac{x^3}{3}-3x-3$

(C) $y=\frac{x^3}{3}-x-3$

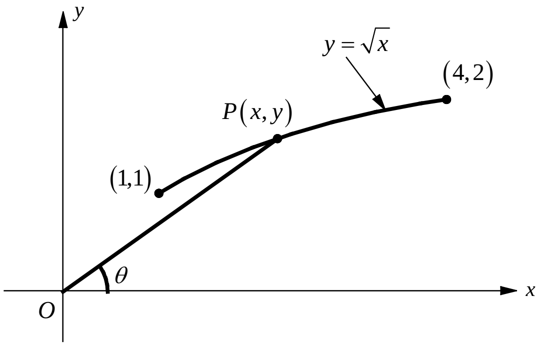
(D) $y=\frac{x^3}{3}+x-3$

---

6. If $F(x)=\int_1^{x^2}\sqrt{t+1}\,dt$, what is the length of the curve of from $x=1$ to $x=2$?

(A) $\frac{8}{3}$ (B) $\frac{10}{3}$ (C) $\frac{15}{3}$ (D) $\frac{17}{3}$

Free Response Questions



7. The figure above shows a point, $P(x, y)$, moving on the curve of $y=\sqrt{x}$, from the point $(1,1)$ to the point $(4,2)$. Let θ be the angle between $\overline{OP}$ and the positive x-axis.

(a) Find the x- and y-coordinates of point P in terms of $\cot\theta$.

(b) Find the length of the curve from the point $(1,1)$ to the point $(4,2)$.

(c) If the angle θ is changing at the rate of -0.1 radian per minute, how fast is the point P moving along the curve at the instant it is at the point $(3,\sqrt{3})$?

Chapter 6
Techniques of Integration

6.1 Basic Integration Rules

In this section, we will study several integration techniques for fitting an integrand into one of the basic integration rules. The basic integration rules are reviewed in Table 6.1 on page 252.

Procedures for Fitting Integrands to Basic Rules

Procedure	**Example**
1. Separating numerator	$\frac{1-2x}{1+x^2}=\frac{1}{1+x^2}-\frac{2x}{1+x^2}$
2. Adding and subtracting terms in numerator	$\frac{1}{1-e^x}=\frac{1-e^x+e^x}{1-e^x}=\frac{1-e^x}{1-e^x}+\frac{e^x}{1-e^x}$
3. Dividing improper fractions	$\frac{x^3-3x}{x^2-1}=x-\frac{2x}{x^2-1}$
4. Completing the square	$\frac{1}{\sqrt{4x-x^2}}=\frac{1}{\sqrt{4-(x-2)^2}}$

Other integration techniques, such as the simple substitution method, were covered in section 4.8. Using trigonometric identities, trigonometric substitution, Method of Partial Fractions and Integration by Parts will be covered later in this chapter.

1. Separating numerator

Example 1 □ Evaluate $\int \frac{1-2x}{1+x^2}\,dx$.

Solution □ $\int \frac{1-2x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{-2x}{1+x^2}\,dx$ Separate the numerator.

$\int \frac{1}{1+x^2}\,dx = \arctan x$ Basic integration rules.

$\int \frac{-2x}{1+x^2}\,dx = \int \frac{-du}{u} = -\ln u$ Let $u=1+x^2$, then $du=2x\,dx$.

Therefore $\int \frac{1-2x}{1+x^2}\,dx = \arctan x - \ln(1+x^2)+C$.

Table 6-1 Differentiation Rules and Basic Integration Rules

Differentiation Rules

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$

3. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$

4. $\frac{d}{dx}f\left[g(x)\right] = f'\left[g(x)\right]g'(x)$

5. $\frac{d}{dx}(e^x) = e^x$

6. $\frac{d}{dx}(a^x) = (\ln a)a^x$

7. $\frac{d}{dx}\ln|x| = \frac{1}{x}$

8. $\frac{d}{dx}\log_a x = \frac{1}{\ln a}\cdot\frac{1}{x}$

9. $\frac{d}{dx}(\sin x) = \cos x$

10. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

11. $\frac{d}{dx}(\cos x) = -\sin x$

12. $\frac{d}{dx}(\sec x) = \sec x \tan x$

13. $\frac{d}{dx}(\tan x) = \sec^2 x$

14. $\frac{d}{dx}(\cot x) = -\csc^2 x$

15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

17. $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

18. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

19. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

20. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Basic Integration Rules

1. $\int du = u + C$

2. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \frac{du}{u} = \ln|u| + C$

4. $\int e^u \, du = e^u + C$

5. $\int a^u \, du = \frac{1}{\ln a}a^u + C$

6. $\int \sin u \, du = -\cos u + C$

7. $\int \cos u \, du = \sin u + C$

8. $\int \sec^2 u \, du = \tan u + C$

9. $\int \csc^2 u \, du = -\cot u + C$

10. $\int \sec u \tan u \, du = \sec u + C$

11. $\int \csc u \cot u \, du = -\csc u + C$

12. $\int \tan u \, du = -\ln|\cos u| + C$

13. $\int \cot u \, du = \ln|\sin u| + C$

14. $\int \sec u \, du = \ln|\sec u + \tan u| + C$

15. $\int \csc u \, du = -\ln|\csc u + \cot u| + C$

16. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

2. Adding and subtracting terms in numerator

Example 2 □ Evaluate $\int \frac{1}{1-e^x}\, dx$.

Solution □ $\int \frac{1}{1-e^x}\, dx = \int \frac{1-e^x+e^x}{1-e^x}\, dx$ Add and subtract e^x in the numerator.

$= \int \frac{1-e^x}{1-e^x}\, dx + \int \frac{e^x}{1-e^x}\, dx$ Separate the numerator.

$= \int dx + \int \frac{e^x}{1-e^x}\, dx$

$= x - \ln(1-e^x) + C$ Use the basic integration rules.

3. Dividing improper fractions

Example 3 □ Evaluate $\int \frac{x^3-3x}{x^2-1}\, dx$.

Solution □ $\int \frac{x^3-3x}{x^2-1}\, dx = \int (x - \frac{2x}{x^2-1})\, dx$ Divide an improper fraction.

$= \int x\, dx - \int \frac{2x}{x^2-1}\, dx$

$= \frac{1}{2}x^2 - \ln(x^2-1) + C$ Use the basic integration rules.

4. Completing the square

Example 4 □ Evaluate $\int \frac{1}{\sqrt{4x-x^2}}\, dx$.

Solution □ $\int \frac{1}{\sqrt{4x-x^2}}\, dx = \int \frac{1}{\sqrt{4-(x-2)^2}}\, dx$ Complete the square.

$= \int \frac{1}{\sqrt{4-u^2}}\, du$ Let $u = x-2$, then $du = dx$.

$= \sin^{-1}(\frac{u}{2}) + C$ Use the basic integration rules.

$= \sin^{-1}(\frac{x-2}{2}) + C$

Exercises - Basic Integration Rules

Multiple Choice Questions

1. $\int \frac{1+\sin x}{\cos^2 x}\, dx =$

(A) $\tan x - \sec x \tan x + C$

(B) $\tan x + \sec x + C$

(C) $\tan x + \sec^2 x + C$

(D) $\ln(1+\cos^2 x) + C$

---

2. $\int \frac{e^{2x}}{1+e^x}\, dx =$

(A) $e^{2x} + \ln(1+e^x) + C$

(B) $e^{2x} - \ln(1+e^x) + C$

(C) $2e^{2x} - \ln(1+e^x) + C$

(D) $e^x - \ln(1+e^x) + C$

---

3. $\int 2\tan x \; \ln(\cos x)\, dx =$

(A) $\cos x \left[\ln(\cos x)\right] + C$

(B) $\sin x\left[\ln(\cos x)\right] + C$

(C) $-\left[\ln(\cos x)\right]^2 + C$

(D) $\left[\ln(\sin x)\right]^2 + C$

4. $\int_2^3 \frac{1}{x^2-4x+5}\,dx =$

(A) $\frac{\pi}{4}$ (B) $1-\frac{\pi}{4}$ (C) $1+\frac{\pi}{6}$ (D) $1+\frac{\pi}{4}$

---

5. $\int \frac{2x}{x^2+2x+1}\,dx =$

(A) $-\operatorname{arc}\cot x-\frac{1}{x+1}+C$

(B) $\arctan x+\frac{1}{x+1}+C$

(C) $2\ln|x+1|-\frac{2}{(x+1)^2}+C$

(D) $2\ln|x+1|+\frac{2}{x+1}+C$

Free Response Questions

6. The region bounded by $y=\frac{\sin x}{\sqrt{\cos x}}$, $x=0$, $x=\frac{\pi}{4}$, and the x-axis is revolved around the x-axis. What is the volume of the resulting solid?

6.2 Trigonometric Integrals

Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad \cot^2 x + 1 = \csc^2 x$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin 2x = 2\sin x \cos x$$

Guidelines for Evaluating $\int \sin^m x \ \cos^n x \ dx$.

1. If m is odd, save one sine factor and use $\sin^2 x = 1-\cos^2 x$ to express the remaining factor in terms of cosine.

Example 1 □ Evaluate $\int \sin^3 x \ \cos^2 x \ dx$.

Solution □

$$\begin{aligned}
&\int \sin^3 x \ \cos^2 x \ dx \\
&= \int \sin^2 x \ \cos^2 x \ (\sin x) \ dx && \text{One sine factor is saved.} \\
&= \int (1-\cos^2 x)\cos^2 x \ (\sin x) \ dx && \sin^2 x = 1-\cos^2 x \\
&= \int (\cos^2 x - \cos^4 x)\sin x \ dx && \text{Multiply.} \\
&= \int (u^2 - u^4)\ (-du) && u = \cos x \,,\ du = -\sin x \ dx \\
&= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C
\end{aligned}$$

2. If n is odd, save one cosine factor and use $\cos^2 x = 1-\sin^2 x$ to express the remaining factor in terms of sine.

Example 2 □ Evaluate $\int \cos^5 x \ dx$.

Solution □

$$\begin{aligned}
&\int \cos^5 x \ dx \\
&= \int \cos^4 x(\cos x) \ dx && \text{One cosine factor is saved.} \\
&= \int (1-\sin^2 x)^2 \cos x \ dx && \cos^2 x = 1-\sin^2 x \\
&= \int (1-u^2)^2 \ du && u = \sin x \,,\ du = \cos x \ dx \\
&= \int (1-2u^2+u^4)\ du && \text{Multiply.} \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \\
&= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C
\end{aligned}$$

3. If both m and n are even, substitute $\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{1+\cos 2x}{2}$ to reduce the integrand to lower powers of $\cos 2x$.

Example 3 □ Evaluate $\int \sin^2 x \cos^2 x \, dx$.

Solution □

$$\begin{aligned}
&\int \sin^2 x \cos^2 x \, dx \\
&= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx && \sin^2 x = \frac{1-\cos 2x}{2},\ \cos^2 x = \frac{1+\cos 2x}{2} \\
&= \frac{1}{4}\int \left(1-\cos^2 2x\right) dx && \text{Multiply.} \\
&= \frac{1}{4}\int \left(1-\frac{1+\cos 4x}{2}\right) dx && \cos^2 2x = \frac{1+\cos 4x}{2} \\
&= \frac{1}{8}\int \left(1-\cos 4x\right) dx && \text{Simplify.} \\
&= \frac{1}{8}x - \frac{1}{32}\sin 4x + C
\end{aligned}$$

Guidelines for Evaluating $\int \tan^m x \sec^n x \, dx$.

1. If m is odd, save a secant-tangent factor and convert the remaining factors to secants, and substitute $\tan^2 x = \sec^2 x - 1$.

Example 4 □ Evaluate $\int \tan^3 2x \sec^2 2x \, dx$.

Solution □

$$\begin{aligned}
&\int \tan^3 2x \sec^2 2x \, dx \\
&= \int \tan^2 2x \sec 2x \,(\tan 2x \sec 2x)\, dx && \text{A secant-tangent factor is saved.} \\
&= \int (\sec^2 2x - 1) \sec 2x \,(\tan 2x \sec 2x)\, dx && \tan^2 x = \sec^2 x - 1 \\
&= \int (\sec^3 2x - \sec 2x)\,(\tan 2x \sec 2x)\, dx && \text{Multiply.} \\
&= \frac{1}{2}\int (u^3 - u)\, du && u = \sec 2x,\ du = 2\sec 2x \tan 2x \, dx \\
&= \frac{1}{2}\left(\frac{1}{4}u^4 - \frac{1}{2}u^2\right) + C \\
&= \frac{1}{8}\sec^4 2x - \frac{1}{4}\sec^2 2x + C
\end{aligned}$$

2. If there are no secant factors and m is even, convert a tangent-squared factor to a secant-squared factor by substituting $\tan^2 x = \sec^2 x - 1$.

Example 5 □ Evaluate $\int \tan^4 x \, dx$.

Solution □ $\int \tan^4 x \, dx$

$= \int \tan^2 x \, (\tan^2 x) \, dx$

$= \int \tan^2 x \, (\sec^2 x - 1) \, dx$ $\quad \tan^2 x = \sec^2 x - 1$

$= \int \tan^2 x \, \sec^2 x \, dx - \int \tan^2 x \, dx$

$= \int \tan^2 x \, \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$

$= \int \underbrace{\tan^2 x \, \sec^2 x \, dx}_{\text{Set } u=\tan x,\ du=\sec^2 x \, dx} - \int \sec^2 x \, dx + \int 1 \, dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

3. If n is even, save a secant-squared factor and convert the remaining factors to tangents, and substitute $\sec^2 x = 1 + \tan^2 x$.

Example 6 □ Evaluate $\int_0^{\pi/4} \sec^4 x \tan^2 x \, dx$.

Solution □ $\int \sec^4 x \tan^2 x \, dx$

$= \int \sec^2 x \tan^2 x \, (\sec^2 x) \, dx$ $\quad$ A secant-squared factor is saved.

$= \int (1 + \tan^2 x) \tan^2 x \, (\sec^2 x) \, dx$ $\quad \sec^2 x = 1 + \tan^2 x$

$= \int (\tan^2 x + \tan^4 x) \, (\sec^2 x) \, dx$ $\quad$ Multiply.

$= \int (u^2 + u^4) \, du$ $\quad u = \tan x, \ du = \sec^2 x \, dx$

$= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$

$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$

Therefore

$\int_0^{\pi/4} \sec^4 x \tan^2 x \, dx$

$= \left[\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x\right]_0^{\pi/4}$

$= \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$

4. If there are no tangent factors and n is odd use integration by parts as illustrated in the section 6.5.

Exercises - Trigonometric Integrals

Multiple Choice Questions

1. $\int \sin^3 nx\, dx =$

(A) $\frac{1}{3n}\sin^3 nx - \frac{1}{n}\sin nx + C$

(B) $\frac{1}{3n}\cos^3 nx - \frac{1}{n}\cos nx + C$

(C) $\frac{1}{3n}\sin^3 nx - \frac{1}{n}\sin nx + C$

(D) $\frac{1}{3n}\cos^3 nx - \frac{1}{n}\sin nx + C$

---

2. $\int \cos^3 x\sqrt{\sin x}\, dx =$

(A) $\frac{1}{3}(\cos x)^3 - \frac{2}{5}(\cos x)^{5/2} + C$

(B) $\frac{2}{3}(\cos x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + C$

(C) $\frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + C$

(D) $\frac{2}{3}(\sin x)^{3/2} - \frac{2}{5}(\cos x)^{5/2} + C$

---

3. $\int_0^{\pi} 4\sin^4 \theta\, d\theta =$

(A) π (B) $\frac{3\pi}{2}$ (C) 2π (D) $\frac{5\pi}{2}$

4. $\int_0^{\pi/4} 4\tan^2\theta\, d\theta =$

(A) $2-\pi$ (B) $2-\frac{\pi}{2}$ (C) $2+\frac{\pi}{2}$ (D) $4-\pi$

---

5. $\int \sec^4 x\, dx =$

(A) $\frac{1}{3}\tan^3 x+\tan x+C$

(B) $\frac{1}{3}\tan^3 x+\sec x+C$

(C) $\frac{1}{3}\tan^3 x+\sec^2 x+C$

(D) $\frac{1}{5}\sec^5 x-\sec x\tan x+C$

Free Response Questions

6. Find the area bounded by the curves $y=\sin x$ and $y=\sin^3 x$ between $x=0$ and $x=\frac{\pi}{2}$.

6.3 Trigonometric Substitutions

Trigonometric Substitution

1. For integrals involving $\sqrt{a^2-u^2}$, let $u = a\sin\theta$. Then

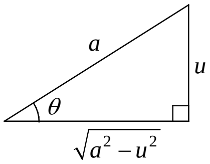
$$\boldsymbol{\sqrt{a^2-u^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta}$$

2. For integrals involving $\sqrt{a^2+u^2}$, let $u = a\tan\theta$. Then

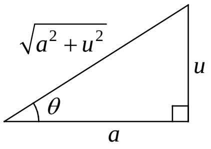
$$\boldsymbol{\sqrt{a^2+u^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta}$$

3. For integrals involving $\sqrt{u^2-a^2}$, let $u = a\sec\theta$. Then

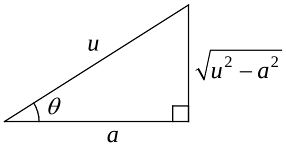
$$\boldsymbol{\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\theta-a^2+} = \sqrt{a^2(\sec^2\theta-1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta}$$



$u = a\sin\theta$



$u = a\tan\theta$



$u = a\sec\theta$

Note: $\operatorname{arc}\sec x = \arccos\frac{1}{x}$ $\qquad$ $\operatorname{arccsc}x = \arcsin\frac{1}{x}$

Example 1 □ Evaluate $\int_0^2 \sqrt{4-x^2}\ dx$.

Solution □

$$\int \sqrt{4-x^2}\ dx$$

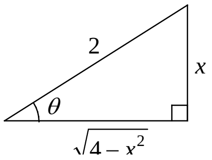
$$= \int \sqrt{4-4\sin^2\theta}\ 2\cos\theta\ d\theta \qquad x = 2\sin\theta,\ dx = 2\cos\theta\ d\theta$$

$$= 4\int \cos^2\theta\ d\theta \qquad \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = 2\cos\theta$$

$$= 4\int \frac{1+\cos 2\theta}{2}\ d\theta \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= 2\int (1+\cos 2\theta)\ d\theta$$

$$= 2(\theta+\frac{1}{2}\sin 2\theta)+C = 2\theta+\sin 2\theta+C$$



$x = 2\sin\theta$

$$= 2\theta+2\sin\theta\cos\theta+C$$

$$= 2\sin^{-1}(\frac{x}{2})+2(\frac{x}{2})(\frac{\sqrt{4-x^2}}{2})+C \qquad x = 2\sin\theta \ \Rightarrow\ \sin\theta = \frac{x}{2} \ \Rightarrow\ \theta = \sin^{-1}(\frac{x}{2})$$

$$= 2\sin^{-1}(\frac{x}{2})+\frac{x\sqrt{4-x^2}}{2}+C$$

Therefore

$$\int_0^2 \sqrt{4-x^2}\ dx = \left[2\sin^{-1}(\frac{x}{2}) + \frac{x\sqrt{4-x^2}}{2}\right]_0^2 = 2\sin^{-1}(1) = 2\cdot\frac{\pi}{2} = \pi$$

Example 2 □ Evaluate $\int \frac{dx}{\sqrt{9+x^2}}$.

Solution □ $\int \frac{dx}{\sqrt{9+x^2}}$

$$= \int \frac{3\sec^2\theta\ d\theta}{\sqrt{9+9\tan^2\theta}}$$

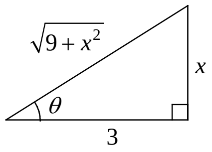
$x = 3\tan\theta\,,\ dx = 3\sec^2\theta\ d\theta$

$$= \int \frac{3\sec^2\theta\ d\theta}{3\sec\theta}$$

$$= \int \sec\theta\ d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$



$x = 3\tan\theta$

Example 3 □ Evaluate $\int \frac{dx}{x^2\sqrt{x^2-4}}$.

Solution □ $\int \frac{dx}{x^2\sqrt{x^2-4}}$

$$= \int \frac{2\sec\theta\tan\theta\ d\theta}{4\sec^2 x\ \sqrt{4\sec^2\theta - 4}}$$

$x = 2\sec\theta\,,\ dx = 2\sec\theta\tan\theta\ d\theta$

$$= \int \frac{2\sec\theta\tan\theta\ d\theta}{4\sec^2\theta\ 2\tan\theta}$$

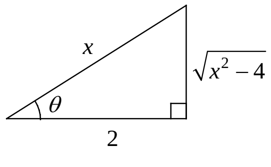
$\sqrt{4\sec^2\theta - 4} = \sqrt{4(\sec^2\theta - 1)} = 2\tan\theta$

$$= \frac{1}{4}\int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{4}\int \cos\theta\ d\theta$$

$$= \frac{1}{4}\sin\theta + C$$

$$= \frac{1}{4}\frac{\sqrt{x^2-4}}{x} + C$$



$x = 2\sec\theta$

Exercises - Trigonometric Substitutions

Multiple Choice Questions

1. If the substitution $x = 2\tan\theta$ is made in $\int \frac{x^3}{\sqrt{x^2+4}}\, dx$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the resulting integral is

   (A) $4\int \tan^2\theta \sec\theta\, d\theta$

   (B) $4\int \tan^2\theta \sec^2\theta\, d\theta$

   (C) $8\int \tan^3\theta\, d\theta$

   (D) $8\int \tan^3\theta \sec\theta\, d\theta$

---

2. $\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\, dx =$

   (A) $\frac{\pi}{18}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

---

3. $\int \frac{1}{x^2\sqrt{25-x^2}}\, dx =$

   (A) $-\frac{\sqrt{25-x^2}}{5x^2} + C$

   (B) $-\frac{\sqrt{25-x^2}}{25} + C$

   (C) $-\frac{\sqrt{25-x^2}}{25x} + C$

   (D) $\frac{\sqrt{25-x^2}}{25x^2} + C$

4. If the substitution $x = \sec\theta$ is made in $\int \frac{\sqrt{x^2 - 1}}{x^4}\, dx$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the resulting integral is

(A) $\int \sec^2\theta \tan\theta \, d\theta + C$

(B) $\int \sec\theta \tan^2\theta \, d\theta + C$

(C) $\int \sin\theta \cos^2\theta \, d\theta + C$

(D) $\int \sin^2\theta \cos\theta \, d\theta + C$

---

5. The average value of $f(x) = \frac{4}{\sqrt{9 + x^2}}$ on the interval $[0, 4]$ is

(A) $\ln 2$ (B) $\ln(\sqrt{2} - 1)$ (C) $\ln 3$ (D) $\ln(\sqrt{2} + 1)$

Free Response Questions

6. Let f be the function given by $f(x) = (9 - x^2)^{3/2}$ on the closed interval $[0, 3]$.

(a) Find the average value of f on the closed interval $[0, 3]$.

(b) Substitute $x = 3\sin\theta$ for f. Set up, but do not integrate, an integral expression in terms of θ for the average value of f on the closed interval $[0, 3]$.

6.4 L'Hospital's Rule

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near $x = c$ (except possibly at c).

If the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x \to c} \frac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}} = \lim_{x \to c} \frac{\boldsymbol{f'(x)}}{\boldsymbol{g'(x)}}$$

provided the limit on the right side exists.

L'Hospital's Rule can be applied only to quotients leading to indeterminate forms such as

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 1^∞, ∞^0, 0^0, and $\infty - \infty$.

L'Hospital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example 1 □ Find $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$.

Solution □ $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$ Indeterminate form $\frac{0}{0}$.

$= \lim_{x \to 0} \frac{e^x}{\cos x}$ L'Hospital's Rule: $\frac{d}{dx}(e^x - 1) = e^x$, $\frac{d}{dx}(\sin x) = \cos x$.

$= 1$ $e^0 = 1$ and $\cos 0 = 1$.

Example 2 □ Find $\lim_{x \to \pi/2} \frac{\sec x + 9}{\tan x}$.

Solution □ $\lim_{x \to \pi/2} \frac{\sec x + 9}{\tan x}$ Indeterminate form $\frac{\infty}{\infty}$.

$= \lim_{x \to \pi/2} \frac{\sec x \tan x}{\sec^2 x}$ L'Hospital's Rule: $\frac{d}{dx}(\sec x + 9) = \sec x \tan x$, $\frac{d}{dx}(\tan x) = \sec^2 x$.

$= \lim_{x \to \pi/2} \frac{\tan x}{\sec x}$ Simplify.

$= \lim_{x \to \pi/2} \sin x$

$= 1$

Example 3 □ Find $\lim\limits_{x\to\infty} x \tan\frac{1}{x}$.

Solution □

$\lim\limits_{x\to\infty} x \tan\frac{1}{x}$	Indeterminate form $\infty \cdot 0$.
$= \lim\limits_{x\to\infty} \frac{\tan(1/x)}{1/x}$	Rewrite $x = \frac{1}{1/x}$.
$= \lim\limits_{x\to\infty} \frac{(-1/x^2)\sec^2(1/x)}{-1/x^2}$	$\frac{d}{dx}(\tan x) = \sec^2 x$
$= \lim\limits_{x\to\infty} \sec^2(\frac{1}{x})$	Simplify.
$=1$	$\sec(0)=1$

Example 4 □ Find $\lim\limits_{x\to 0} \frac{e^x}{x}$.

Solution □

$\lim\limits_{x\to 0} \frac{e^x}{x} = \lim\limits_{x\to 0} \frac{e^x}{1} = 1$	Incorrect use of L'Hospital's Rule

This application is incorrect since $\frac{e^0}{0} = \frac{1}{0}$ is not an indeterminate form.

$\lim\limits_{x\to 0} \frac{e^x}{x} = \frac{1}{0}$	Make direct substitution.
$= \infty$	Answer

Exercises - L'Hospital's Rule

Multiple Choice Questions

1. $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2} =$

(A) 0 (B) $\dfrac{1}{2}$ (C) 1 (D) ∞

---

2. $\lim\limits_{x \to 0} \dfrac{\sin^{-1} x}{x} =$

(A) $-\infty$ (B) 0 (C) $\dfrac{\pi}{2}$ (D) 1

---

3. $\lim\limits_{\theta \to \pi} \dfrac{\sin \theta}{\theta - \pi} =$

(A) -1 (B) $-\dfrac{1}{2}$ (C) 0 (D) $\dfrac{1}{2}$

---

4. $\lim\limits_{x \to 0^+} (\tan x)^x =$

(A) $-\infty$ (B) 0 (C) $\dfrac{\pi}{4}$ (D) 1

---

5. $\lim\limits_{x \to \infty} (x)^{1/x} =$

(A) $-\infty$ (B) e (C) 1 (D) ∞

6. $\lim_{x\to 0}\left(\frac{1}{\tan x}-\frac{1}{x}\right)=$

(A) $-\infty$ (B) $-\frac{1}{2}$ (C) 0 (D) ∞

---

7. $\lim_{x\to 1^-}\left(\frac{2}{x^2-1}-\frac{x}{x-1}\right)=$

(A) $-\infty$ (B) $-\frac{3}{2}$ (C) -1 (D) ∞

Free Response Questions

8. Use L'Hospital's Rule to find the exact value of $\lim_{x\to\infty} x\left[\ln(x+3)-\ln x\right]$. Show the work that leads to your answer.

---

9. Use L'Hospital's Rule to find the exact value of $\lim_{x\to\infty}\left[x-\sqrt{x^2+x}\right]$. Show the work that leads to your answer.

6.5 Integration by Partial Fractions $\boxed{BC}$

A method for rewriting a rational function into the sum of simpler rational functions is called the **method of partial fractions**. For instance, the rational function $\dfrac{5x+1}{x^2+x-2}$ can be written as $\dfrac{5x+1}{x^2+x-2}=\dfrac{2}{x-1}+\dfrac{3}{x+2}$.

We call the fractions $\dfrac{2}{x-1}$ and $\dfrac{3}{x+2}$ **partial fractions** because their denominators are only part of the original denominator. To integrate the rational function $\dfrac{5x+1}{x^2+x-2}$, we simply sum the integrals of the partial fractions.

$$\int \frac{5x+1}{x^2+x-2}\,dx=\int \frac{2}{x-1}\,dx+\int \frac{3}{x+2}\,dx=2\ln|x-1|+3\ln|x+2|+C$$

Example 1 □ Evaluate $\displaystyle\int \frac{x^3}{x^2-1}\,dx$.

Solution □ If the degree of the numerator is greater than or equal to the degree of the denominator, **divide numerator by the denominator** to get a polynomial plus a proper fraction.

$$\begin{array}{rl} & \quad x \\ x^2-1 & \big)\overline{\,x^3\qquad} \\ - & \underline{x^3-x} \\ & \qquad x \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{x^3}{x^2-1}=x+\frac{x}{x^2-1}$$

Write $\dfrac{x}{x^2-1}=\dfrac{x}{(x+1)(x-1)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}$.

Multiplying both sides by $(x+1)(x-1)$, we have $x=A(x-1)+B(x+1)$

Let $x=1$. Then $1=A(1-1)+B(1+1) \Rightarrow B=\dfrac{1}{2}$.

Let $x=-1$. Then $-1=A(-1-1)+B(-1+1) \Rightarrow A=\dfrac{1}{2}$.

Thus, $\displaystyle\int \frac{x^3}{x^2-1}\,dx=\int x\,dx+\frac{1}{2}\int \frac{1}{x+1}\,dx+\frac{1}{2}\int \frac{1}{x-1}\,dx$

$$=\frac{1}{2}x^2+\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C.$$

Example 2 □ Evaluate $\int \frac{x+10}{(x-4)(x+3)}\,dx$.

Solution □ If the denominator is a product of **distinct linear factors**, then the partial fraction decomposition has the form

$$\frac{x+10}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}.$$

To find the values of A and B, multiply both sides of the equation by the least common denominator and get
$x+10 = A(x+3) + B(x-4)$.

To solve for A, choose $x = 4$, to eliminate the term $B(x-4)$.
$4+10 = A(4+3) + B(4-4) \;\Rightarrow\; 14 = 7A \;\Rightarrow\; A = 2$

To solve for B, choose $x = -3$, to eliminate the term $A(x+3)$.
$-3+10 = A(-3+3) + B(-3-4) \;\Rightarrow\; 7 = -7B \;\Rightarrow\; B = -1$

$$\begin{aligned}\int \frac{x+10}{(x-4)(x+3)}\,dx &= \int \frac{2}{(x-4)}\,dx + \int \frac{-1}{(x+3)}\,dx \\ &= 2\ln|x-4| - \ln|x+3| + C\end{aligned}$$

Exercises - Integration by Partial Fractions $\boxed{BC}$

Multiple Choice Questions

1. $\int \frac{dx}{x^2+x-6} =$

(A) $\frac{1}{5}\ln\left|\frac{x-1}{x+6}\right|+C$

(B) $\frac{1}{5}\ln\left|\frac{x+3}{x-2}\right|+C$

(C) $\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|+C$

(D) $\frac{1}{5}\ln\left|(x-2)(x+3)\right|+C$

---

2. $\int_4^7 \frac{5}{(x-2)(2x+1)}\,dx =$

(A) $\ln\frac{9}{10}$ (B) $\ln\frac{10}{9}$ (C) $\ln\frac{3}{2}$ (D) $\ln\frac{9}{4}$

---

3. $\int \frac{x}{x^2+5x+6}\,dx =$

(A) $-2\ln|x+2|+3\ln|(x+3)|+C$

(B) $2\ln|x+2|+3\ln|(x+3)|+C$

(C) $2\ln|(x+3)|-3\ln|x+2|+C$

(D) $-2\ln|(x+3)|-3\ln|x+2|+C$

4. $\int \dfrac{2e^{2x}}{(e^x-1)(e^x+1)}\,dx =$

(A) $\ln\left|e^x(e^{2x}-1)\right|+C$

(B) $\ln\left|2e^x(e^{2x}-1)\right|+C$

(C) $\ln\left|\dfrac{1}{e^{2x}-1}\right|+C$

(D) $\ln\left|(e^x-1)(e^x+1)\right|+C$

Free Response Questions

5. Let f be the function given by $f(\theta)=\int \dfrac{\sin\theta}{\cos\theta(\cos\theta-1)}\,d\theta$.

(a) Substitute $x=\cos\theta$ and write an integral expression for f in terms of x.

(b) Use the method of partial fractions to find $f(\theta)$.

6.6 Integration by Parts $\boxed{BC}$

Integration by Parts Formula

If u and v are functions of x and have continuous derivatives, then

$$\int \boldsymbol{u\,dv} = \boldsymbol{uv} - \int \boldsymbol{v\,du}\,.$$

Guidelines for Integration by Parts

1. For integrals of the form

$\int x^n e^{ax}\,dx$, $\int x^n \sin ax\,dx$, or $\int x^n \cos ax\,dx$

let $u = x^n$ and let $dv = e^{ax}\,dx$, $\sin ax\,dx$, or $\cos ax\,dx$.

2. For integrals of the form

$\int x^n \ln x\,dx$, $\int x^n \arcsin ax\,dx$, $\int x^n \arccos ax\,dx$, or $\int x^n \arctan ax\,dx$

let $u = \ln x$, $\arcsin ax$, $\arccos ax$, or $\arctan ax$ and let $dv = x^n\,dx$.

3. For integrals of the form

$\int e^{ax} \sin bx\,dx$ or $\int e^{ax} \cos bx\,dx$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax}\,dx$.

Example 1 □ Find $\int x \sin x\,dx$.

Solution □ Let $u = x$ and $dv = \sin x\,dx$.

Then $du = dx$ and $v = \int \sin x\,dx = -\cos x$.

$$\int \overbrace{x}^{u}\overbrace{\sin x\,dx}^{dv} = \overbrace{x}^{u}\overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\,\overbrace{dx}^{du}$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

Example 2 □ Evaluate $\int \arctan x\,dx$.

Solution □ Let $u = \arctan x$ and $dv = dx$.

Then $du = \dfrac{1}{1+x^2}\,dx$ and $v = \int dx = x$

$$\int \overbrace{\arctan x}^{u}\ \overbrace{dx}^{dv} = \overbrace{\arctan x}^{u}\cdot\overbrace{x}^{v} - \int \overbrace{x}^{v}\cdot\overbrace{\frac{1}{1+x^2}\,dx}^{du}$$

$$= x\arctan x - \int \frac{x}{1+x^2}\,dx$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

Tabular Method

In problems involving repeated applications of integration by parts, a tabular method can help to organized the work. This method works well for integrals of the form $\int x^n e^{ax}\,dx$, $\int x^n \sin ax\,dx$, and $\int x^n \cos ax\,dx$.

Example 3 □ Evaluate $\int x^2 e^x\,dx$.

Solution □ Let $u = x^2$ and $dv = e^x\,dx$.

$$\begin{tikzcd}
\underline{\text{u and its Derivatives}} & & \underline{\text{v' and its Antiderivatives}} & \\
x^2 \arrow[drr, "+"] & & e^x & \\
2x \arrow[drr, "-"] & & e^x \arrow[r] & +x^2e^x \\
2 \arrow[drr, "+"] & & e^x \arrow[r] & -2xe^x \\
0 & & e^x \arrow[r] & +2e^x
\end{tikzcd}$$

Differentiate until you obtain 0 as aderivative.

Hence, $\int x^2 e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$

Example 4 □ Evaluate $\int e^{2x} \cos x\,dx$.

Solution □ Let $u = \cos x$ and $dv = e^{2x}\,dx$.

$$\begin{tikzcd}
\underline{\text{u and its Derivatives}} & & \underline{\text{v' and its Antiderivatives}} & \\
\cos x \arrow[drr, "+"] & & e^{2x} & \\
-\sin x \arrow[drr, "-"] & & \frac{1}{2}e^{2x} \arrow[r] & +\frac{1}{2}e^{2x}\cos x \\
-\cos x \arrow[r, "+"] & {} & \frac{1}{4}e^{2x} \arrow[r] & -\frac{1}{4}e^{2x}(-\sin x)
\end{tikzcd}$$

Stop here. The variable part of the last row is same as the first row.

We stop differentiating and integrating as soon as we reach a row that is the same as the first row except for multiplicative constants. The table is interpreted as follow.

$$\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\cos x - \frac{1}{4}e^{2x}(-\sin x) + \int \overbrace{\frac{1}{4}e^{2x}(-\cos x)}^{\text{Product of the last row}}\,dx$$

By adding $\int \frac{1}{4}e^{2x}(\cos x)\,dx$ on each side we get,

$$\frac{5}{4}\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}(\sin x).$$

Therefore, $\int e^{2x}\cos x\,dx = \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x + C$.

Exercises - Integration by Parts $\boxed{BC}$

Multiple Choice Questions

1. $\int x\sin(2x)\,dx =$

(A) $-x\cos(2x)+\frac{1}{2}\sin(2x)+C$

(B) $\frac{x}{2}\cos(2x)-\frac{1}{4}\sin(2x)+C$

(C) $-\frac{x}{2}\cos(2x)+\frac{1}{4}\sin(2x)+C$

(D) $\frac{x}{2}\cos(2x)+\frac{1}{4}\sin(2x)+C$

---

2. $\int_0^2 xe^x\,dx =$

(A) e^2-1 (B) e^2+1 (C) $e-1$ (D) $e+1$

---

3. If $\int x^2\cos(3x)\,dx = f(x)-\frac{2}{3}\int x\sin(3x)\,dx$, then $f(x)=$

(A) $\frac{2}{3}x\sin(3x)$

(B) $\frac{1}{3}x^2\sin(3x)$

(C) $\frac{2}{3}x\cos(3x)$

(D) $\frac{1}{3}x\sin(3x)-\frac{2}{3}\cos(3x)$

4. $\int x^2 \ln x \, dx =$

(A) $\dfrac{x^2 \ln x}{2} - \dfrac{x^2}{4} + C$

(B) $x^3 \ln x - \dfrac{x^3}{3} + C$

(C) $\dfrac{x^3 \ln x}{3} - \dfrac{x^3}{9} + C$

(D) $\dfrac{x(\ln x)^2}{2} - \dfrac{x^3}{3} + C$

---

5. $\int_0^{\pi/4} x \sec^2 x \, dx =$

(A) $\dfrac{\pi}{4} - \ln 2$ (B) $\dfrac{\pi}{4} + \ln 2$ (C) $\dfrac{\pi}{4} - \dfrac{\ln 2}{2}$ (D) $\dfrac{\pi}{4} + \dfrac{\ln 2}{2}$

---

6. $\int \sec^3 x \, dx =$

(A) $\dfrac{1}{4}\sec^4 x + C$

(B) $\dfrac{1}{2}\sec^2 x \tan x + \dfrac{1}{2}\ln\left|\sec x\right| + C$

(C) $\dfrac{1}{2}\sec^2 x \tan x + \dfrac{1}{2}\ln\left|\tan x\right| + C$

(D) $\dfrac{1}{2}\sec x \tan x + \dfrac{1}{2}\ln\left|\sec x + \tan x\right| + C$

7. $\int f(x)\cos(nx)\,dx =$

(A) $\frac{1}{n}f(x)\sin(nx)-\frac{1}{n}\int f'(x)\sin(nx)\,dx$

(B) $\frac{1}{n}f(x)\cos(nx)-\frac{1}{n}\int f'(x)\cos(nx)\,dx$

(C) $n\,f(x)\cos(nx)+\frac{1}{n}\int f'(x)\sin(nx)\,dx$

(D) $n\,f(x)\cos(nx)-\frac{1}{n}\int f'(x)\cos(nx)\,dx$

---

8. If $\int \operatorname{arc}\cos x\,dx = x\ \arccos x+\int f(x)\,dx$, then $f(x)=$

(A) $-x\sqrt{1-x^2}$ (B) $x\sqrt{1-x^2}$ (C) $-\frac{1}{\sqrt{1-x^2}}$ (D) $\frac{x}{\sqrt{1-x^2}}$

---

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	–2	3	4	–1
3	2	–1	–3	5

9. The table above gives values of f, f', g, and g' for selected values of x.
If $\int_1^3 f(x)g'(x)\,dx=8$, then $\int_1^3 f'(x)g(x)\,dx=$

(A) –4 (B) –1 (C) 5 (D) 8

Free Response Questions

10. Find the area of the region bounded by $y = \arcsin x$, $y = 0$, and $x = 1$. Show the work that leads to your answer.

6.7 Improper Integrals $\boxed{BC}$

Improper Integrals with Infinite Integration Limits

1. If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty \boldsymbol{f(x)\,dx} = \lim_{b\to\infty}\int_a^b \boldsymbol{f(x)\,dx}\,.$$

2. If $f(x)$ is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^b \boldsymbol{f(x)\,dx} = \lim_{a\to-\infty}\int_a^b \boldsymbol{f(x)\,dx}\,.$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty \boldsymbol{f(x)\,dx} = \int_{-\infty}^c \boldsymbol{f(x)\,dx} + \int_c^\infty \boldsymbol{f(x)\,dx}\,,$$

where c is any real number.

Improper Integrals with Infinite Discontinuities

1. If $f(x)$ is continuous on $[a,b)$ and has an infinite discontinuity at b, then

$$\int_a^b \boldsymbol{f(x)\,dx} = \lim_{c\to b^-}\int_a^c \boldsymbol{f(x)\,dx}\,.$$

2. If $f(x)$ is continuous on $(a,b]$ and has an infinite discontinuity at a, then

$$\int_a^b \boldsymbol{f(x)\,dx} = \lim_{c\to a^+}\int_c^b \boldsymbol{f(x)\,dx}\,.$$

3. If $f(x)$ is continuous on $[a,b]$, except for some number c in (a,b) at which f has an infinite discontinuity, then

$$\int_a^b \boldsymbol{f(x)\,dx} = \int_a^c \boldsymbol{f(x)\,dx} + \int_c^b \boldsymbol{f(x)\,dx}\,,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the value of the improper integral. If the limit fails to exists, the improper integral **diverges**.

Example 1 □ Evaluate $\int_0^\infty xe^{-x^2}\,dx$.

Solution □ $\int_0^\infty xe^{-x^2}\,dx$

$$= \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^b$$

$$= -\frac{1}{2}\lim_{b\to\infty}\left[e^{-b^2} - e^0\right] = -\frac{1}{2}(0-1) = \frac{1}{2}$$

Example 2 □ Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution □
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$
$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{\mathrm{b}} \frac{dx}{1+x^2}$$
$$= \lim_{a\to-\infty} \left[\tan^{-1} x\right]_a^0 + \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b$$
$$= \lim_{a\to-\infty} \left(\tan^{-1} 0 - \tan^{-1} a\right) + \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right)$$
$$= \left(0-(-\frac{\pi}{2})\right) + \left(\frac{\pi}{2} - 0\right) = \pi$$

Example 3 □ Evaluate $\int_{1}^{5} \frac{dx}{\sqrt{x-1}}$.

Solution □
$$\int_{1}^{5} \frac{dx}{\sqrt{x-1}}$$
$$= \lim_{b\to1^+} \int_{b}^{5} \frac{dx}{\sqrt{x-1}} \ = \lim_{b\to1^+} \left[2\sqrt{x-1}\right]_b^5$$
$$= \lim_{b\to1^+} \left[2\sqrt{4} - 2\sqrt{b-1}\right]$$
$$= 4$$

Example 4 □ Find $\int_{0}^{1} \frac{dx}{1-x}$.

Solution □
$$\int_{0}^{1} \frac{dx}{1-x}$$
$$= \lim_{b\to1^-} \int_{0}^{b} \frac{dx}{1-x} \ = \lim_{b\to1^-} \left[-\ln|1-x|\right]_0^b$$
$$= \lim_{b\to1^-} \left[-\ln|1-b| + \ln 1\right]$$
$$= \infty$$

Exercises - Improper Integrals $\boxed{BC}$

Multiple Choice Questions

1. $\int_2^{\infty} \frac{1}{\sqrt{x-1}}\,dx =$

(A) $-\infty$ (B) -2 (C) 1 (D) ∞

---

2. $\int_0^{\infty} \frac{1}{(x+3)(x+4)}\,dx =$

(A) $-\ln\frac{4}{3}$ (B) $-\ln\frac{3}{4}$ (C) 0 (D) $\ln 4$

---

3. $\int_0^{4} \frac{dx}{(x-1)^{2/3}} =$

(A) $3\sqrt[3]{3}$ (B) $3(1-\sqrt[3]{3})$ (C) $3(1+\sqrt[3]{3})$ (D) divergent

---

4. $\int_0^{\infty} x^2 e^{-x^3} =$

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) divergent

5. $\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx =$

(A) -6 (B) -4 (C) -2 (D) divergent

---

6. If $\int_0^1 \frac{ke^{-\sqrt{x}}}{\sqrt{x}}\, dx = 1$, what is the value of k ?

(A) $-\frac{1}{2}$ (B) $\frac{e}{2}$ (C) $\frac{1}{2}$ (D) There is no such value of k

---

Free Response Questions

7. Let f be the function given by $f(x) = \frac{x}{\sqrt{x^2+1}}\, dx$.

(a) Show that the improper integral $\int_1^\infty f(x)\, dx$ is divergent.

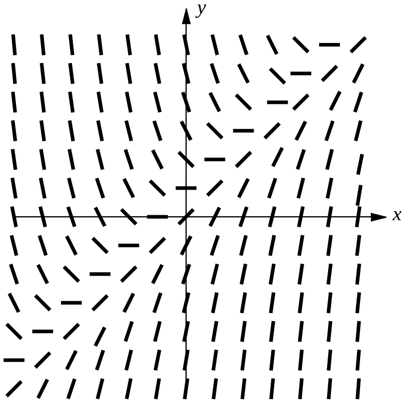
(b) Find the average value of f on the interval $[1, \infty)$.

Chapter 7
Further Applications of Integration

7.1 Slope Field

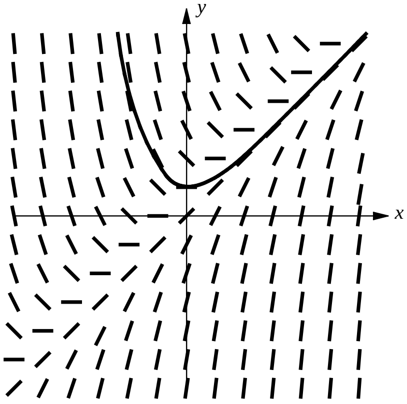
A first order differential equation of the form $\boldsymbol{y' = f(x, y)}$ says that the slope of a solution curve at a point (x, y) on the curve is $f(x, y)$. If we draw short line segments with slope $f(x, y)$ at several points (x, y), the result is called a **slope field**.

Figure 7-1 shows a slope field for the differential equation $y' = x - y + 1$
Figure 7-2 shows a particular solution curve through the point $(0,1)$.



Slope field for $y' = x - y + 1$

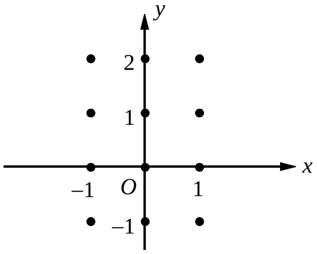
Figure 7-1



Particular solution for $y' = x - y + 1$ passing through (0,1)

Figure 7-2

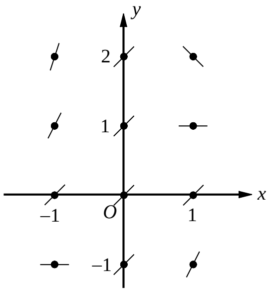
Example 1 □ On the axes provided, sketch a slope field for the differential equation $y' = 1 - xy$.



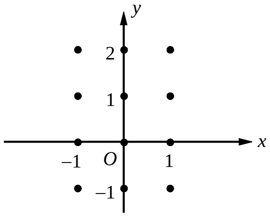
Solution □ Make a table showing the slope at the points shown on the graph.

x	–1	–1	–1	–1	0	0	0	0	1	1	1	1
y	–1	0	1	2	–1	0	1	2	–1	0	1	2
$y' = 1 - xy$	0	1	2	3	1	1	1	1	2	1	0	–1

Draw the line segments at the points with their respective slopes.



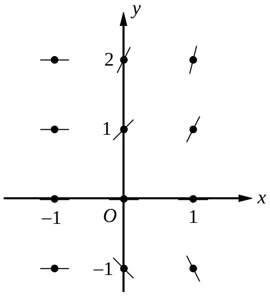
Example 2 □ On the axes provided, sketch a slope field for the differential equation $y' = y + xy$.



Solution □ Make a table showing the slope at the points shown on the graph.

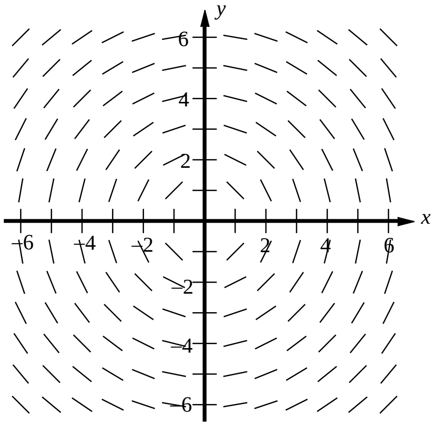
x	–1	–1	–1	–1	0	0	0	0	1	1	1	1
y	–1	0	1	2	–1	0	1	2	–1	0	1	2
$y' = y + xy$	0	0	0	0	–1	0	1	2	–2	0	2	4

Draw the line segments at the points with their respective slopes.



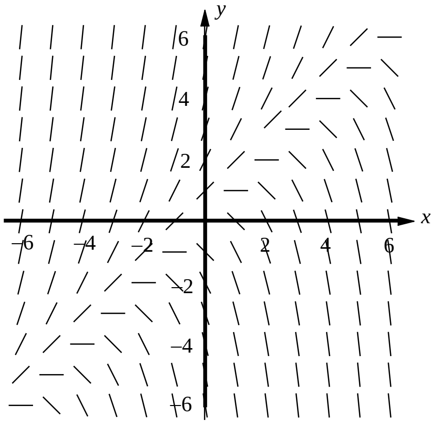
Exercises - Slope Field

Multiple Choice Questions



1. Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = \frac{x}{y}$ (B) $\frac{dy}{dx} = -\frac{x}{y}$ (C) $\frac{dy}{dx} = \frac{x^2}{y}$ (D) $\frac{dy}{dx} = -\frac{x^2}{y}$

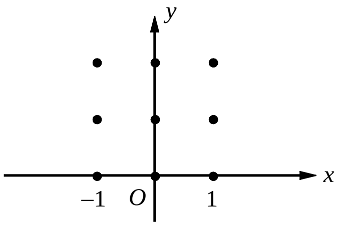


2. Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = x + y$ (B) $\frac{dy}{dx} = x - y$ (C) $\frac{dy}{dx} = -x + y$ (D) $\frac{dy}{dx} = x^2 - y$

Free Response Questions

3. On the axis provided, sketch a slope field for the differential equation $\frac{dy}{dx} = y - x^2$.



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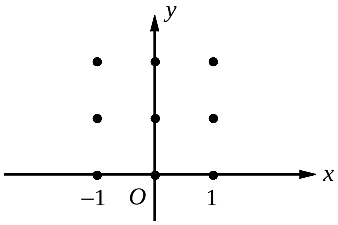
4. On the axis provided, sketch a slope field for the differential equation $\frac{dy}{dx} = x^2 + y^2$.



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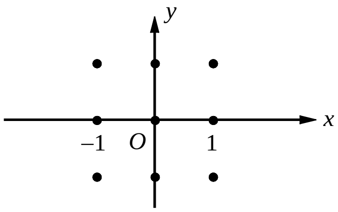
5. On the axis provided, sketch a slope field for the differential equation $\frac{dy}{dx} = (x+1)(y-2)$.



7.2 Separable Differential Equations

The equation $y' = f(x, y)$ is a separable equation if all x terms can be collected with dx and all y terms with dy. The differential equation then has the form

$$\boldsymbol{\frac{dy}{dx} = f(x)g(y)} \quad \text{or} \quad \boldsymbol{\frac{dy}{dx} = \frac{f(x)}{h(y)}}.$$

To solve the first equation we could rewrite it in the form $\frac{dy}{g(y)} = f(x)dx$, and integrate both sides of the equation:

$$\boldsymbol{\int \frac{dy}{g(y)} = \int f(x)dx}.$$

To solve the second equation we could rewrite it in the form $h(y)dy = f(x)dx$ and integrate both sides of the equation:

$$\boldsymbol{\int h(y)\,dy = \int f(x)\,dx}.$$

Example 1 ☐ Find the general solution of $(x+3)y' = 2y$.

Solution ☐

$(x+3)y' = 2y$	
$(x+3)\dfrac{dy}{dx} = 2y$	Rewrite y' as $\dfrac{dy}{dx}$.
$\dfrac{dy}{y} = \dfrac{2}{x+3}\,dx$	Separate the variables.
$\displaystyle\int \frac{dy}{y} = \int \frac{2}{x+3}\,dx$	Integrate.
$\ln\lvert y\rvert = 2\ln\lvert x+3\rvert + C_1$	
$= \ln(x+3)^2 + \ln C$	Let $C_1 = \ln C$.
$y = C(x+3)^2$	General solution

Example 2 ☐ Find the general solution of $\dfrac{dy}{dx} = -\dfrac{2x}{y}$.

Solution ☐

$\dfrac{dy}{dx} = -\dfrac{2x}{y}$	
$y\,dy = -2x\,dx$	Separate the variables.
$\displaystyle\int y\,dy = \int -2x\,dx$	Integrate.
$\dfrac{1}{2}y^2 = -x^2 + C_1$	
$2x^2 + y^2 = C$	General solution, $C = 2C_1$

Exercises - Separable Differential Equations

Multiple Choice Questions

1. The solution to the differential equation $\frac{dy}{dx} = \frac{3x^2}{2y}$, where $y(3) = 4$, is

(A) $y = \sqrt{\frac{x^3}{3} + 1}$ (B) $y = 7 - \sqrt{\frac{x^3}{3}}$ (C) $y = \sqrt{x^3 - 9}$ (D) $y = \sqrt{x^3 - 11}$

---

2. If $\frac{dy}{dx} = \frac{x + \sec^2 x}{y}$ and $y(0) = 2$, then $y =$

(A) $\sqrt{x^2 + 2\sec x + 2}$

(B) $\sqrt{x^2 + 2\tan x + 4}$

(C) $\sqrt{x^2 + \sec^2 x + 2}$

(D) $\sqrt{x^2 + \tan^2 x + 4}$

---

3. At each point (x, y) on a certain curve, the slope of the curve is xy. If the curve contains the point $(0, -1)$, which of the following is the equation for the curve?

(A) $y = x^2 - 2$ (B) $y = 3x^2 - 4$ (C) $y = -e^{\frac{x^2}{2}}$ (D) $y = -e^{(x^2 - 1)}$

---

4. If $\frac{dy}{dx} = (y - 4)\sec^2 x$ and $y(0) = 5$, then $y =$

(A) $e^{\tan x} + 4$ (B) $6e^{\tan x} - 1$ (C) $2e^{\tan x} + 2$ (D) $4\sec x + 1$

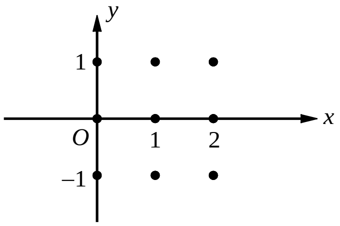
5. What is the value of $m+b$, if $y=mx+b$ is a solution to the differential equation $\frac{dy}{dx}=\frac{1}{4}x-y+1$?

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{5}{4}$

Free Response Questions

6. Consider the differential equation $\frac{dy}{dx}=\frac{x+1}{y}$.

(a) On the axis provided sketch a slope field for the given differential equation at the nine points indicated.



(b) Let $y=f(x)$ be the particular solution to the differential equation with the initial condition $y(1)=\sqrt{3}$. Write an equation for the line tangent to the graph of f at $(1,\sqrt{3})$ and use it to approximate $f(1.2)$.

(c) Find the particular solution $y=f(x)$ to the differential equation with the initial condition $y(1)=\sqrt{3}$.

(d) Use your solution from part (c) to find $f(1.2)$.

7. Consider the differential equation $\frac{dy}{dx} = \frac{2x+3}{e^y}$.

(a) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(0) = 2$. Write an equation for the line tangent to the graph of f at $(0, 2)$.

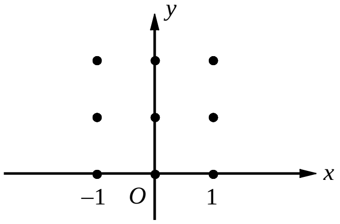
(b) Find $f''(0)$ with the initial condition $y(0) = 2$.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = \frac{2x+3}{e^y}$ with the initial condition $y(0) = 2$.

---

8. Consider the differential equation $\frac{dy}{dx} = \frac{y^2(1-2x)}{3}$.

(a) On the axis provided sketch a slope field for the given differential equation at the nine points indicated.



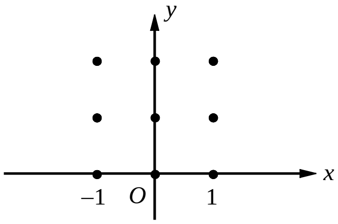
(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(\frac{1}{2}) = 4$. Does f have a relative minimum, a relative maximum, or neither at $x = \frac{1}{2}$? Justify your answer.

(d) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $y(\frac{1}{2}) = 4$.

9. Consider the differential equation $\frac{dy}{dx} = -2x + y + 1$.

   (a) On the axis provided sketch a slope field for the given differential equation at the nine points indicated.



   (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy- plane in which all the solution curves to the differential equation are concave down.

   (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = -1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

   (d) Find the value of the constants m and b, for which $y = mx + b$ is a solution to the differential equation.

7.3 Exponential Growth and Decay

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. If y is a function of time t, the proportion can be written as follows.

Rate of change of *y* is proportional to *y*.

$$\boldsymbol{\frac{dy}{dt} = ky}$$

The Law of Exponential Change

If y is a differentiable function of t such that $y > 0$ and $y' = ky$, for some constant k , then

$$\boldsymbol{y = y_0 e^{kt}} \text{, where } y_0 \text{ is the initial value of } y\text{.}$$

Exponential growth occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.
The number k is the **rate constant** of the equation.

Example 1 □ The number of bacteria in a culture increases at a rate proportional to the number present. If the number of bacteria was 600 after 3 hours and 19,200 after 8 hours, when will the population reach 120,000?

Solution □ Since the growth rate is proportional to population size, we use the equation $y = y_0 e^{kt}$.

$$600 = y_0 e^{k \cdot 3} \qquad y = 600 \text{ and } t = 3$$

$$y_0 = \frac{600}{e^{3k}} \qquad \text{Solve for } y_0 .$$

$$19,200 = y_0 e^{k \cdot 8} \qquad y = 19,200 \text{ and } t = 8$$

$$19,200 = \frac{600}{e^{3k}} e^{8k} \qquad \text{Substitution}$$

$$32 = e^{5k} \qquad \text{Simplify.}$$

$$k = \frac{1}{5} \ln 32 \approx 0.693$$

Therefore, the exponential growth model is $y = y_0 e^{0.693t}$.
To solve for y_0, substitute $y = 600$ when $t = 3$ and obtain

$$600 = y_0 e^{0.693(3)} .$$

$$y_0 = \frac{600}{e^{0.693(3)}} \approx 75$$

So the model is $y = 75e^{0.693t}$.

$$120,000 = 75e^{0.693t} \;\Rightarrow\; \frac{120,000}{75} = e^{0.693t}$$

$$\Rightarrow\; e^{0.693t} = 1600 \;\Rightarrow\; 0.693t = \ln 1600$$

$$\Rightarrow\; t \approx 10.646$$

Exercises - Exponential Growth and Decay

Multiple Choice Questions

1. Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles every four hours, in how many hours will the number of bacteria triple?

(A) $\ln(\frac{27}{2})$ (B) $\ln(\frac{81}{2})$ (C) $\frac{4\ln 2}{\ln 3}$ (D) $\frac{4\ln 3}{\ln 2}$

---

2. Population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 15 years what is the value of k?

(A) 0.035 (B) 0.046 (C) 0 .069 (D) 0.078

---

3. A baby weighs 6 pounds at birth and 9 pounds three months later. If the weight of baby increasing at a rate proportional to its weight, then how much will the baby weigh when she is 6 months old?

(A) 11.9 (B) 12.8 (C) 13.5 (D) 14.6

---

4. Temperature F changes according to the differential equation $\frac{dF}{dt} = kF$, where k is a constant and t is measured in minutes. If at time $t = 0$, $F = 180$ and at time $t = 16$, $F = 120$, what is the value of k?

(A) -0.025 (B) -0.032 (C) -0.045 (D) -0.058

Free Response Questions

5. The rate at which the amount of coffee in a coffeepot changes with time is given by the differential equation $\frac{dV}{dt} = kV$, where V is the amount of coffee left in the coffeepot at any time t seconds. At time $t = 0$ there were 16 ounces of coffee in the coffeepot and at time $t = 80$ there were 8 ounces of coffee remaining in the pot.

   (a) Write an equation for V, the amount of coffee remaining in the pot at any time t.

   (b) At what rate is the amount of coffee in the pot decreasing when there are 4 ounces of coffee remaining?

   (c) At what time t will the pot have 2 ounces of coffee remaining?

7.4 Logistic Equations $\boxed{BC}$

The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$$

is called a **logistic equation**.

In this equation $P(t)$ is the size of the population at time t, A is the **carrying capacity** (the maximum population that the environment is capable of sustaining in the long run), and k is a constant.

If $0 < P < A$, then $(1 - P/A)$ is positive so, $dP/dt > 0$ and the population increases.
If $P > A$, then $(1 - P/A)$ is negative, so $dP/dt < 0$ the population decreases.

In logistic equations

1. $\lim\limits_{t \to \infty} \frac{dP}{dt} = 0$
2. $\lim\limits_{t \to \infty} P(t) = A$.
3. The population is growing the fastest when $P = \frac{A}{2}$.
   (When P is half the carrying capacity.)
4. The graph of $P(t)$ has a point of inflection at the point where $P = \frac{A}{2}$.

Figure 7-3 displays typical logistic curves.

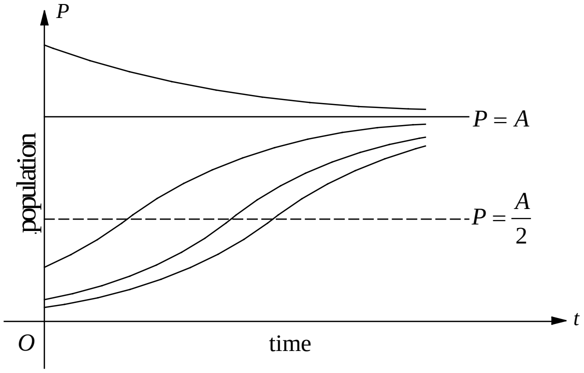


Figure 7-3

Solution curves for the logistic equations
with different initial conditions

Example 1 □ A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt}=\frac{P}{2}\left(3-\frac{P}{20}\right)$, where the initial population $P(0)=100$ and t is the time in years.

(a) What is $\lim_{t\to\infty} P(t)$?

(b) For what values of P is the population growing the fastest?

(c) Find the slope of the graph of P at the point of inflection.

Solution □ (a) Write the differential equation in the standard form.

$$\frac{dP}{dt}=\frac{P}{2}\left(3-\frac{P}{20}\right)=\frac{3P}{2}\left(1-\frac{P}{60}\right)$$

$$\lim_{t\to\infty} P(t) = A = 60$$

(b) The population is growing the fastest when $P=\frac{A}{2}$.

$$P=\frac{A}{2}=\frac{60}{2}=30$$

(c) The graph of P has a point of inflection at $P=\frac{A}{2}$.

So, when $P=30$,

$$\left.\frac{dP}{dt}\right|_{P=30}=\frac{30}{2}\left(3-\frac{30}{20}\right)=22.5$$

Exercises - Logistic Equations $\boxed{BC}$

Multiple Choice Questions

1. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = 3P - 0.0006P^2$, where the initial population is $P(0) = 1000$ and t is the time in years. What is $\lim_{t \to \infty} P(t)$?

(A) 1000 (B) 2000 (C) 3000 (D) 5000

---

2. A healthy population $P(t)$ of animals satisfies the logistic differential equation $\frac{dP}{dt} = 5P(1 - \frac{P}{240})$, where the initial population is $P(0) = 150$ and t is the time in years. For what value of P is the population growing the fastest?

(A) 48 (B) 60 (C) 120 (D) 240

---

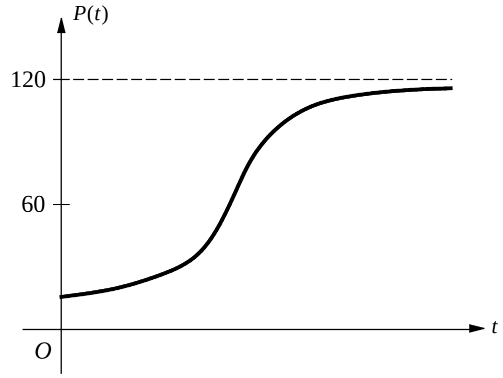
3. A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{150}\right)$, where the initial population is $P(0) = 800$ and t is the time in years. What is the slope of the graph of P at the point of inflection?

(A) 5 (B) 7.5 (C) 10 (D) 12.5

---

4. A certain rumor spreads in a small town at the rate $\frac{dy}{dt} = y(1 - 3y)$, where y is the fraction of the population that has heard the rumor at any time t. What fraction of the population has heard the rumor when it is spreading the fastest?

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$



5. Which of the following differential equations for population P could model the logistic growth shown in the figure above

(A) $\frac{dP}{dt} = 0.03P^2 - 0.0005P$

(B) $\frac{dP}{dt} = 0.03P^2 - 0.000125P$

(C) $\frac{dP}{dt} = 0.03P - 0.001P^2$

(D) $\frac{dP}{dt} = 0.03P - 0.00025P^2$

Free Response Questions

6. Let f be a function with $f(2)=1$, such that all points (t, y) on the graph of f satisfy the differential equation $\dfrac{dy}{dt}=2y\left(1-\dfrac{t}{4}\right)$.

   Let g be a function with $g(2)=2$, such that all points (t, y) on the graph of g satisfy the logistic differential equation $\dfrac{dy}{dt}=y\left(1-\dfrac{y}{5}\right)$.

   (a) Find $y=f(t)$.

   (b) For the function found in part (a), what is $\lim_{t\to\infty} f(t)$?

   (c) Given that $g(2)=2$, find $\lim_{t\to\infty} g(t)$ and $\lim_{t\to\infty} g'(t)$.

   (d) For what value of y does the graph of g have a point of inflection? Find the slope of the graph of g at the point of inflection.

7.5 Euler's Method $\boxed{BC}$

Euler's Method is a numerical approach to approximate the particular solution of the differential equation $y' = f(x, y)$ with an initial condition $y(x_0) = y_0$.

Using a small step h and (x_0, y_0) as a starting point, move along the tangent line until you arrive at the point (x_1, y_1), where

$$\boldsymbol{x_1 = x_0 + h} \quad \text{and} \quad \boldsymbol{y_1 = y_0 + h\left[\frac{dy}{dx}\right]_{(x_0, y_0)}},$$

as shown in Figure 7-4.

Repeat the process with the same step size h at a new starting point (x_1, y_1). The values of x_i and y_i are as follows.

$$\begin{array}{ll}
\boldsymbol{x_1 = x_0 + h} & \boldsymbol{y_1 = y_0 + h\left[\frac{dy}{dx}\right]_{(x_0, y_0)}} \\
\boldsymbol{x_2 = x_1 + h} & \boldsymbol{y_2 = y_1 + h\left[\frac{dy}{dx}\right]_{(x_1, y_1)}} \\
\vdots & \vdots \\
\boldsymbol{x_n = x_{n-1} + h} & \boldsymbol{y_n = y_{n-1} + h\left[\frac{dy}{dx}\right]_{(x_{n-1}, y_{n-1})}}
\end{array}$$

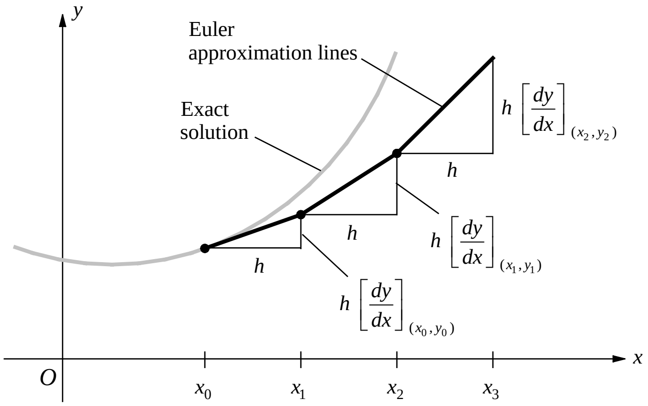


Figure 7-4

Example 1 □ Let f be the function whose graph goes through the point $(1, -1)$ and whose derivative is given $y' = 2 - \dfrac{y}{x}$. Use Euler's method starting at $x = 1$ with a step size of 0.5 to approximate $f(3)$.

Solution □ Given $\dfrac{dy}{dx} = 2 - \dfrac{y}{x}$, $x_0 = 1$, $y_0 = -1$, and $h = 0.5$.

$$f(1.5) \approx y_1 = y_0 + h\left[\frac{dy}{dx}\right]_{(x_0,y_0)}$$

$$= -1 + (0.5)(2 - \frac{-1}{1}) = 0.5 \qquad x_0 = 1,\ y_0 = -1$$

$$f(2) \approx y_2 = y_1 + h\left[\frac{dy}{dx}\right]_{(x_1,y_1)}$$

$$= 0.5 + (0.5)(2 - \frac{0.5}{1.5}) = \frac{4}{3} \qquad x_1 = x_0 + h = 1 + 0.5 = 1.5,\ y_1 = 0.5$$

$$f(2.5) \approx y_3 = y_2 + h\left[\frac{dy}{dx}\right]_{(x_2,y_2)}$$

$$= \frac{4}{3} + (0.5)(2 - \frac{4/3}{2}) = 2 \qquad x_2 = x_1 + h = 1.5 + 0.5 = 2,\ y_2 = 4/3$$

$$f(3) \approx y_4 = y_3 + h\left[\frac{dy}{dx}\right]_{(x_3,y_3)}$$

$$= 2 + (0.5)(2 - \frac{2}{2.5}) = 2.6 \qquad x_3 = x_2 + h = 2 + 0.5 = 2.5,\ y_3 = 2$$

Example 2 □ Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x - y + 2$ with the initial condition $f(0) = 2$. Use Euler's method starting at $x = 0$ with a step size of 0.5 to approximate $f(2)$.

Solution □ Given $\frac{dy}{dx} = x - y + 2$, $x_0 = 0$, $y_0 = 2$, and $h = 0.5$.

$$f(0.5) \approx y_1 = y_0 + h\left[\frac{dy}{dx}\right]_{(x_0,y_0)}$$

$$= 2 + (0.5)(0 - 2 + 2) = 2 \qquad x_0 = 0,\ y_0 = 2$$

$$f(1) \approx y_2 = y_1 + h\left[\frac{dy}{dx}\right]_{(x_1,y_1)}$$

$$= 2 + (0.5)(0.5 - 2 + 2) = 2.25 \qquad x_1 = 0 + 0.5 = 0.5,\ y_1 = 2$$

$$f(1.5) \approx y_3 = y_2 + h\left[\frac{dy}{dx}\right]_{(x_2,y_2)}$$

$$= 2.25 + (0.5)(1 - 2.25 + 2) = 2.625 \qquad x_2 = 0.5 + 0.5 = 1,\ y_2 = 2.25$$

$$f(2) \approx y_4 = y_3 + h\left[\frac{dy}{dx}\right]_{(x_3,y_3)}$$

$$= 2.625 + (0.5)(1.5 - 2.625 + 2) = 3.0625 \qquad x_3 = 1 + 0.5 = 1.5,\ y_3 = 2.625$$

Exercises - Euler's Method $\boxed{BC}$

Multiple Choice Questions

1. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = 1 + 2x - y$ with the initial condition $f(1) = 2$. What is the approximation for $f(2)$ if Euler's method is used, starting at $x = 1$ with a step size of 0.5?

(A) 2.5 (B) 2.75 (C) 3.25 (D) 3.75

---

2. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x - xy$ with the initial condition $f(0.5) = 0$. What is the approximation for $f(2)$ if Euler's method is used, starting at $x = 0.5$ with a step size of 0.5?

(A) 0.825 (B) 0.906 (C) 1.064 (D) 1.178

---

3. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = \arctan(xy)$ with the initial condition $f(0) = 1$. What is the approximation for $f(2)$ if Euler's method is used, starting at $x = 0$ with a step size of 1?

(A) $\frac{\pi}{2}$ (B) $1 + \frac{\pi}{4}$ (C) $1 + \frac{\pi}{2}$ (D) π

---

x	-1	-0.6	-0.2	0.2	0.6
$f'(x)$	1	2	-0.5	-1.5	1.2

4. The table above gives selected values for the derivative of a function f on the interval $-1 \le x \le 0.6$. If $f(-1) = 1.5$ and Euler's method is used to approximate $f(0.6)$ with step size of 0.8, what is the resulting approximation?

(A) 1.9 (B) 2.1 (C) 2.3 (D) 2.5

$x_0 = 0$	$f(x_0) = 1$
$x_1 = 0.5$	$f(x_1) \approx 1.5$
$x_2 = 1$	$f(x_2) \approx 3$

5. Consider the differential equation $\frac{dy}{dx} = kx + y - 2x^2$, where k is a constant. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. Euler's method, starting at $x = 0$ with step size of 0.5, is used to approximate $f(1)$. Steps from this approximation are shown in the table above. What is the value of k?

(A) 2.5 (B) 3 (C) 3.5 (D) 4

Free Response Questions

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x - y - \frac{1}{2}$.

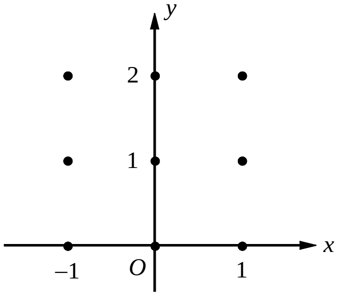
(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(0, -\frac{1}{2})$. Does the graph of f have relative minimum, a relative maximum, or neither at the point $(0, -\frac{1}{2})$? Justify your answer.

(c) Let $y = g(x)$ be another solution to the given differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 0.5, gives the approximation $g(1) \approx 1$.Find the value of k.

7. Consider the differential equation $\frac{dy}{dx} = 2x + y$.

(a) On the axis provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(1,1)$.



(b) Let f be the function that satisfies the given differential equation with the initial condition $f(1) = 1$. Use Euler's method, starting at $x = 1$ with a step size of 0.1, to approximate $f(1.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = -2x + b$ is a solution to the given differential equation. Show the work that leads to your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(1) = -2$. Does the graph of g have a local extremum at the point $(1, -2)$? If so, is the point a local maximum or a local minimum? Justify your answer.

Chapter8
Parametric Equations, Vectors, and Polar Coordinates

8.1 Slopes and Tangents for the Parametric Curves

If x and y are both given as functions of a third variable t, then the equations

$$\boldsymbol{x = f(t)}, \quad \boldsymbol{y = g(t)}$$

are called **parametric equations**, and t is called the **parameter**.

The set of points $(x, y) = (f(t), g(t))$ defined by the parametric equations is called the **parametric curve**.

When the points in a parametric curve are plotted in order of increasing values of t, the curve is traced out in a specific direction. This is called the **direction of path (or motion)** of the curve.

Parametric Formula for dy/dx

If the equation $x = f(t)$, $y = g(t)$ define y as a differentiable function of x and $dx/dt \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ where } \frac{dx}{dt} \neq 0.$$

Parametric Formula for $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Horizontal Tangent

If $dy/dt = 0$ and $dx/dt \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$.

Vertical Tangent

If $dx/dt = 0$ and $dy/dt \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$.

Example 1 □ A curve in the plane is defined parametrically by the equations $x = t^2$ and $y = t^3 - 2t$.

(a) Sketch the curve in the xy- plane for $-2 \le t \le 2$. Indicate the direction in which the curve is traced as t increases.

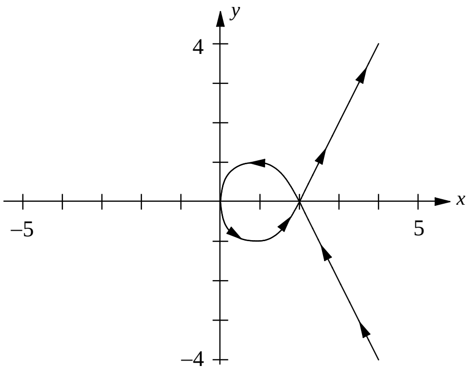
(b) For what values of t does the curve have a vertical tangent?

(c) For what values of t does the curve have a horizontal tangent?

(d) Find the equation of the tangent lines to the curve at $t = \pm\sqrt{2}$.

Solution □ (a) Use a graphing calculator to draw a parametric curve.
Set the graphing calculator as follows.

- ○ Mode: Parametric mode
- ○ Window: $T_{\min} = -2$, $T_{\max} = 2$,
  $X_{\min} = -5$, $X_{\max} = 5$
  $Y_{\min} = -4$, $Y_{\max} = 4$
- ○ $Y =$: $X_{1T} = T^2$, $Y_{1T} = T^3 - 2T$



(b) $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2 - 2$

$\dfrac{dx}{dt} = 0 \;\Rightarrow\; t = 0$

The curve has a vertical tangent when $t = 0$.

(c) $\dfrac{dy}{dt} = 0 \;\Rightarrow\; 3t^2 - 2 = 0 \;\Rightarrow\; t = \pm\dfrac{\sqrt{2}}{\sqrt{3}} = \pm\dfrac{\sqrt{6}}{3}$

The curve has a horizontal tangent when $t = \pm\dfrac{\sqrt{6}}{3}$

(d) If $t = \sqrt{2}$, $\dfrac{dy}{dx} = \dfrac{3(\sqrt{2})^2 - 2}{2(\sqrt{2})} = \dfrac{4}{2\sqrt{2}} = \sqrt{2}$

The equation of the tangent line is

$y - 0 = \sqrt{2}(x - 2)$.

If $t = -\sqrt{2}$, $\dfrac{dy}{dx} = \dfrac{3(-\sqrt{2})^2 - 2}{2(-\sqrt{2})} = \dfrac{4}{-2\sqrt{2}} = -\sqrt{2}$

The equation of the tangent line is

$y - 0 = -\sqrt{2}(x - 2)$.

Example 2 □ A particle moves in the xy- plane so that its position at any time t, $0 \le t \le 4$, is given by the equations $x(t) = \cos t + t\sin t$ and $y(t) = \sin t - t\cos t$.

(a) Sketch the curve in the xy- plane for $0 \le t \le 4$. Indicate the direction in which the curve is traced as t increases.

(b) At what time t, $0 < t < 4$, does the line tangent to the path of the particle have a slope of -1?

(c) At what time t, $0 < t < 4$, does $x(t)$ attain its maximum value? What is the position $(x(t), y(t))$ of the particle at this time?

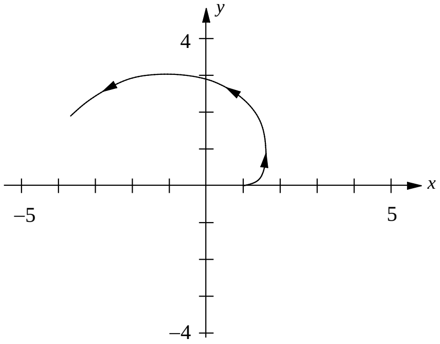
(d) At what time t, $0 < t < 4$, is the particle on the y-axis?

Solution □ (a) Use a graphing calculator to draw the parametric curve.

○ Mode: Parametric mode, Radian mode

○ Window: $T_{\min} = 0$, $T_{\max} = 4$,
$X_{\min} = -5$, $X_{\max} = 5$
$Y_{\min} = -4$, $Y_{\max} = 4$

○ $Y =$: $X_{1T} = \cos T + T\sin T$,
$Y_{1T} = \sin T - T\cos T$



(b) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t - (-t\sin t + \cos t)}{-\sin t + (t\cos t + \sin t)} = \dfrac{t\sin t}{t\cos t} = \tan t$

So $\dfrac{dy}{dx} = \tan t = -1 \;\Rightarrow\; t = \tan^{-1}(-1) = 3\pi/4$.

(c) $x'(t) = -\sin t + (\sin t + t\cos t) = t\cos t$
$x'(t) = 0 \;\Rightarrow\; t = \pi/2$, for $0 < t < 4$.
$x(t)$ attains its maximum value when $t = \pi/2$.

$$x(\frac{\pi}{2}) = \cos\frac{\pi}{2} + \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$

$$y(\frac{\pi}{2}) = \sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2} = 1$$

The position when $t = \pi/2$ is $(\frac{\pi}{2}, 1)$.

(d) The particle is on the y- axis when $x(t) = 0$.
$x(t) = \cos t + t\sin t = 0$
Use a graphing calculator (in function mode) to find the value of t which makes $\cos t + t\sin t = 0$.
For $0 < t < 4$, $x(t) = 0$ when $t = 2.798$.

Exercises - Slopes and Tangents to the Parametric Curves

Multiple Choice Questions

1. If $x = te^t$ and $y = t + e^t$, then $\dfrac{dy}{dx}$ at $t = 0$ is

(A) 0 (B) $\dfrac{1}{2}$ (C) 1 (D) 2

---

2. If $x = \tan t$ and $y = \sin t$, then $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{6}$ is

(A) $-\dfrac{9}{11}$ (B) $-\dfrac{27}{32}$ (C) $\dfrac{13}{16}$ (D) $\dfrac{7}{8}$

---

3. A curve C is defined by the parametric equations $x = t^3 - 3$ and $y = 2t^2$. Which of the following is the equation for the line tangent to the graph of C at the point $(5,8)$?

(A) $y = \dfrac{1}{3}x + \dfrac{8}{3}$ (B) $y = 2x - \dfrac{8}{3}$ (C) $y = \dfrac{2}{3}x + \dfrac{14}{3}$ (D) $y = 3x + 8$

---

4. If $x = \cos t$ and $y = 2\sin^2 t$, then $\dfrac{dy}{dx}$ at $t = 1$ is

(A) $-2\cos 1$ (B) $-4\cos 1$ (C) $-2\tan 1$ (D) $-2\sin 1$

5. For what value(s) of t does the curve defined by the parametric equations $x = \frac{5t}{1+t^3}$ and $y = \frac{2t^2}{1+t^3}$ have a horizontal tangent?

   (A) 0 only

   (B) $\sqrt[3]{2}$ only

   (C) 0 and 4 only

   (D) 0 and $\sqrt[3]{2}$ only

---

6. A point (x, y) is moving along a curve $y = f(x)$. At the instant when the slope of the curve is $\frac{3}{4}$, the x-coordinate of the point is decreasing at the rate of $\frac{2}{5}$ units per second. The rate of change, in units per second, of the y-coordinate of the point is

   (A) $-\frac{15}{8}$ (B) $-\frac{3}{5}$ (C) $-\frac{3}{10}$ (D) $\frac{3}{10}$

---

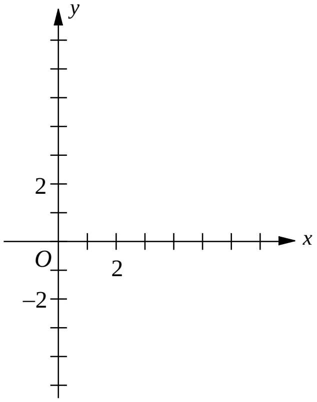
7. An object moving along a curve in the xy-plane is in position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 2 - \sin(t^2)$. At time $t = 3$, the object is at position $(2, 7)$. What is the x-coordinate of the position of the object at time $t = 6$?

   (A) 8.135 (B) 9.762 (C) 10.375 (D) 11.308

Free Response Questions

8. A particle moves in the xy- plane so that its position at any time t, $0 \le t \le 2\pi$, is given by $x(t) = t - \sin t$ and $y(t) = t\cos t$.

   (a) Sketch the path of the particle in the xy- plane below. Indicate the direction of motion along the path.



   (b) At what time t, $0 \le t \le 2\pi$, does $y(t)$ attain its minimum value? What is the position $(x(t), y(t))$ of the particle at this time?

   (c) Write an equation for the line tangent to the curve at time $t = \pi$.

---

9. A particle moving along the curve is defined by the equation $y = x^3 - 4x^2 + 4$. The x-coordinate of the particle, $x(t)$, satisfies the equation $\dfrac{dx}{dt} = \dfrac{t}{\sqrt{t^2+9}}$, for $t \ge 0$ with initial condition $x(0) = 1$.

   (a) Find $x(t)$ in terms of t.

   (b) Find $\dfrac{dy}{dt}$ in terms of t.

   (c) Find the location of the particle at time $t = 4$.

   (d) Write an equation for the line tangent to the curve at time $t = 4$.

8.2 Arc Length (Distance Traveled Along a Curve) in Parametric Form

If a curve C is given by the parametric equations $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ then the **arc length** C, or the **distance traveled by a particl**e **along the curve** is given by

$$\boldsymbol{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}\,.$$

The **displacement** of a particle is the distance between its initial and final positions. The displacement of a particle between time $t = a$ and $t = b$ is given by

$$\textbf{Dispalcement} = \sqrt{\left[\int_a^b \boldsymbol{x'(t)\, dt}\right]^2 + \left[\int_a^b \boldsymbol{y'(t)\, dt}\right]^2}$$

$$= \sqrt{[\boldsymbol{x(b) - x(a)}]^2 + [\boldsymbol{y(b) - y(a)}]^2}\,.$$

Example 1 □ A particle moves in the xy-plane so that its position at any time t, for $0 \le t$, is given by $x(t) = e^t$ and $y(t) = 2\cos(t)$.

(a) Find the distance traveled by the particle from $t = 0$ to $t = 2$.

(b) Find the magnitude of the displacement of the particle between time $t = 0$ and $t = 2$.

Solution □ (a) $\dfrac{dx}{dt} = e^t$, $\dfrac{dy}{dt} = -2\sin(t)$

Distance traveled by the particle $= \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^2 \sqrt{\left(e^t\right)^2 + \left(-2\sin t\right)^2}\, dt$$

≈ 7.035 Use a graphing calculator (in function mode) to find the value of the definite integral.

(b) Displacement $= \sqrt{\left[\displaystyle\int_a^b x'(t)\, dt\right]^2 + \left[\displaystyle\int_a^b y'(t)\, dt\right]^2}$

$$= \sqrt{\left[\int_0^2 e^t\, dt\right]^2 + \left[\int_0^2 (-2\sin t)\, dt\right]^2}$$

$$= \sqrt{(6.389)^2 + (-2.832)^2}$$

$$\approx \sqrt{48.8395}$$

≈ 6.988 Use a graphing calculator.

Exercises - Arc Length (Distance Traveled Along a Curve) in Parametric Form

Multiple Choice Questions

1. The position of a particle at any time $t \geq 0$ is given by $x = t - t^2$ and $y = \frac{4}{3}t^{3/2}$. What is the total distance traveled by the particle from $t = 1$ to $t = 3$?

   (A) 7.165 (B) 8.268 (C) 9.431 (D) 10.346

---

2. The position of particle at any time $t \geq 0$ is given by $x(t) = a(\cos t + t\sin t)$ and $y(t) = a(\sin t - t\cos t)$. What is the total distance traveled by the particle from $t = 0$ to $t = \pi$?

   (A) $\frac{1}{2}\pi a$ (B) πa^2 (C) $\frac{1}{2}\pi^2 a$ (D) $\frac{1}{2}\pi^2 a^2$

---

3. The length of the path described by the parametric equations $x = \sin t + \ln(\cos t)$ and $y = \cos t$, for $\frac{\pi}{6} \leq t \leq \frac{\pi}{3}$, is given by

   (A) $\int_{\pi/6}^{\pi/3} \sqrt{\cos^2 t + 2\sin t + 2}\ dt$

   (B) $\int_{\pi/6}^{\pi/3} \sqrt{\sin^2 t + 2\cos t + 2}\ dt$

   (C) $\int_{\pi/6}^{\pi/3} \sqrt{\cot^2 t + 2\cos t}\ dt$

   (D) $\int_{\pi/6}^{\pi/3} \sqrt{\sec^2 t - 2\sin t}\ dt$

4. A particle moving in the xy- plane has velocity vector given by $v(t) = \left\langle e^t - t,\ t\sin t \right\rangle$ for time $t \geq 0$. What is the magnitude of the displacement of the particle between time $t = 0$ and $t = 2$?

   (A) 4.722 (B) 4.757 (C) 4.933 (D) 5.109

Free Response Questions

5. A particle moving along a curve in the xy-plane is at position $(x(t),\ y(t))$ at any time t, where $\frac{dx}{dt} = 2\sin(t^2)$ and $\frac{dy}{dt} = \cos(t^3)$. At time $t = 1$, the object is at position $(3, 2)$.

   (a) Write an equation for the line tangent to the curve at $(3, 2)$.

   (b) Find the total distance traveled by the particle from $t = 1$ to $t = 3$.

   (c) Find the position of the particle at time $t = 3$.

   (d) Find the magnitude of the displacement of the particle between $t = 1$ and $t = 3$.

8.3 Vector valued Functions

Vector Valued Function, Velocity Vector, Acceleration Vector, and Speed

If a particle moves in the *xy*-plane so that at time $t > 0$ its position vector is given by

$$\mathbf{r}(t) = < x(t), y(t) >,$$

then the velocity vector, acceleration vector, and speed at time t are as follows.

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = < x'(t), y'(t) >$$

$$\textbf{Accleration} = \mathbf{a}(t) = \mathbf{v}'(t) = < x''(t), y''(t) >$$

$$\textbf{Speed} = |\mathbf{v}(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

If $x(t)$ is increasing $\frac{dx}{dt}$ is positive and if $x(t)$ is decreasing $\frac{dx}{dt}$ is negative.

If $y(t)$ is increasing $\frac{dx}{dt}$ is positive and if $y(t)$ is decreasing $\frac{dx}{dt}$ is negative.

Example 1 □ A particle moving in the *xy*-plane is defined by the vector-valued function $f(t) = < t - \sin t, 1 - \cos t >$, for $0 \le t \le \pi$.

(a) Find the velocity vector for the particle at any time t.

(b) Find the speed of the particle when $t = \frac{\pi}{3}$.

(c) Find the acceleration vector for the particle at any time t.

(d) Find the average speed of the particle from time $t = 0$ to time $t = \pi$.

Solution □ (a) $x(t) = t - \sin t$, $y(t) = 1 - \cos t$

$x'(t) = 1 - \cos t$, $y'(t) = \sin t$

$\text{v}(t) = < x'(t), y'(t) > = < 1 - \cos t, \sin t >$

(b) $x'(\frac{\pi}{3}) = 1 - \cos\frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2}$

$y'(\frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\text{Speed} = \left|\text{v}(\frac{\pi}{3})\right| = \sqrt{\left[x'(\frac{\pi}{3})\right]^2 + \left[y'(\frac{\pi}{3})\right]^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

(c) $x''(t) = \sin t$, $y''(t) = \cos t$

$\text{a}(t) = < x''(t), y''(t) > = < \sin t, \cos t >$

(d) $$\text{Average speed} = \frac{1}{\pi - 0}\int_0^{\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2}\, dt = \frac{4}{\pi}$$

Exercises - Vector Valued Functions

Multiple Choice Questions

1. If a particle moves in the *xy*-plane so that at time $t > 0$ its position vector is $(t^3 - 1, \ln\sqrt{t^2+1})$, then at time $t = 1$, its velocity vector is

(A) $(0, \frac{1}{2})$ (B) $(1, \frac{1}{2})$ (C) $(3, \frac{1}{2})$ (D) $(3, \frac{1}{4})$

---

2. A particle moves in the *xy*-plane so that at any time t its coordinates are $x = t^3 - t^2$ and $y = t + \ln t$. At time $t = 2$, its acceleration vector is

(A) $(4, \frac{1}{2})$ (B) $(6, \frac{1}{4})$ (C) $(8, \frac{3}{4})$ (D) $(10, -\frac{1}{4})$

---

3. A particle moves in the *xy*-plane so that its position at time $t > 0$ is given by $x(t) = e^t \cos t$ and $y(t) = e^t \sin t$. What is the speed of the particle when $t = 2$?

(A) $\sqrt{2}e$ (B) $\sqrt{2}e^2$ (C) $2e$ (D) $2e^2$

---

4. If f is a vector-valued function defined by $f(t) = \left(\ln(\sin t),\ t^2 + e^{-t}\right)$, then the acceleration vector is

(A) $(-\csc^2 t,\ 2 + e^{-t})$

(B) $(\sec^2 t,\ 2 + e^{-t})$

(C) $(\csc^2 t,\ 2 - e^{-t})$

(D) $(-\csc^2 t \cdot \cot t,\ 2 + e^{-t})$

5. A particle moves on the curve $y = x + \sqrt{x}$ so that the x-component has velocity $x'(t) = \cos t$ for $t \geq 0$. At time $t = 0$, the particle is at the point $(1, 0)$. At time $t = \frac{\pi}{2}$, the particle is at the point

(A) $(0, 0)$ (B) $(1,\ 2)$ (C) $(\frac{\pi}{2},\ \frac{\pi}{2} + \sqrt{\frac{\pi}{2}})$ (D) $(2,\ 2 + \sqrt{2})$

---

6. In the xy- plane, a particle moves along the curve defined by the equation $y = 2x^4 - x$ with a constant speed of 20 units per second. If $\frac{dy}{dt} > 0$, what is the value of $\frac{dx}{dt}$ when the particle is at the point $(1,\ 1)$

(A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4

Free Response Questions

7. An object moving along a curve in the xy-plane is at position $(x(t),\ y(t))$ at time t, where $\frac{dx}{dt} = 1 + \cos(e^t)$. and $\frac{dy}{dt} = e^{(2-t^2)}$ for $t \geq 0$.

(a) At what time t is the speed of the object 3 units per second?

(b) Find the acceleration vector at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 4$.

(d) Find the magnitude of the displacement of the object over the time interval $1 \leq t \leq 4$.

8. An object moving along a curve in the *xy*-plane has position $(x(t),\ y(t))$ at time $t \geq 0$, with $\frac{dx}{dt} = t - \sin(e^t)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 1$, the value of $\frac{dy}{dt}$ is 3 and the object is at position $(1, 4)$.

   (a) Find the *x*-coordinate of the position of the object at time $t = 5$.

   (b) Write an equation for the line tangent to the curve at the point $(x(1),\ y(1))$.

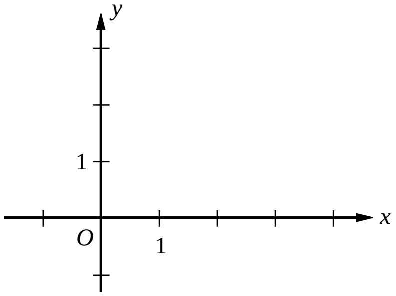
   (c) Find the speed of the object at time $t = 1$.

   (d) Suppose the line tangent to the curve at $(x(t),\ y(t))$ has a slope of $(t-2)$ for $t \geq 0$. Find the acceleration vector of the object at time $t = 3$.

---

9. The position of a particle moving in the *xy*-plane is given by the parametric equations $x(t) = t - \sin(\pi t)$ and $y(t) = 1 - \cos(\pi t)$ for $0 \leq t \leq 2$.

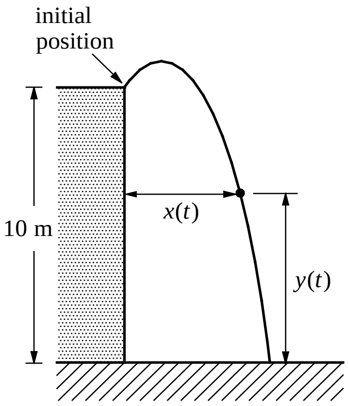
   (a) On the axis provided below, sketch the graph of the path of the particle from $t = 0$ to $t = 2$. Indicate the direction of the particle along its path.



   (b) Find the position of the particle when $t = 1$.

   (c) Find the velocity vector for the particle at any time t.

   (d) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance traveled of the particle from $t = 0$ to $t = 2$.



Note: Figure not drawn to scale.

10. An object is thrown upward into the air 10 meters above the ground. The figure above shows the initial position of the object and the position at a later time. At time t seconds after the object is thrown upward, the horizontal distance from the initial position is given by $x(t)$ meters, and the vertical distance from the ground is given by $y(t)$ meters, where $\frac{dx}{dt}=1.4$ and $\frac{dy}{dt}=4.2-9.8t$, for $t \geq 0$.

(a) Find the time t when the object reaches its maximum height.

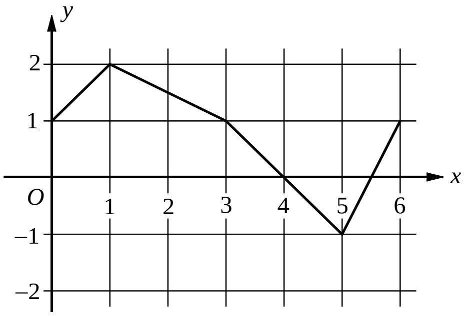
(b) Find the maximum vertical distance from the ground to the object.

(c) Find the time t when the object hit the ground.

(d) Find the total distance traveled by the object from time $t=0$ until the object hit the ground.

(e) Find the magnitude of the displacement of the object from time $t=0$ until the object hit the ground.

(f) Find the angle θ, $0<\theta<\frac{\pi}{2}$, between the path of the object and the ground at the instance the object hit the ground.



11. At time t, the position of particle moving in the xy- plane is given by the parametric functions $\left(x(t),\ y(t)\right)$, where $\dfrac{dx}{dt} = e^{\sqrt{x}} - \cos(x^2)$. The graph of y consisting of four line segments, is shown in the figure above. At time $t = 0$, the particle is at position $(2,1)$.

(a) Find the position of the particle at $t = 2$.

(b) Find the slope of the line tangent to the path of the particle at $t = 2$.

(c) Find the magnitude of the velocity vector at $t = 2$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 3$.

8.4 Polar Coordinates and Slopes of Curves

Coordinate-System Conversion Formula

If a point P has rectangular coordinates (x, y) and polar coordinates (r, θ), then

$$\boldsymbol{x = r\cos\theta}, \qquad \boldsymbol{y = r\sin\theta},$$

$$\boldsymbol{r^2 = x^2 + y^2}, \qquad \boldsymbol{\tan\theta = \frac{y}{x}(x \neq 0)}.$$

The **slope** of the tangent line to the graph of $r = f(\theta)$ at point (r, θ) is

$$\boldsymbol{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}, \text{ where } \frac{dx}{d\theta} \neq 0 \text{ at } (r, \theta).$$

If $r > 0$ and $\frac{dr}{d\theta} < 0$ for $\alpha < \theta < \beta$, then r is decreasing on this interval, which means the curve is getting closer to the origin.

If $r > 0$ and $\frac{dr}{d\theta} > 0$ for $\alpha < \theta < \beta$, then r is increasing on this interval, which means the curve is getting farther from the origin.

If $\frac{dr}{d\theta} = 0$, the curve is either closest to the origin or farthest from the origin.

Example 1 □ A curve is defined by the polar equation $r = 4\sin(2\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$.

(a) Graph the curve.

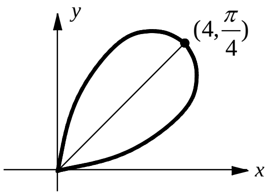
(b) Find the slope of the curve at the point where $\theta = \pi/4$.

(c) Find an equation in terms of x and y for the line tangent to the curve at the point where $\theta = \frac{\pi}{4}$.

(d) Find an interval where the curve is getting closer to the origin.

(e) Find the value of θ in the interval $0 \leq \theta \leq \frac{\pi}{2}$ such that the point on the curve has the greatest distance from the origin.

Solution □ (a)



(b) $x = r\cos\theta = 4\sin(2\theta)\cos\theta$, $\dfrac{dx}{d\theta} = 4(-\sin(2\theta)\sin\theta + 2\cos(2\theta)\cos\theta)$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{4}} = 4(-\sin\frac{\pi}{2}\sin\frac{\pi}{4} + 2\cos\frac{\pi}{2}\cos\frac{\pi}{4}) = -2\sqrt{2}$$

$y = r\sin\theta = 4\sin(2\theta)\sin\theta$, $\dfrac{dy}{d\theta} = 4(\sin(2\theta)\cos\theta + 2\cos(2\theta)\sin\theta)$

$$\left.\frac{dy}{d\theta}\right|_{\theta=\frac{\pi}{4}} = 4(\sin\frac{\pi}{2}\cos\frac{\pi}{4} + 2\cos\frac{\pi}{2}\sin\frac{\pi}{4}) = 2\sqrt{2}$$

The slope of a tangent line to the polar curve is $\dfrac{dy}{dx}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{4}} = \frac{2\sqrt{2}}{-2\sqrt{2}} = -1$$

(c) $x = r\cos\theta = 4\sin(2\theta)\cos\theta$, $x(\frac{\pi}{4}) = 4\sin(\frac{\pi}{2})\cos(\frac{\pi}{4}) = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$.

$y = r\sin\theta = 4\sin(2\theta)\sin\theta$, $y(\frac{\pi}{4}) = 4\sin(\frac{\pi}{2})\sin\frac{\pi}{4} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = -1$$

The equation of tangent line is $y - 2\sqrt{2} = -1(x - 2\sqrt{2})$ or
$y = -x + 4\sqrt{2}$.

(d) The curve is getting closer to the origin when $\dfrac{dr}{d\theta} < 0$.

$$\frac{dr}{d\theta} = 8\cos(2\theta) < 0 \Rightarrow \cos(2\theta) < 0 \Rightarrow \frac{\pi}{2} < 2\theta < \pi$$

$$\Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2}$$

(e) The point on the curve is farthest from the origin when $\dfrac{dr}{d\theta} = 0$.

$$\frac{dr}{d\theta} = 0 \Rightarrow \cos(2\theta) = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

Exercises - Polar Coordinates and Slopes of Curves

Multiple Choice Questions

1. If $r = \theta - 3\sin\theta$ then $\frac{dr}{d\theta}$ at (π, π) is

(A) 2 (B) π (C) 4 (D) 2π

---

2. If $r = \frac{2}{1-\cos\theta}$ then $\frac{dr}{d\theta}$ at $(2, \frac{\pi}{2})$ is

(A) -2 (B) -1 (C) 0 (D) 1

---

3. If $r = 3\sin\theta$ then $\frac{dy}{dx}$ at the point where $\theta = \frac{\pi}{3}$ is

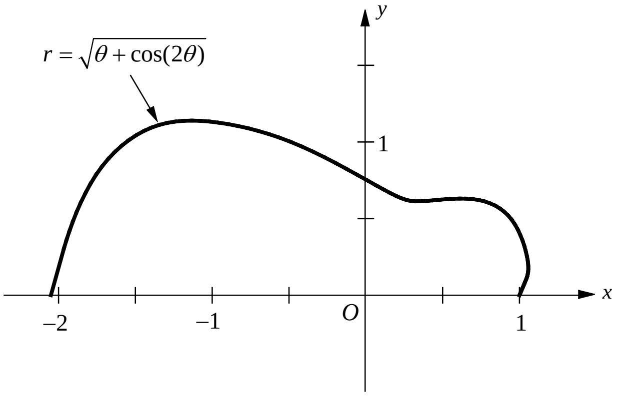
(A) -2 (B) $-\sqrt{3}$ (C) -1 (D) $\frac{\sqrt{3}}{3}$

---

4. The equation of the polar curve is given by $r = \frac{8}{1-\cos\theta}$. What is the angle θ that corresponds to the point on the curve with *x*-coordinate -3?

(A) 1.248 (B) 1.356 (C) 1.596 (D) 2.214

Free Response Questions



5. The polar curve $r = \sqrt{\theta + \cos(2\theta)}$, for $0 \le \theta \le \pi$, is drawn in the figure above.

(a) Find $\frac{dr}{d\theta}$, the derivative of r with respect to θ.

(b) Find the angle θ that corresponds to the point on the curve with *x*-coordinate 0.5.

(c) For $\frac{\pi}{12} < \theta < \frac{5\pi}{12}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r ? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that correspond to the point on the curve in the first quadrant with the least distance from the origin. Justify your answer.

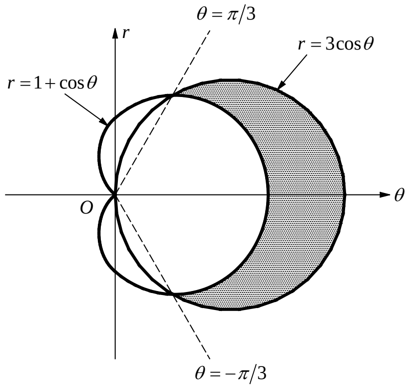
8.5 Areas in Polar Coordinates

Area in Polar Coordinates

The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$\boldsymbol{A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta}$$
$$\boldsymbol{= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta}$$

Example 1 □ Find the area of the region that lies inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.



Solution □ Find the points of intersection of the two curves.

$$3\cos\theta = 1 + \cos\theta$$
$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$$

The desired area can be found by subtracting the area inside the cardioid between $\theta = -\pi/3$ and $\theta = \pi/3$ from the area inside the circle between $\theta = -\pi/3$ and $\theta = \pi/3$.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(3\cos\theta)^2 - (1 + \cos\theta)^2\right] d\theta$$
$$= 2 \cdot \frac{1}{2}\int_{0}^{\pi/3} \left[(3\cos\theta)^2 - (1 + \cos\theta)^2\right] d\theta$$

The region is symmetric about the horizontal axis $\theta = 0$.

$$= \int_{0}^{\pi/3} \left[(8\cos^2\theta - 2\cos\theta - 1)\right] d\theta$$
$$= \int_{0}^{\pi/3} \left[8(\frac{1+\cos 2\theta}{2}) - 2\cos\theta - 1\right] d\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

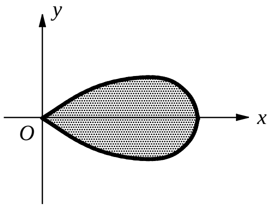
$$= \int_{0}^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta$$
$$= \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_{0}^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

Exercises - Areas in Polar Coordinates

Multiple Choice Questions

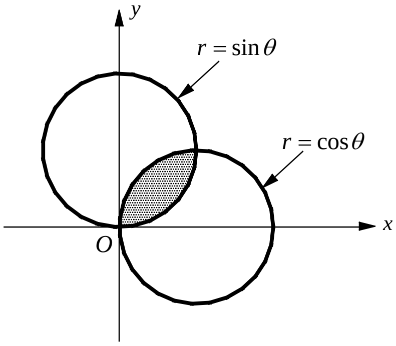
1. The area of the region enclosed by the polar curve $r^2 = 6\sin(2\theta)$ is

(A) 2 (B) 4 (C) 6 (D) 12



2. What is the area of the region enclosed by the loop of the graph of the polar curve $r = 2\cos(2\theta)$ shown in the figure above?

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{2}$ (C) $\dfrac{3\pi}{4}$ (D) π

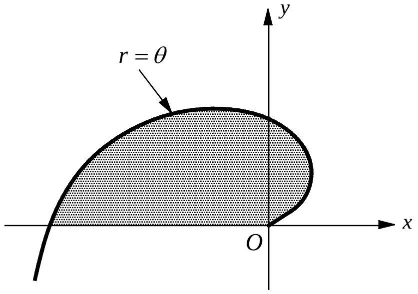


3. The area of the shaded region that lies inside the polar curves $r = \sin\theta$ and $r = \cos\theta$ is

(A) $\dfrac{1}{8}(\pi - 2)$ (B) $\dfrac{1}{4}(\pi - 2)$ (C) $\dfrac{1}{2}(\pi - 2)$ (D) $\dfrac{1}{8}(\pi - 1)$

4. The area of the region enclosed by the polar curve $r = 2 + \sin\theta$ is

(A) 3π (B) $\frac{7\pi}{2}$ (C) 4π (D) $\frac{9\pi}{2}$



5. The area of the shaded region bounded by the polar curve $r = \theta$ and the x-axis is

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^3}{6}$ (C) $\frac{\pi^3}{3}$ (D) $\frac{\pi^3}{2}$

---

6. Which of the following gives the area of the region inside the polar curve $r = 1 + \cos\theta$ and outside the polar curve $r = 2\cos\theta$?

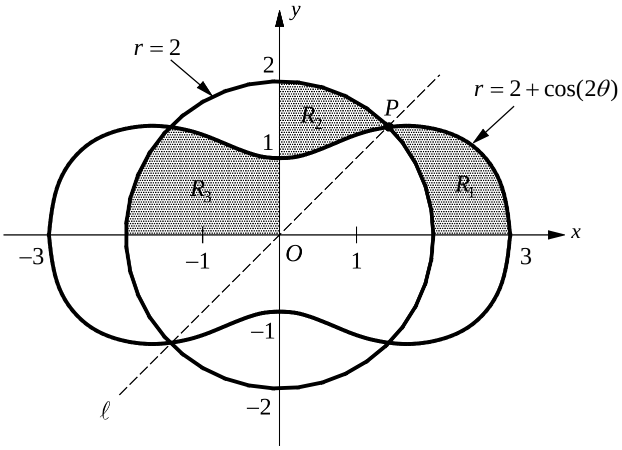
(A) $\frac{1}{2}\int_0^{2\pi} (1 - \cos\theta)^2 \, d\theta$

(B) $\int_0^{2\pi} (1+2\cos\theta)^2 \, d\theta$

(C) $\int_0^{2\pi} (1+\cos\theta)^2 \, d\theta \; - \pi$

(D) $\frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 \, d\theta \; - \pi$

Free Response Questions



7. The figure above shows the graphs of the polar curves $r = 2 + \cos(2\theta)$ and $r = 2$. Let R_1 be the shaded region in the first quadrant bounded by the two curves and the x-axis, and R_2 be the shaded region in the first quadrant bounded by the two curves and the y-axis. The graphs intersect at point P in the first quadrant.

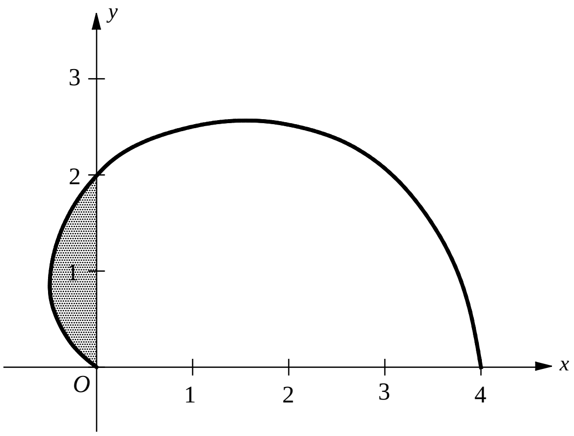
(a) Find the polar coordinates of point P and write the polar equation for the line ℓ.

(b) Set up, but do not integrate, an integral expression that represents the area of R_1.

(c) Set up, but do not integrate, an integral expression that represents the area of R_2.

(d) Let R_3 be the shaded region in the second quadrant bounded by the two curves and the coordinate axis. Find the area of R_3.

(e) The distance between the two curves changes for $0 < \theta < \frac{\pi}{4}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{6}$.



8. The graph of the polar curve $r = 2 + 2\cos(\theta)$ for $0 \le \theta \le \pi$ is shown above.

(a) Write an integral expression for the area of the shaded region.

(b) Write expressions for $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the curve at the point where $\theta = \dfrac{\pi}{2}$.

Chapter 9
Infinite Sequences and Series

9.1 Sequences and Series

A **sequence** is an ordered list of numbers

$$a_1,\ a_2,\ a_3,\ \dots\ ,\ a_n,\ \dots$$

whereas a **series** is an infinite sum of a list of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

Mathematically, a sequence is defined as a function whose domain is the set of positive integers.

If a sequence $\{a_n\}$ has the **limit** L, where L is a real number, it is written as

$$\boldsymbol{\lim_{n\to\infty} a_n = L},$$

and we say the sequence **converges** to L. If the sequence does not have a limit we say the sequence **diverges**.

Given a sequence of numbers of $\{a_n\}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**.

For the infinite series $\sum a_n$, the ***n*th partial sum** is given by

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n .$$

If the sequence of partial sums converges to a limit L, we say that the series **converges**, and its sum is L. If the series does not converge, we say that the series **diverges**.

The **geometric series** $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to the sum

$$\boldsymbol{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}},\ \text{if } |r| < 1.$$

If $|r| \geq 1$, the series diverges.

A series is a **telescoping series** if it is in the form

$$\boldsymbol{(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_5) +}\ \cdot\ \cdot\ \cdot.$$

Since the *n*th partial sum of the series is $S_n = a_1 - a_{n+1}$, a telescoping series converges if and only if $\lim_{n\to\infty} a_n = L$, where L is a real number.

The sum of the series is $\boldsymbol{S = a_1 - L}$.

Limit of *n*th Term of a Convergent Series

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The converse of this theorem is not true in general. But the contrapositive of this theorem provides a useful test for divergent series.

***n*th Term Test for Divergence**

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1 □ Find the sum of the series.

(a) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution □ (a) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{2\cdot 2^n}{3^n} = 2\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is an infinite geometric series with $a = \frac{2}{3}$ and $r = \frac{2}{3}$.

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = 2\cdot\frac{2/3}{1-2/3} = 4$$

(b) $a_n = \frac{1}{n(n+1)} = \left(\frac{1}{n} - \frac{1}{n+1}\right)$ Partial fractions

$$S_n = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}$$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (1-\frac{1}{n+1}) = 1$ Sum of the telescoping series

Example 2 □ Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{n}{2n+3}$ (b) $\sum_{n=1}^{\infty} 2^{-n}5^n$

Solution □ (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

Therefore, the series diverges by the *n*th Term Test for Divergence.

(b) $\sum_{n=1}^{\infty} 2^{-n}5^n = \sum_{n=1}^{\infty} \frac{5^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$ is an infinite geometric series with $r = \frac{5}{2}$.

Since $|r| = |5/2| \geq 1$, the series diverges.

Exercises - Sequences and Series

Multiple Choice Questions

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(3)^{n+1}}{5^n} =$

(A) $\frac{3}{5}$ (B) $\frac{5}{2}$ (C) $\frac{9}{2}$ (D) The series diverges

---

2. If $f(x)=\displaystyle\sum_{n=1}^{\infty}(\tan x)^n$, then $f(1)=$

(A) -2.794 (B) -0.61 (C) 0.177 (D) The series diverges

---

3. $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n^2-1} =$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$

---

4. The sum of the geometric series $\frac{2}{21}+\frac{4}{63}+\frac{8}{189}+\ldots$ is

(A) $\frac{5}{21}$ (B) $\frac{2}{7}$ (C) $\frac{4}{7}$ (D) The series diverges

5. If $S_n = \left(\dfrac{3^{n-1}}{(4+n)^{20}} \right) \left(\dfrac{(7+n)^{20}}{3^n} \right)$, to what number does the sequence $\{S_n\}$ converge?

(A) $\dfrac{1}{3}$ (B) $\dfrac{7}{4}$ (C) $\left(\dfrac{7}{4}\right)^{20}$ (D) Diverges

---

6. Which of the following <u>sequences</u> converge?

I. $\left\{ \dfrac{\cos^2 n}{(1.1)^n} \right\}$ II. $\left\{ \dfrac{e^n - 3}{3^n} \right\}$ III. $\left\{ \dfrac{n}{9+\sqrt{n}} \right\}$

(A) I only (B) II only (C) III only (D) I and II only

---

7. Which of the following series converge?

I. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10(n+1)}$ II. $\displaystyle\sum_{n=1}^{\infty} \arctan n$ III. $\displaystyle\sum_{n=1}^{\infty} \frac{-6}{(-5)^n}$

(A) I only (B) II only (C) III only (D) II and III only

Free Response Questions

8. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{1}{7^n} \right)$.

9.2 The Integral Test and *p*-Series

The Integral Test

If f is positive, continuous, and decreasing on $[1,\infty)$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In other words:

1. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\int_1^{\infty} f(x)\,dx$ is convergent.
2. If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\int_1^{\infty} f(x)\,dx$ is divergent.

Example 1 □ Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$

Solution □ (a) The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1,\infty)$ so we can use the Integral Test:

$$\int_1^{\infty}\frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^{b}\frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\left[\tan^{-1}x\right]_1^b$$

$$= \lim_{b\to\infty}\left[\tan^{-1}b - \tan^{-1}1\right] = \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

So the series converges.

Note : The fact that the integral converges to $\pi/4$ does not imply that the infinite series converges to $\pi/4$.

(b) The function $f(x) = \ln x/x$ is continuous, positive, and decreasing on $[1,\infty)$ so we can use the Integral Test:

$$\int_1^{\infty}\frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_1^{b}\frac{\ln x}{x}\,dx = \lim_{b\to\infty}\left[\frac{\ln^2 x}{2}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{\ln^2 b}{2}-\frac{\ln^2 1}{2}\right] = \infty - 0 = \infty$$

So the series diverges.

p- Series and Harmonic Series

The p- series $\sum_{\boldsymbol{n}=1}^{\infty}\frac{1}{\boldsymbol{n}^{\boldsymbol{p}}}=\frac{1}{1^{\boldsymbol{p}}}+\frac{1}{2^{\boldsymbol{p}}}+\frac{1}{3^{\boldsymbol{p}}}+\frac{1}{4^{\boldsymbol{p}}}+\cdots$

is convergent if $p>1$ and divergent if $0<p\leq 1$.

For $p=1$, the series $\sum_{\boldsymbol{n}=1}^{\infty}\frac{1}{\boldsymbol{n}}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ is called **harmonic series**.

A **general harmonic series** is of the form $\sum\frac{1}{(\boldsymbol{an}+\boldsymbol{b})}$.

Example 2 □ Determine whether the series is convergent or divergent.

(a) $1+\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{9}}+\frac{1}{\sqrt[3]{16}}+\frac{1}{\sqrt[3]{25}}+\cdots$

(b) $\sum_{n=1}^{\infty}n^{1-\pi}$

Solution □ (a) $1+\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{9}}+\frac{1}{\sqrt[3]{16}}+\frac{1}{\sqrt[3]{25}}+\cdots=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2}}=\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$

The p- series is divergent since $p=2/3<1$.

(b) $\sum_{n=1}^{\infty}n^{1-\pi}=\sum_{n=1}^{\infty}n^{-(\pi-1)}=\sum_{n=1}^{\infty}\frac{1}{n^{\pi-1}}$

The p- series is convergent since $p=\pi-1\approx 2.14>1$.

Exercises - The Integral Test and *p*-Series

Multiple Choice Questions

1. If $\int_1^\infty \frac{dx}{x^2+1} = \frac{\pi}{4}$, then which of the following must be true?

I. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ diverges.

II. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

III. $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{4}$

(A) none (B) I only (C) II only (D) II and III only

---

2. What are all values of p for which $\int_1^\infty \frac{1}{\sqrt[3]{x^p}}$ converges?

(A) $P < -3$
(B) $P < -1$
(C) $P > 1$
(D) $P > 3$

---

3. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ II. $\sum_{n=1}^{\infty} ne^{-n^2}$ III. $\sum_{n=2}^{\infty} \frac{1}{x \ln x}$

(A) I only (B) II only (C) III only (D) I and II only

4. What are all values of p for which $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^p + 1}$ converges?

(A) $p > 0$ (B) $p > \frac{1}{2}$ (C) $p > 1$ (D) $p > \frac{3}{2}$

---

5. What are all values of k for which the series $1 + (\sqrt{2})^k + (\sqrt{3})^k + (\sqrt{4})^k + \cdots + (\sqrt{n})^k + \cdots$ converges?

(A) $k < -2$ (B) $k < -1$ (C) $k > 1$ (D) $k > 2$

Free Response Questions

6. Determine whether the following series converge or diverge.

(a) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$

(b) $1 + \frac{1}{(\sqrt[3]{2})^2} + \frac{1}{(\sqrt[3]{3})^2} + \frac{1}{(\sqrt[3]{4})^2} + \cdots$

9.3 The Comparison Test

Direct Comparison Test

Let $0 < a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive, then both series either converge or both diverge.

Note: When choosing a series for comparison, you can disregard all but the highest powers of n in both the numerator and denominator.

Example 1 □ Determine whether the series is convergent or divergent.

(a) $\sum_{n=2}^{\infty} \frac{n}{n^2 - 3}$ (b) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{\sqrt{n^3} + 1}$

Solution □ (a) $\frac{n}{n^2-3} > \frac{n}{n^2} = \frac{1}{n}$ for all $n \ge 2$.

$\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent harmonic series.

Therefore $\sum_{n=2}^{\infty} \frac{n}{n^2-3}$ is divergent by the Direct Comparison Test.

(b) $\frac{\sin^2 n}{\sqrt{n^3}+1} = \frac{\sin^2 n}{n^{3/2}+1} \le \frac{1}{n^{3/2}+1} < \frac{1}{n^{3/2}}$ for $n \ge 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent because it is a p- series with $p = 3/2 > 1$.

Therefore $\sum_{n=1}^{\infty} \frac{\sin^2 n}{\sqrt{n^3}+1}$ is convergent by the Direct Comparison Test.

Example 2 □ Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$ (b) $\sum_{n=3}^{\infty} \frac{2^n}{3^n+1}$

Solution □ (a) Let $a_n = \frac{1}{\sqrt{n^2+4}}$ and $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^2+4}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+4}} = 1$$

Since $\sum b_n = \sum 1/n$ is a divergent (harmonic series), the given series diverges by the Limit Comparison Test.

(b) Let $a_n = \frac{2^n}{3^n+1}$ and $b_n = \frac{2^n}{3^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n/(3^n+1)}{2^n/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n+1} = 1$$

Since $\sum b_n = \sum \frac{2^n}{3^n} = \sum (\frac{2}{3})^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

Exercises - The Comparison Tests

Multiple Choice Questions

1. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+3}$ II. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+2}$ III. $\sum_{n=1}^{\infty} \frac{1+4^n}{3^n}$

(A) I only (B) II only (C) III only (D) I and II only

---

2. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} \frac{1}{n!}$ II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2}$ III. $\sum_{n=1}^{\infty} \sin(\frac{1}{\text{n}})$

(A) I only (B) II only (C) II and III only (D) I, II, and III

3.Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n^{3/2}}{3n^3+7}$ II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4+1}}$ III. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(A) I only (B) I and II only (C) I and III only (D) I, II, and III

4. Which of the following series cannot be shown to converge using the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$?

(A) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(B) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(C) $\sum_{n=1}^{\infty} \frac{2n}{2^{n+1}\sqrt{n^2+1}}$

(D) $\sum_{n=1}^{\infty} \frac{2n^2 - 3n}{2^n (n^2 + n - 100)}$

Free Response Questions

5. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{\cos(2n)}{1 + (1.6)^n}$

(b) $\sum_{n=1}^{\infty} \frac{4^n}{2^n + 3^n}$

9.4 Alternating Series and Error Bound

Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \le a_n$, for all n greater than some integer N .

Alternating Series Estimation Theorem (Error Bound)

If S_n is a partial sum and $S = \sum_{n=1}^{\infty} (-1)^n a_n$ is the sum of a convergent alternating series that satisfies the condition $a_{n+1} \le a_n$, then the remainder $R_n = S - S_n$ is smaller than a_{n+1} , which is the absolute value of the first neglected term.

$$\left|R_n\right| = \left|S - S_n\right| \le a_{n+1}$$

Definition of Absolute and Conditional Convergence

1. $\sum a_n$ is **absolutely convergent** if $\sum \left|a_n\right|$ converges.
2. $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converge but $\sum \left|a_n\right|$ diverges.

Example 1 □ Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n-1}$

Solution □ (a) 1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ 2. $a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$

So the series is convergent by the Alternating Series Test.

(b) 1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} \ne 0$

So the series is divergent by the *n*th Term Test for Divergence.

Example 2 □ Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt[n]{e}}{n^2}$ (b) $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-2/3}$

Solution □ (a) Since $0 \le \frac{\sqrt[n]{e}}{n^2} \le \frac{e}{n^2} = e(\frac{1}{n^2})$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p- series $(p = 2 > 1)$, $\sum_{n=1}^{\infty} \frac{\sqrt[n]{e}}{n^2}$ converges, and so $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt[n]{e}}{n^2}$ is absolutely convergent.

(b) 1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{2/3}} = 0$

2. $a_{n+1} = \frac{1}{(n+1)^{2/3}} < \frac{1}{(n)^{2/3}} = a_n$

So the series $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-2/3}$ is convergent by the Alternating Series Test.

Now consider the series of absolute values.

$\sum_{n=1}^{\infty} \left|(-1)^{n+1} n^{-2/3}\right| = \sum_{n=1}^{\infty} n^{-2/3}$ is a divergent p- series $(p = \frac{2}{3} < 1)$.

Thus, $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-2/3}$ is conditionally convergent.

Example 3 □ Let $f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + - \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$.

Use the alternating series error bound to show that $1 - \frac{1}{2!} + \frac{1}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{500}$.

Solution □ $f(1) = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots + \frac{(-1)^n}{(2n)!} + \cdots$

Since series is alternating, with terms convergent to 0 and decreasing in absolute value, the error is less than the first neglected term.

So, $\left| f(1) - (1 - \frac{1}{2!} + \frac{1}{4!}) \right| \le \frac{1}{6!} = \frac{1}{720} < \frac{1}{500}$.

Exercises - Alternating Series Tests

Multiple Choice Questions

1. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n}$ II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ III. $\sum_{n=1}^{\infty} \cos(n\pi)$

(A) I only (B) II only (C) III only (D) I and II only

---

2. Which of the following series converge?

I. $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{\pi}{n})$ II. $\sum_{n=1}^{\infty} \sin(\frac{2n-1}{2})\pi$ III. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n^2+1}$

(A) I only (B) II only (C) III only (D) I and II only

---

3. For what integer k, $k > 1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{n^2\sqrt{n}}{n^k+1}$ converge?

(A) 3 (B) 4 (C) 5 (D) 6

---

4. Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ and s_n be the sum of the first n terms of the series. If $|s - s_n| < \frac{1}{500}$ what is the smallest value of n?

(A) 6 (B) 7 (C) 8 (D) 9

5. Which of the following series converge?

I. $\sum_{n=2}^{\infty} (-1)^n \sqrt[n]{3}$ II. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{\pi^n}$ III. $\sum_{n=1}^{\infty} (\tan^{-1}(n+1) - \tan^{-1}(n))$

(A) I only (B) II only (C) III only (D) II and III only

---

6. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+3)}{n^2}$ is true?

(A) The series converges conditionally.

(B) The series converges absolutely.

(C) The series converges but neither conditionally nor absolutely.

(D) The series diverges.

---

7. Which of the following series is absolutely convergent?

(A) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n}$

(B) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 \sqrt{n}}$

(C) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 - \sqrt{n}}$

(D) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n^2+1)}{n^3}$

8. An alternating series is given by $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+3}$. Let S_3 be the sum of the first three terms of the given alternating series. Of the following, which is the smallest number M for which the alternating series error bound guarantees that $\left|S - S_3\right| \le M$?

(A) $\frac{1}{4}$ (B) $\frac{1}{7}$ (C) $\frac{1}{19}$ (D) $\frac{1}{28}$

Free Response Questions

9. Let $f(x) = 1 - \frac{3x}{2!} + \frac{9x^2}{4!} - \frac{27x^3}{6!} + \cdots + \frac{(-1)^n (3x)^n}{(2n)!} + \cdots$.

Use the alternating series error bound to show that $1 - \frac{3}{2!} + \frac{9}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{20}$.

9.5 The Ratio Test

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Example 1 □ Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty}\frac{3^n}{n!}$ (b) $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{5^n}$ (c) $\sum_{n=1}^{\infty}\frac{3^n}{2^n-1}$

Solution □ (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}/(n+1)!}{3^n/n!}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right|$

$$= \lim_{n\to\infty}\left|\frac{3}{n+1}\right| = 0 < 1$$

Thus, by the Ratio Test, the series converges.

(b) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^3/5^{n+1}}{(-1)^n n^3/5^n}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{5^{n+1}}\cdot\frac{5^n}{n^3}$

$$= \lim_{n\to\infty}\frac{1}{5}\cdot\left(\frac{n+1}{n}\right)^3 = \frac{1}{5} < 1$$

Thus, by the Ratio Test, the series converges.

(c) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}/(2^{n+1}-1)}{3^n/(2^n-1)}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{2^{n+1}-1}\cdot\frac{2^n-1}{3^n}\right|$

$$= \lim_{n\to\infty}\left|\frac{3(2^n-1)}{2^{n+1}-1}\right| = \lim_{n\to\infty}\frac{3(1-1/2^n)}{2-1/2^n} = \frac{3}{2} > 1$$

Thus, by the Ratio Test, the series diverges.

Example 2 □ Determine whether the series is conditionally convergent or absolute convergent.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+3}$ (b) $\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n!}$

Solution □ (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\sqrt{n+1}\big/(n+4)}{(-1)^n\sqrt{n}\big/(n+3)}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}}{n+4}\cdot\frac{n+3}{\sqrt{n}}$

$$= \lim_{n\to\infty}\sqrt{\frac{n+1}{n}}\cdot\frac{n+3}{n+4} = 1$$

The Ratio Test is in conclusive. Try a different test.

In this case, you can apply the Alternating Series Test.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+3} = 0$

2. $a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+3} < \frac{\sqrt{n}}{n+3} = a_n$ for $n \geq 3$

So, the series is conditionally convergent.

(b) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}/(n+1)!}{e^n/n!}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n}\right|$

$$= \lim_{n\to\infty}\frac{e}{n+1} = 0$$

So, the series is absolutely convergent.

Summary of Tests for Series

Test	Series	Conditions of Convergence or Divergence
n th-Term	$\sum_{n=1}^{\infty} a_n$	The series is divergent if $\lim_{x\to\infty} a_n \neq 0$. Test is inconclusive if $\lim_{x\to\infty} a_n = 0$.
Geometric Series	$\sum_{n=1}^{\infty} ar^{n-1}$	The series is convergent if $\lvert r\rvert<1$, divergent if $\lvert r\rvert \geq 1$. $S=\dfrac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (a_n - a_{n+1})$	The series is convergent if $\lim_{x\to\infty} a_n = L$. $S = a_1 - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	The series is convergent if $p>1$, divergent if $p \leq 1$.
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	The series is convergent if $\lim_{x\to\infty} a_n = 0$ and $0 < a_{n+1} \leq a_n$.
Integral	$\sum_{n=1}^{\infty} a_n$	If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then the series converges if $\int_1^{\infty} f(x)\,dx$ converges, diverges if $\int_1^{\infty} f(x)\,dx$ diverges.
Ratio	$\sum_{n=1}^{\infty} a_n$	The series is convergent if $\lim_{n\to\infty}\left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert<1$, divergent if $\lim_{n\to\infty}\left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert>1$, inconclusive if $\lim_{n\to\infty}\left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert=1$.
Direct Comparison	$\sum_{n=1}^{\infty} a_n$	Let $0<a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
Limit Comparison	$\sum_{n=1}^{\infty} a_n$	Suppose that $a_n>0$, $b_n>0$, and $\lim_{n\to\infty}(\dfrac{a_n}{b_n}) = L$. Then $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} a_n$ diverges if $\sum_{n=1}^{\infty} b_n$ diverges.

Exercises - The Ratio Tests

Multiple Choice Questions

1. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ II. $\sum_{n=1}^{\infty} \frac{n}{3^n}$ III. $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$

(A) I only (B) II only (C) II and III only (D) I, II, and III

---

2. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ II. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ III. $\sum_{n=1}^{\infty} \frac{n^9}{9^n}$

(A) I only (B) II only (C) I and II only (D) I, II, and III

Free Response Questions

3. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n!}{n\ 2^n}$

(b) $\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$

(c) $\sum_{k=1}^{\infty} \frac{3^k k!}{(k+3)!}$

9.6 Convergence of Power Series

Power Series

A **power series about $\boldsymbol{x = 0}$** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdot\cdot\cdot + a_n x^n + \cdot\cdot\cdot$$

where x is a variable and the a_n's are constants.

More generally, a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdot\cdot\cdot + a_n(x-c)^n + \cdot\cdot\cdot$$

is called a **power series centered at *c***, or a **power series about *c*.**

Convergence of a Power Series

For a power series centered at c there are only three possibilities:

1. The series converges only at $x = c$.
2. The series converges for all x.
3. There exists a real number $R > 0$ such that the series converges for $|x-c| < R$, and diverges for $|x-c| > R$.

The number R is called the **radius of convergence** of the power series.

In most cases, R can be found by using the Ratio Test. The Ratio Test always fails when x is an endpoint of the interval of convergence, so each endpoint must be tested separately for convergence or divergence.

If the series converges only at c, the radius of convergence is $R = 0$.
If the series converges for all x, the radius of convergence is $R = \infty$.
The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Example 1 □ Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+3}}$.

Solution □ Let $a_n = (-2)^n x^n \big/ \sqrt{n+3}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}x^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{(-2)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-2)x\sqrt{n+3}}{\sqrt{n+4}}\right| = 2|x|$$

By the ratio test, the given series converges if $2|x| < 1$ or $|x| < \dfrac{1}{2}$.

Thus the radius of convergence is $R = \dfrac{1}{2}$.

The inequality $|x| < \dfrac{1}{2}$ can be written as $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

We must now test for convergence at the end points of this interval.

When $x = -\dfrac{1}{2}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-2)^n (-1/2)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}} = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \ \cdot \ \cdot \ \cdot$$

which is a p-series and diverges since $p = 1/2 < 1$.

When $x = \dfrac{1}{2}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-2)^n (1/2)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}} = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \ \cdot \ \cdot \ \cdot$$

which converges by the Alternating Series Test.

So the interval of convergence is $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

Example 2 □ Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} n!(2x)^n$.

Solution □ Let $a_n = n!(2x)^n$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!(2x)^{n+1}}{n!(2x)^n}\right| = \lim_{n\to\infty} |(n+1)(2x)|$$

$= |2x| \lim\limits_{n\to\infty} |n+1| = \infty$ if $x \neq 0$, so $R = 0$ and the series converges only for $x = 0$.

Example 3 □ Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!}$.

Solution □ Let $a_n = \dfrac{n^2 x^n}{n!}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2 x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2 x}{n^2 (n+1)}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{n+1}{n^2}\right| = 0$$

The series converges for all x, so $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

Exercises - Convergence of Power Series

Multiple Choice Questions

1. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n^3}$ converges?

(A) $-1 < x < 1$ (B) $-1 \le x \le 1$ (C) $-1 < x \le 1$ (D) $-1 \le x < 1$

---

2. What are all values of x for which the series $\sum_{n=0}^{\infty} \frac{n(x-2)^n}{3^n}$ converges?

(A) $-1 < x < 5$ (B) $-1 < x \le 5$ (C) $-2 \le x < 4$ (D) $-2 < x \le 4$

---

3. What are all values of x for which the series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$ converges?

(A) $0 < x < 2$ (B) $0 \le x < 2$ (C) $-1 < x \le 2$ (D) All real x

---

4. What are all values of x for which the series $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{2^n \sqrt{n}}$ converges?

(A) $-2 < x < 2$ (B) $-2 \le x < 2$ (C) $-2 < x \le 2$ (D) All real x

5. What are all values of x for which the series $\sum_{n=1}^{\infty} n!(3x-2)^n$ converges?

(A) No values of x (B) $(-\infty, \frac{2}{3}]$ (C) $x = \frac{2}{3}$ (D) $[\frac{2}{3}, \infty)$

Free Response Questions

6. Find the radius of convergence and the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} x^{n+1}.$$

9.7 Representations of Functions as Power Series

Geometric Power Series

A power series representation for $f(x) = \dfrac{1}{1-x}$ can be obtained from the sum of a geometric series $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$, if you let $a = 1$ and $r = x$.

Therefore $\dfrac{1}{1-\boldsymbol{x}} = \sum_{\boldsymbol{n}=0}^{\infty} \boldsymbol{x}^{\boldsymbol{n}} = 1 + \boldsymbol{x} + \boldsymbol{x}^2 + \boldsymbol{x}^3 + \cdots, \quad |x| < 1$.

Example 1 □ Find the first four nonzero terms and the general terms for the power series expansion of the function given by $f(x) = \dfrac{2}{1+x^2}$.

Solution □ $f(x) = \dfrac{2}{1+x^2} = \dfrac{2}{1-(-x^2)}$ is a geometric series with $a = 2$ and $r = -x^2$.

Therefore, $f(x) = 2 - 2x^2 + 2x^4 - 2x^6 + \cdots + (-1)^n 2x^{2n} + \cdots$.

Example 2 □ Find the first four nonzero terms and the general terms for the power series expansion of the function given by $f(x) = \dfrac{x^2}{1+x}$.

Solution □ $f(x) = \dfrac{x^2}{1+x} = \dfrac{x^2}{1-(-x)}$ is a geometric series with $a = x^2$ and $r = -x$.

Therefore, $f(x) = x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^{n+2} + \cdots$.

Differentiation and Integration of Power Series

If the function given by

$$f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n$$
$$=a_0+a_1(x-c)+a_2(x-c)^2+\cdot\cdot\cdot+a_n(x-c)^n+\cdot\cdot\cdot$$

is differentiable, the derivative and antiderivative of f are as follows.

1. $f'(x)=\sum_{n=0}^{\infty}na_n(x-c)^{n-1}$

$$=a_1+2a_2(x-c)+3a_3(x-c)^2+\cdot\cdot\cdot$$

2. $\int f(x)=C+\sum_{n=0}^{\infty}a_n\frac{(x-c)^{n+1}}{n+1}$

$$=C+a_0(x-c)+a_1\frac{(x-c)^2}{2}+a_2\frac{(x-c)^3}{3}+\cdot\cdot\cdot$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. However the interval of convergence may differ as a result of the behavior at the endpoints.

Example 3 ◻ Let f be a function given by $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n(x-2)^n}{n}$.

Find the interval of convergence for each of the following.

(a) $f(x)$ (b) $f'(x)$ (c) $\int f(x)\,dx$.

Solution ◻ (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{(-1)^n(x-2)^n}\right|$

$$=\lim_{n\to\infty}\left|(x-2)\frac{n}{n+1}\right|=\left|x-2\right|$$

By the Ratio Test, the series converges if $\left|x-2\right|<1$.

$\left|x-2\right|<1 \;\Rightarrow\; 1<x<3$.

Now check the endpoints.

If $x=1$, $f(x)=\frac{(-1)^n(1-2)^n}{n}=\frac{1}{n}$, which diverges by the p-Series Test.

If $x=3$, $f(x)=\frac{(-1)^n(3-2)^n}{n}=\frac{(-1)^n}{n}$, which converges by the Alternating Series Test.

The interval of convergence is $\left(1,3\right]$.

(b) $f'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n n(x-2)^{n-1}}{n}=\sum_{n=1}^{\infty}(-1)^n(x-2)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^n}{(-1)^n(x-2)^{n-1}}\right|=|x-2|$$

$|x-2|<1 \Rightarrow 1<x<3$. Check endpoints.

If $x=1$, $f'(x)=\sum_{n=1}^{\infty}(-1)^n(1-2)^{n-1}=\sum_{n=1}^{\infty}(-1)^{2n-1}$,
which diverges by the *n*th-Term Test.

If $x=3$, $f'(x)=\sum_{n=1}^{\infty}(-1)^n(3-2)^{n-1}=\sum_{n=1}^{\infty}(-1)^n$,
which also diverges by the *n*th-Term Test.

The interval of convergence is $(1,3)$.

(c) $\int f(x)\,dx=C+\sum_{n=1}^{\infty}\frac{(-1)^n(x-2)^{n+1}}{n(n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^{n+2}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{(-1)^n(x-2)^{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n(x-2)}{n+2}\right|=|x-2|$$

$|x-2|<1 \Rightarrow 1<x<3$. Check endpoints.

If $x=1$, $\int f(x)\,dx=C+\sum_{n=1}^{\infty}\frac{(-1)^n(1-2)^{n+1}}{n(n+1)}=C+\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n(n+1)}$,

which convergent because $\frac{1}{n(n+1)}<\frac{1}{n^2}$.

If $x=3$, $\int f(x)\,dx=C+\sum_{n=1}^{\infty}\frac{(-1)^n(3-2)^{n+1}}{n(n+1)}=C+\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}$,

which also converges by the Alternating Series Test.

The interval of convergence is $[1,3]$.

Exercises - Representations of Functions as Power Series

Multiple Choice Questions

1. The power series expansion for $\dfrac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which of the following is a power series expansion for $\dfrac{1}{1+x^3}$?

(A) $1+x^2+x^4+x^6+\cdots$

(B) $1-x^3+x^6-x^9+\cdots$

(C) $1+\dfrac{x^3}{3}+\dfrac{x^6}{6}+\dfrac{x^9}{9}+\cdots$

(D) $1-\dfrac{x^3}{3}+\dfrac{x^6}{6}-\dfrac{x^9}{9}+\cdots$

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2. The power series expansion for $\dfrac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which of the following is a power series expansion for $\dfrac{1}{2-x}$?

(A) $1+\dfrac{x}{2}+\dfrac{x^2}{4}+\dfrac{x^3}{8}+\cdots$

(B) $1-\dfrac{x}{2}+\dfrac{x^2}{4}-\dfrac{x^3}{8}+\cdots$

(C) $\dfrac{1}{2}+\dfrac{x}{4}+\dfrac{x^2}{8}+\dfrac{x^3}{16}+\cdots$

(D) $\dfrac{1}{2}-\dfrac{x}{4}+\dfrac{x^2}{8}-\dfrac{x^3}{16}+\cdots$

3. If $f(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n!}=(x-2)-\frac{(x-2)^2}{2!}+\frac{(x-2)^3}{3!}-\frac{(x-2)^4}{4!}+\cdots$, which of the following represents $f'(x)$?

(A) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^{n-1}}{n!}$

(B) $\sum_{n=1}^{\infty}(-1)^{n}\frac{(x-2)^{n-1}}{(n+1)!}$

(C) $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-2)^{n-1}}{n!}$

(D) $\sum_{n=0}^{\infty}(-1)^{n}\frac{(x-2)^{n}}{n!}$

Free Response Questions

4. A power series expansion for $f(x)=\frac{1}{1-x}$ can be obtained from the sum of the geometric series $\sum_{n=0}^{\infty}ar^n=\frac{a}{1-r}$, if you let $a=1$ and $r=x.$. Let $g(x)$ be defined as $g(x)=\frac{1}{1+x}$.

(a) Write the first four terms and the general term of the power series expansion of $g(x)$.

(b) Write the first four terms and the general term of the power series expansion of $g(x^2)$.

(c) Write the first four terms and the general term of the power series expansion of h, where $h(x)=\int g(x^2)\,dx$ and $h(0)=0$.

(d) Find the value of $h(1)$.

5. The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n)!}=1-\frac{3x^2}{2!}+\frac{5x^4}{4!}-\frac{7x^6}{6!}+\cdots+(-1)^n\frac{(2n+1)x^{2n}}{(2n)!}+\cdots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x=0$. Give a reason for your answer.

(b) Show that $1-\frac{3}{2!}+\frac{5}{4!}$ approximates $f(1)$ with an error less than $\frac{1}{100}$.

(c) Let g be the function given by $g(x)=\int_0^x f(t)\,dt$. Write the first four terms and the general term of the power series expansion of $\frac{g(x)}{x}$.

9.8 Taylor Polynomial and Lagrange Error Bound

*n*th Taylor Polynomial and *n*th Maclaurin Polynomial

If a function f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdot\ \cdot\ \cdot\ + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the **nth Taylor polynomial for f at c**. If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdot\ \cdot\ \cdot\ + \frac{f^{(n)}(0)}{n!}x^n$$

is called the **Maclaurin polynomial for f**.

Guidelines for Finding a Taylor Polynomial

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c),\ f'(c),\ f''(c),\ f'''(c),\ \cdot\ \cdot\ \cdot,\ f^{(n)}(c)$$

2. Use the sequence developed in the first step to form the **Taylor coefficients**

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Example 1 □ Let f be the function given by $f(x) = \ln(2-x)$. Write the third-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(1.2)$.

Solution □

$$\begin{array}{ll} f(x) = \ln(2-x) & f(1) = \ln(2-1) = \ln 1 = 0 \\ f'(x) = \dfrac{-1}{2-x} & f'(1) = \dfrac{-1}{2-1} = -1 \\ f''(x) = \dfrac{-1}{(2-x)^2} & f''(1) = \dfrac{-1}{(2-1)^2} = -1 \\ f'''(x) = \dfrac{-2}{(2-x)^3} & f'''(1) = \dfrac{-2}{(2-1)^2} = -2 \end{array}$$

$$\begin{aligned} P_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= -(x-1) - \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 \end{aligned}$$

$$\begin{aligned} f(1.2) \approx P_3(1.2) &= -(1.2-1) - \frac{1}{2}(1.2-1)^2 - \frac{1}{3}(1.2-1)^3 \\ &\approx -.222666 \end{aligned}$$

Lagrange Error Bound

If f has $n+1$ derivatives at c and $R_n(x)$ is the remainder term of the Taylor polynomial $P_n(x)$, then $f(x)=P_n(x)+R_n(x)$.

So $R_n(x)=f(x)-P_n(x)$ and the absolute value of $R_n(x)$ satisfies the following inequality.

$$\boldsymbol{\left|R_n(x)\right|=\left|f(x)-P_n(x)\right|\le \max\left|f^{(n+1)}(k)\right|\cdot\frac{\left|x-c\right|^{n+1}}{(n+1)!}},$$

where $\max\left|f^{(n+1)}(k)\right|$ is the maximum value of $f^{(n+1)}(k)$ between *x* and *c*.

The remainder $R_n(x)$ is called the **Lagrange Error Bound** (or **Lagrange form of the remainder**).

Example 2 □ Let f be the function given by $f(x)=\sin(3x-\frac{\pi}{6})$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x=0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left|f(0.2)-P(0.2)\right|<\frac{1}{100}$.

Solution □ (a) $f(x)=\sin(3x-\frac{\pi}{6})$ $\qquad f(0)=\sin(-\frac{\pi}{6})=-\frac{1}{2}$

$f'(x)=3\cos(3x-\frac{\pi}{6})$ $\qquad f'(0)=3\cos(-\frac{\pi}{6})=\frac{3\sqrt{3}}{2}$

$f''(x)=-9\sin(3x-\frac{\pi}{6})$ $\qquad f''(0)=-9\sin(-\frac{\pi}{6})=\frac{9}{2}$

$f'''(x)=-27\cos(3x-\frac{\pi}{6})$ $\qquad f'''(0)=-27\cos(-\frac{\pi}{6})=-\frac{27\sqrt{3}}{2}$

$$\begin{aligned}P(x)&=-\frac{1}{2}+\frac{3\sqrt{3}}{2}x+\frac{9/2}{2!}x^2+\frac{-27\sqrt{3}/2}{3!}x^3\\&=-\frac{1}{2}+\frac{3\sqrt{3}}{2}x+\frac{9}{4}x^2-\frac{9\sqrt{3}}{4}x^3\end{aligned}$$

(b) $f^{(4)}(x)=81\sin(3x-\frac{\pi}{6})\Rightarrow \max\limits_{0\le k\le 0.2}\left|f^{(4)}(k)\right|=81$ since the maximum value of sine and cosine functions is 1. Therefore

$$\left|f(0.2)-P(0.2)\right|\le \max_{0\le k\le 0.2}\left|f^{(3+1)}(k)\right|\frac{(0.2-0)^{(3+1)}}{(3+1)!}$$

The polynomial was created at $x=0$ and the approximation is made at $x=0.2$.

$$\le 81\cdot\frac{(0.2)^4}{4!}=81\cdot\frac{0.0016}{24}$$

$$=0.0054<\frac{1}{100}$$

Exercises - Taylor Polynomial and Lagrange Error Bound

Multiple Choice Questions

1. Let $P(x)=\frac{1}{3}-\frac{2}{3}x+\frac{2}{3}x^2-\frac{4}{9}x^3+\frac{2}{9}x^4$ be the fourth-degree Taylor polynomial for the function f about $x=0$. What is the value of $f^{(4)}(0)$?

(A) $-\frac{32}{3}$ (B) $-\frac{4}{3}$ (C) $\frac{8}{9}$ (D) $\frac{16}{3}$

---

2. Let $P(x)=4-3x^2+\frac{13}{12}x^4-\frac{121}{360}x^6$ be the sixth-degree Taylor polynomial for the function f about $x=0$. What is the value of $f'''(0)$?

(A) $-\frac{121}{15}$ (B) $-\frac{3}{2}$ (C) 0 (D) $\frac{121}{15}$

---

3. Let f be a function that has derivatives of all orders for all real numbers. If $f(1)=2$, $f'(1)=-3$, $f''(1)=4$, and $f'''(1)=-9$, which of the following is the third-degree Taylor polynomial for f about $x=1$?

(A) $P(x)=2-3(x-1)+2(x-1)^2-\frac{3}{2}(x-1)^3$

(B) $P(x)=2-3(x+1)+2(x+1)^2-\frac{3}{2}(x+1)^3$

(C) $P(x)=2-3(x-1)+4(x-1)^2-9(x-1)^3$

(D) $P(x)=2-3(x+1)+2(x+1)^2-3(x+1)^3$

4. The third-degree Taylor polynomial of xe^x about $x = 0$ is

(A) $P_3(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3$

(B) $P_3(x) = x + x^2 + \frac{1}{2}x^3$

(C) $P_3(x) = x + x^2 - \frac{1}{3}x^3$

(D) $P_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

---

5. The second-degree Taylor polynomial of $\sec x$ about $x = \frac{\pi}{4}$ is

(A) $P_2(x) = 1 + \sqrt{2}(x - \frac{\pi}{4}) + \sqrt{2}(x - \frac{\pi}{4})^2$

(B) $P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) + \frac{3\sqrt{2}}{3!}(x - \frac{\pi}{4})^2$

(C) $P_2(x) = \sqrt{2} + \sqrt{2}(x - \frac{\pi}{4}) + \frac{3\sqrt{2}}{2!}(x - \frac{\pi}{4})^2$

(D) $P_2(x) = 1 + \sqrt{2}(x - \frac{\pi}{4}) + \frac{3\sqrt{2}}{3!}(x - \frac{\pi}{4})^2$

---

6. A function f has derivatives of all orders at $x = 0$. Let P_n denote the nth- degree Taylor polynomial for f about $x = 0$. It is known that $f(0) = \frac{1}{3}$ and $f''(0) = \frac{4}{3}$. If $P_2(\frac{1}{2}) = \frac{1}{8}$, what is the value of $f'(0)$?

(A) $-\frac{3}{8}$ (B) $-\frac{3}{4}$ (C) $-\frac{5}{4}$ (D) $-\frac{3}{2}$

Free Response Questions

7. Let $P(x) = 3 - 2(x-2) + 5(x-2)^2 - 12(x-2)^3 + 3(x-2)^4$ be the fourth-degree Taylor polynomial for the function f about $x = 2$. Assume f has derivatives of all orders for all real numbers.

   (a) Find $f(2)$ and $f'''(2)$.

   (b) Write the third-degree Taylor polynomial for f' about 2 and use it to approximate $f'(2.1)$.

   (c) Write the fourth-degree Taylor polynomial for $g(x) = \int_2^x f(t)\,dt$ about 2.

   (d) Can $f(1)$ be determined from the information given? Justify your answer.

---

8. Let f be the function given by $f(x) = \sin(2x) + \cos(2x)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

   (a) Find $P(x)$.

   (b) Find the coefficient of x^{19} in the Taylor series for f about $x = 0$.

   (c) Use the Lagrange error bound to show that $\left| f(\frac{1}{5}) - P(\frac{1}{5}) \right| < \frac{1}{100}$

   (d) Let h be the function given by $h(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for h about $x = 0$.

9.9 Taylor Series and Maclaurin Series

Taylor Series and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at c**. Moreover, if $c = 0$, then the series is called the **Maclaurin series for f**.

Example 1 □ Find the Maclaurin series for the function $f(x) = \ln(1+x)$.

Solution □

$$f(x) = \ln(1+x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = 2$$

$$f^{(4)}(x) = -\frac{6}{(1+x)^4} \qquad f^{(4)}(0) = -6$$

$$\ln(1+x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 0 + 1\cdot x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 + \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n + \cdots$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + \frac{(-1)^{n-1}}{n}x^n + \cdots$$

Example 2 □ Let f be a function having derivatives of all orders. The fourth degree Taylor polynomial for f about $x = 1$ is given

$T(x) = 4 + 3(x-1) - 6(x-1)^2 + 7(x-1)^3 - 4(x-1)^4$.

Find $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$ and $f^{(4)}(1)$.

Solution □

$$f(1) = T(1) = 4 \qquad f'(1) = 3$$

$$\frac{f''(1)}{2!} = -6 \Rightarrow f''(1) = -12 \qquad \frac{f'''(1)}{3!} = 7 \Rightarrow f'''(1) = 42$$

$$\frac{f^{(4)}(1)}{4!} = -4 \Rightarrow f^{(4)}(1) = -96$$

Direct computation of the Taylor or Maclaurin coefficients is usually a tedious procedure. The easiest way to find a Taylor or Maclaurin series is to develop a power series from a list of elementary functions. From the list of power series for elementary functions, you can develop power series for other functions by the operations of addition, subtraction, multiplication, division, differentiation, integration, or composition with known power series.

Power Series for Elementary Functions

Function	Interval of Convergence
$\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n+\cdots$	$0<x<2$
$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots+x^n+\cdots$	$-1<x<1$
$\ln x = (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots$	$0<x\le 2$
$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$	$-\infty<x<\infty$
$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\tan^{-1} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$	$-1\le x\le 1$

Multiplication of Power Series

Power series can be multiplied the way we multiply polynomials. We usually find only the first few terms because the calculations for the later terms become tedious and the initial terms are the most important ones.

Example 3 □ Find the Maclaurin series for the function $f(x)=\cos x^2$.

Solution □ $g(x)=\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$

$$\begin{aligned} f(x) &= \cos x^2 \\ &= g(x^2) \\ &= 1-\frac{(x^2)^2}{2!}+\frac{(x^2)^4}{4!}-\frac{(x^2)^6}{6!}+\cdots+\frac{(-1)^n(x^2)^{2n}}{(2n)!}+\cdots \\ &= 1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\cdots+\frac{(-1)^n x^{4n}}{(2n)!}+\cdots \end{aligned}$$

Example 4 □ Find the Maclaurin series for the function $f(x) = x^2e^x - x^2$.

Solution □ Use the series e^x .

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot\cdot\cdot + \frac{x^n}{n!} + \cdot\cdot\cdot$$

Multiply e^x by x^2 .

$$\begin{aligned} x^2e^x &= x^2(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot\cdot\cdot + \frac{x^n}{n!} + \cdot\cdot\cdot) \\ &= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdot\cdot\cdot + \frac{x^{n+2}}{n!} + \cdot\cdot\cdot \end{aligned}$$

Subtract x^2 from each side.

$$\begin{aligned} &x^2e^x - x^2 \\ &= (x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdot\cdot\cdot + \frac{x^{n+2}}{n!} + \cdot\cdot\cdot) - x^2 \\ &= x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdot\cdot\cdot + \frac{x^{n+2}}{n!} + \cdot\cdot\cdot \end{aligned}$$

Example 5 □ Find the first three nonzero terms in the Maclaurin series for $e^x \cos x$.

Solution □ Use the power series for e^x and $\cos x$ in the table .

$$\begin{aligned} e^x \cos x &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) \\ &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)(1) + (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)(-\frac{x^2}{2!}) + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &\qquad - \frac{x^2}{2!} - \frac{x^3}{2!} - \cdots \\ &= 1 + x - \frac{x^3}{3} + \cdots \end{aligned}$$

Exercises - Taylor Series and Maclaurin Series

Multiple Choice Questions

1. A series expansion of $\frac{\arctan x}{x}$ is

(A) $1-\frac{x}{3}+\frac{x^3}{5}-\frac{x^5}{7}+\cdots$

(B) $1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\cdots$

(C) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

(D) $x-\frac{x^3}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\cdots$

---

2. The coefficient of x^3 in the Taylor series for e^{-2x} about $x=0$ is

(A) $-\frac{4}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) $\frac{4}{3}$

---

3. A function f has a Maclaurin series given by $-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\cdots+\frac{(-1)^n x^{2n+2}}{(2n+1)!}+\cdots$.

Which of the following is an expression for $f(x)$?

(A) $x^3e^x-x^2$

(B) $x\ln x-x^2$

(C) $\tan^{-1}x-x$

(D) $x\sin x-x^2$

4. A series expansion of $\frac{x-\sin x}{x^2}$ is

(A) $\frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}+\cdots+\frac{(-1)^{n+1}x^{2n-2}}{(2n)!}+\cdots$

(B) $\frac{x}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}+\cdots+\frac{(-1)^{n+1}x^{2n+1}}{(2n)!}+\cdots$

(C) $\frac{x}{3!}-\frac{x^3}{5!}+\frac{x^5}{7!}+\cdots+\frac{(-1)^{n+1}x^{2n-1}}{(2n+1)!}+\cdots$

(D) $\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}+\cdots+\frac{(-1)^{n+1}x^{2n}}{(2n+1)!}+\cdots$

---

5. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!}$ is the Taylor series about zero for which of the following functions?

(A) $x\sin x$ (B) $x\cos x$ (C) x^2e^{-x} (D) $x\ln(x+1)$

---

6. The graph of the function represented by the Maclaurin series $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$ intersects the graph of $y=e^{-x}$ at $x=$

(A) 0.495 (B) 0.607 (C) 1.372 (D) 2.166

7. What is the coefficient of x^4 in the Taylor series for $\cos^2 x$ about $x = 0$?

(A) $\frac{1}{12}$ (B) $\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

---

8. The fifth-degree Taylor polynomial for $\tan x$ about $x = 0$ is $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$. If f is a function such that $f'(x) = \tan(x^2)$, then the coefficient of x^7 for $f(x)$ about $x = 0$ is

(A) $\frac{1}{21}$ (B) $\frac{3}{42}$ (C) 0 (D) $\frac{1}{7}$

---

9. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{n+1} x^n = \frac{1}{2}x - \frac{2}{3}x^2 + x^3 - \cdots + \frac{(-2)^{n-1}}{n+1}x^n + \cdots$. Which of the following is the third-degree Taylor polynomial for $g(x) = \cos x \cdot f(x)$ about $x = 0$?

(A) $x - \frac{1}{2}x^2 - \frac{2}{3}x^3$

(B) $1 - \frac{1}{2}x^2 + \frac{2}{3}x^3$

(C) $\frac{1}{2}x - \frac{2}{3}x^2 + \frac{3}{4}x^3$

(D) $\frac{1}{2}x - \frac{11}{12}x^2 + x^3$

10. The Maclaurin series for the function f is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots+\frac{(-1)^n x^{2n}}{(2n+1)!}+\cdots$$ on its interval of convergence.

Which of the following statements about f must be true?

(A) f has a relative minimum at $x=0$.

(B) f has a relative maximum at $x=0$.

(C) f does not have a relative maximum or a relative minimum at $x=0$.

(D) f has a point of inflection at $x=0$.

Free Response Questions

11. Let f be the function given by $f(x)=e^{-x}$.

(a) Write the first four terms and the general term of the Taylor series for f about $x=0$.

(b) Use the result from part (a) to write the first four nonzero terms and the general term of the series expansion about $x=0$ for $g(x)=\dfrac{1-x-f(x)}{x}$.

(c) For the function g in part (b), find $g'(-1)$ and use it to show that $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(n+1)!}=1$.

12. The Maclaurin series for $f(x)$ is given by $f(x) = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} + \cdots + \frac{(-1)^{n+1} x^{2n-1}}{(2n)!} + \cdots$.

The Maclaurin series for $g(x)$ is given by $g(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots + \frac{(-1)^n x^n}{n+1} + \cdots$.

(a) Find $f'''(0)$ and $f^{(15)}(0)$.

(b) Find the interval of convergence of the Maclaurin series for $g(x)$.

(c) The graph of $y = f(x) + g(x)$ passes through the point $(0,1)$. Find $y'(0)$ and $y''(0)$ and determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

| PART B |

Practice Tests

Calculus AB Practice Test 1

Calculus AB Practice Test 2

Calculus BC Practice Test 1

Calculus BC Practice Test 2

AP Calculus AB
Practice Test 1

Answer Sheet

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CALCULUS AB
SECTION I, Part A
Time — 60 minutes
Number of questions — 30

No calculator is allowed for problems on this part of the exam.

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the choices given, decide which is the best answer choice and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

1. If $f(x) = \begin{cases} \sqrt{x}, & x \neq 4 \\ 3, & x = 4 \end{cases}$, then $\lim\limits_{x \to 4} f(x) =$

(A) 1 (B) 2 (C) 3 (D) nonexistent

---

2. If $f(x) = \sqrt{\dfrac{x^2 - 2}{x^2 + 2}}$, then $f'(2) =$

(A) $\dfrac{2\sqrt{3}}{9}$ (B) $\dfrac{\sqrt{3}}{6}$ (C) $\dfrac{2\sqrt{3}}{3}$ (D) $\dfrac{3\sqrt{3}}{2}$

3. A curve has slope $2x+\sin x$ at each point (x, y) on the curve. Which of the following is an equation for this curve if it passes through the point $(0, 2)$?

(A) $y = x^2 + \cos x + 2$

(B) $y = x^2 - \cos x + 2$

(C) $y = x^2 - \cos x + 3$

(D) $y = x^2 + \cos x + 1$

---

4. $\lim\limits_{x \to 1} \dfrac{1 - x + \ln x}{1 + \cos \pi x} =$

(A) $-\infty$ (B) $-\dfrac{1}{\pi^2}$ (C) $-\pi^2$ (D) ∞

---

5. Let f and g be continuous functions with $f(2) = 5$. If $\lim\limits_{x \to 2}\left[f(x) + \dfrac{1}{2}g(x)\right] = 12$, then $\lim\limits_{x \to 2} g(x) =$

(A) 10 (B) 12 (C) 14 (D) 16

---

6. Using a right Riemann sum with four subintervals $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$ and $[2, 3]$, what is the approximation of $\int_1^3 \dfrac{1}{x}\,dx$?

(A) $\dfrac{5}{9}$ (B) $\dfrac{7}{12}$ (C) $\dfrac{7}{10}$ (D) $\dfrac{19}{20}$

7. If $y = \dfrac{1 + x \cdot f(x)}{\sqrt{x}}$, then $y' =$

(A) $\dfrac{2x^2 f'(x) + x\, f(x) + 1}{x\sqrt{x}}$

(B) $\dfrac{2x^2 f'(x) + x\, f(x) - 1}{2x\sqrt{x}}$

(C) $\dfrac{2x^2 f'(x) + 2x\, f(x) - 1}{2x\sqrt{x}}$

(D) $\dfrac{x^2 f'(x) - x\, f(x) - 1}{2x\sqrt{x}}$

---

8. If $y^2 = 3x^3 - x^4$, then the value of $\dfrac{dy}{dx}$ at the point $(2, 2\sqrt{2})$ is

(A) $\dfrac{\sqrt{2}}{2}$ (B) $\sqrt{2}$ (C) $\dfrac{3\sqrt{2}}{2}$ (D) $2\sqrt{2}$

---

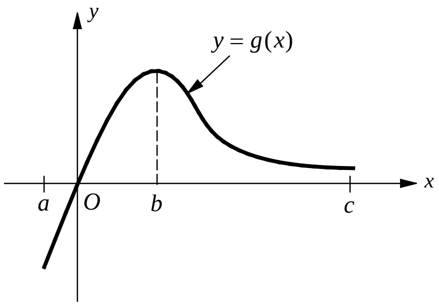
9. The closed interval $[a,b]$ is partitioned into n equal subintervals, each of width Δx, by the numbers $x_0, x_1, ..., x_n$ where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. What is $\lim_{n\to\infty} \sum_{i=1}^{n} (2x_i - 1)\, \Delta x$?

(A) $(b-a)(b+a-1)$

(B) $(b+a)(b-a-1)$

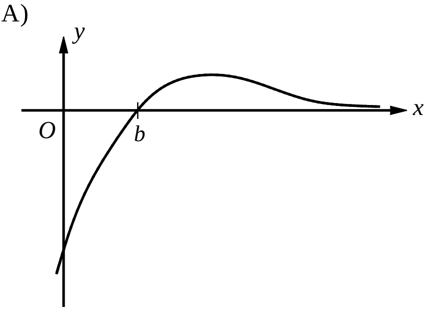
(C) $b^2 - a^2 - 1$

(D) $2(b^2 - a^2)$

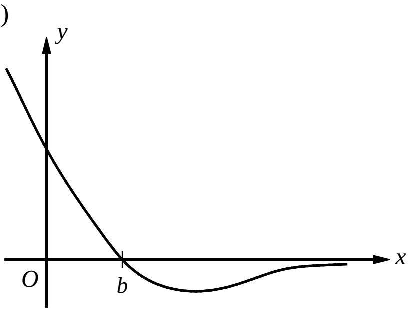


10. Let $g(x) = \int_a^x f(t)\, dt$, where $a \le x \le c$. The figure above shows the graph of g on $[a, c]$. Which of the following could be the graph of f on $[a, c]$?

(A)



(B)



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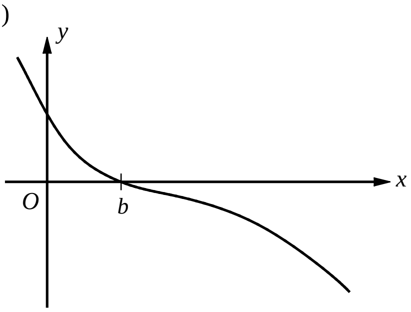
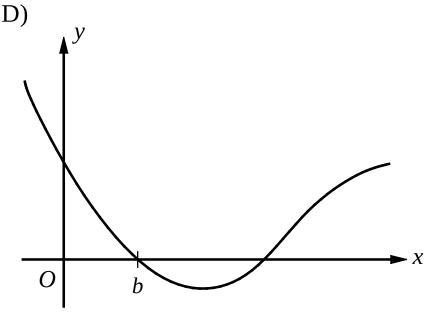


(D)



11. If $h(1) = 2$ and $h'(1) = -3$, what is the value of $\dfrac{d}{dx}\left(\dfrac{\sqrt{x}}{h(x)}\right)$ at $x = 1$?

(A) -2 (B) -1 (C) 0 (D) 1

---

12. If $7x + y = k$ is the equation of a line tangent to the graph of $y = 9x + \dfrac{1}{x^2}$, what is the value of k?

(A) $-\dfrac{17}{2}$ (B) $-\dfrac{13}{2}$ (C) 12 (D) 14

---

13. The graph of $f(x) = x(\ln x)^2$ has a point of inflection at

(A) $\dfrac{1}{e^2}$ (B) $\dfrac{1}{e}$ (C) e (D) e^2

---

14. $\int \cos^2 x \, dx =$

(A) $x + \dfrac{1}{2}\sin 2x + C$

(B) $\dfrac{1}{2}x + \dfrac{1}{4}\sin 2x + C$

(C) $\dfrac{1}{2}x - \dfrac{1}{4}\sin 2x + C$

(D) $\dfrac{1}{2}x + \cos 2x + C$

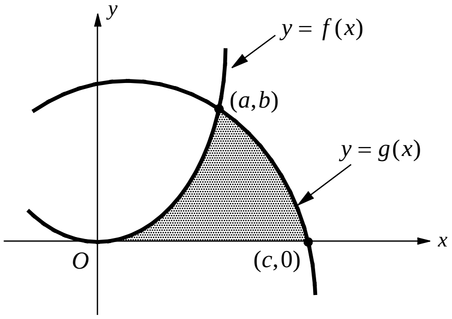
15. If the point $(-1,5)$ is a point of inflection of the curve $y = x^3 + ax^2 + bx - 2$, what is the value of $a + b$?

(A) -2 (B) -1 (C) 3 (D) 6

---

16. $\int_{-1}^{3} f(x)\,dx + \int_{3}^{6} f(x)\,dx + \int_{2}^{-1} f(x)\,dx =$

(A) $\int_{2}^{6} f(x)\,dx$ (B) $\int_{2}^{3} f(x)\,dx$ (C) $\int_{-1}^{6} f(x)\,dx$ (D) $\int_{3}^{6} f(x)\,dx$



17. The curves $y = f(x)$ and $y = g(x)$ shown in the figure above intersect at point (a,b). The area of the shaded region enclosed by these curves and the x-axis is given by

(A) $\int_{0}^{c} \left[f(x) - g(x)\right] dx$

(B) $\int_{0}^{c} \left[g(x) - f(x)\right] dx$

(C) $\int_{0}^{c} g(x)\,dx - \int_{a}^{c} f(x)\,dx$

(D) $\int_{0}^{a} f(x)\,dx + \int_{a}^{c} \text{g}(x)\,dx$

18. $\int_{e}^{e^2} \dfrac{dx}{x\ln x} =$

(A) $\dfrac{1}{e}-1$ (B) $\dfrac{1}{e^2}-\dfrac{1}{e}$ (C) $\ln 2-1$ (D) $\ln 2$

---

19. The region enclosed by the graph of $y=\sin x$ and the lines $y=\dfrac{1}{2}$, $x=\dfrac{\pi}{6}$, and $x=\dfrac{5\pi}{6}$ is rotated about the x-axis. What is the volume of the solid generated?

(A) $\dfrac{\sqrt{3}\pi+\pi^2}{6}$ (B) $\dfrac{\sqrt{3}\pi-\pi^2}{4}$ (C) $\dfrac{\pi^2}{6}+\dfrac{\sqrt{3}}{4}\pi$ (D) $-\dfrac{\pi^2}{6}+\dfrac{\sqrt{3}}{4}\pi$

---

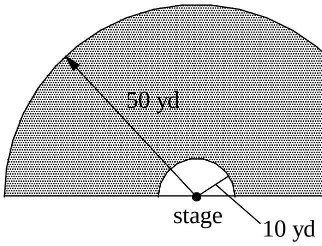
20. $\int_{\pi/4}^{\pi/3} \dfrac{\sec^2\theta}{\tan\theta}\,d\theta =$

(A) $\ln 6$ (B) $\ln\sqrt{3}$ (C) 1 (D) $\ln\sqrt{3}-1$

---

21. If $f(x)=\sqrt[3]{x^2+7}\cdot e^x$, then what is the value of $f'(1)$?

(A) $\dfrac{3e}{4}$ (B) $\dfrac{7e}{6}$ (C) $\dfrac{3e}{2}$ (D) $\dfrac{13e}{6}$



22. At a musical concert the audience stands inside a semicircular area of radius 50 yards. The stage is also a semicircular shape of radius 10 yards. If the density of the audience at r yards from the center of the stage is given by $f(r)$ people per square yard, which of the following expressions gives the number of people at the concert?

(A) $\frac{\pi}{2}\int_{10}^{50} r^2 f(r)\,dr$

(B) $\pi\int_{10}^{50} r^2 f(r)\,dr$

(C) $\pi\int_{10}^{50} r f(r)\,dr$

(D) $2\pi\int_{10}^{50} r f(r)\,dr$

---

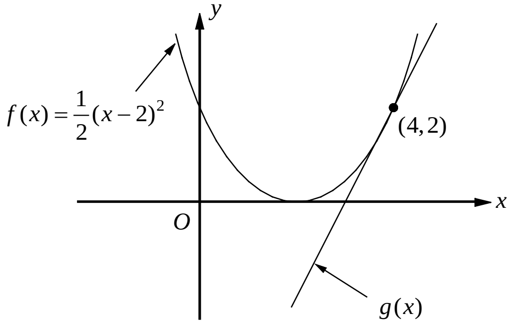
23. If four subdivisions of $[1,3]$ are used, what is the trapezoidal approximation of $\int_1^3 \sqrt{x}\cos x\,dx$?

(A) $\frac{1}{2}(\cos 1+\sqrt{1.5}\cos 1.5+\sqrt{2}\cos 2+\sqrt{2.5}\cos 2.5+\sqrt{3}\cos 3)$

(B) $\frac{1}{2}(\cos 1+2\sqrt{1.5}\cos 1.5+2\sqrt{2}\cos 2+2\sqrt{2.5}\cos 2.5+\sqrt{3}\cos 3)$

(C) $\frac{1}{4}(\cos 1+\sqrt{1.5}\cos 1.5+\sqrt{2}\cos 2+\sqrt{2.5}\cos 2.5+\sqrt{3}\cos 3)$

(D) $\frac{1}{4}(\cos 1+2\sqrt{1.5}\cos 1.5+2\sqrt{2}\cos 2+2\sqrt{2.5}\cos 2.5+\sqrt{3}\cos 3)$



24. The figure above shows the graph of the function $f=\frac{1}{2}(x-2)^2$ and the graph of g which is tangent to the graph of f at the point $(4,2)$. If $h(x)=f(g(x))$, what is $h'(4)$?

(A) –4 (B) –2 (C) 0 (D) 2

---

25. $\lim_{h\to 0}\frac{1}{h}\int_{1}^{1+h}\sqrt{3+x^2}\,dx=$

(A) 0 (B) 1 (C) 2 (D) 3

---

26. $\int_{-3}^{2}\left(\frac{|x|}{x}+3\right)dx=$

(A) –5 (B) –3 (C) 7 (D) 14

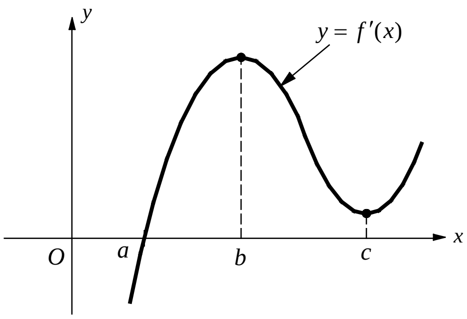
27. If f is a continuous function and if $F'(x) = f(x)$ for all real numbers x, then $\int_1^4 \frac{f(\sqrt{x})}{\sqrt{x}}\,dx =$

(A) $2\left[F(2) - F(1)\right]$

(B) $2\left[F(4) - F(1)\right]$

(C) $\frac{1}{2}\left[F(2) - F(1)\right]$

(D) $\frac{1}{2}\left[F(4) - F(1)\right]$



28. The graph of f' is shown in the figure above. Which of the following statements about f are true?

I. f has a relative minimum at $x = a$.
II. f has a relative maximum at $x = b$.
III. f is decreasing on the interval $b < x < c$.

(A) None

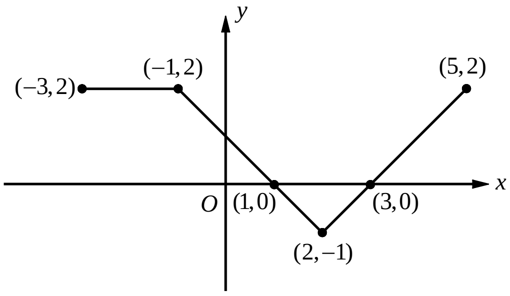
(B) I only

(C) I and III only

(D) II and III only

29. If $f(x) = 3x^3 - x + 4$, then $(f^{-1})'(6) =$

(A) $\frac{1}{8}$ (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) 3



Graph of f

30. The graph of the function f shown above consists of three line segments. If g is the function defined by $g(x) = \int_2^x f(t)\,dt$, then $g(-3) =$

(A) $-\frac{13}{2}$ (B) $-\frac{11}{2}$ (C) $-\frac{9}{2}$ (D) $\frac{11}{2}$

CALCULUS AB
SECTION I, Part B
Time — 45 minutes
Number of questions — 15

A graphing calculator is required for some problems on this part of the exam.

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the choices given, decide which is the best answer choice and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

(1) The exact numerical value of the correct answer does not always appear among the choices given. If this occurs, select the number that best approximates the exact numerical value from the choices given.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(3) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

31. The first derivative of the function f is given by $f'(x) = x - \sin(e^x)$ and the function g is given by $g(x) = \frac{3}{2}x$. At how many points on the graph of $f(x)$ is the tangent line parallel to $g(x)$?

(A) One (B) Two (C) Three (D) More than three

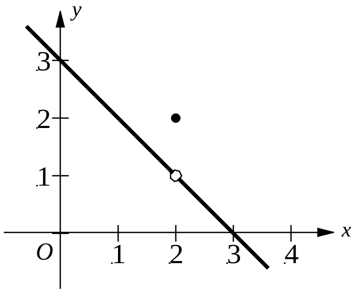
---

32. The velocity, in ft /sec , of a particle moving along the x-axis is given by the function $v(t) = te^{\sin t}$. What is the average velocity of the particle from time $t = 0$ to time $t = 4$?

(A) 2.955 (B) 4.432 (C) 5.909 (D) 8.864

33. Let f be the function defined by $f(x) = x\sqrt{5-x}$. The approximate value of f at $x = 1.1$, obtained from the tangent to the graph at $x = 1$, is

(A) 2.036 (B) 2.085 (C) 2.162 (D) 2.175



34. The graph of the function f is shown in the figure above. The value of $\lim_{x\to 2} \arctan(f(x))$ is

(A) 0 (B) 0.524 (C) 0.785 (D) 1.107

---

35. The position of a particle moving along a line is given by $s(t) = 2t^3 - 27t^2 + 84t + 11$ for $t \geq 0$. For what value of t is the speed of the particle decreasing?

(A) $0 < t < \frac{9}{2}$ only

(B) $2 < t < 7$ only

(C) $0 < t < 2$ and $\frac{9}{2} < t < 7$

(D) $2 < t < \frac{9}{2}$ and $t > 7$

36. A particle moves along the *y*-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = 2\sin(t^2 - t) - 1$. The position of the particle is 4 at time $t = 0$. What is the position of the particle when its velocity is first equal to 0?

(A) 0.922 (B) 1.534 (C) 2.026 (D) 2.466

---

37. Let f be a twice differentiable function with $f'(x) < 0$ and $f''(x) < 0$ for all x, in the closed interval $[0, 6]$. Which of the following could be a table of values for f?

(A)

x	$f(x)$
0	1
2	2
4	4
6	7

(B)

x	$f(x)$
0	1
2	5
4	8
6	10

(C)

x	$f(x)$
0	9
2	5
4	2
6	0

(D)

x	$f(x)$
0	9
2	8
4	5
6	0

---

38. Let f be the function given by $f(x) = \dfrac{9x}{4} - \pi\sin^2 x$. What are all values of c that satisfy Roll's Theorem on the closed interval $\left[0, \dfrac{\pi}{3}\right]$?

(A) 0.128 (B) 0.247 (C) 0.399 (D) 0.524

39. Let C be a function defined by $C(v) = 35 - 7\sqrt[4]{v}$, where v is the velocity of the particle moving along a curve. At time $t = 0$, the velocity is $v = 15$. If the velocity of the particle increases at a constant rate of 6 units per second, what is the rate of change of C with respect to time t, at $t = 8$ seconds.

(A) -0.765 (B) -0.469 (C) -0.374 (D) -0.287

---

40. The base of the solid is the region enclosed by the graph of $y = \ln x$, the x-axis, and the line $x = 2$. If the cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?

(A) 0.074 (B) 0.286 (C) 0.385 (D) 0.493

---

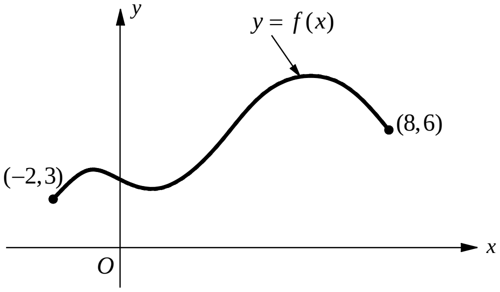
41. Which of the following is an equation of the curve that intersects every curve of the family $y = \ln x + 2$ at right angles?

(A) $y = x$ (B) $y = -x$ (C) $y = \frac{1}{2}x^2$ (D) $y = -\frac{1}{2}x^2$

---

42. If $f(x) = 2\sin(e^{t/2})$ and g is the antiderivative of f such that $g(-1) = -2$, then $g(3) =$

(A) -2.725 (B) -1.436 (C) 1.026 (D) 2.255



43. Let f be a twice differentiable function whose graph is shown in the figure above. Which of the following must be true for the function f on the closed interval $[-2, 8]$.

I. The average rate of change of f is $\frac{3}{10}$.

II. The average value of f is $\frac{9}{2}$.

III. The average value of f' is $\frac{3}{10}$.

(A) None

(B) I and II only

(C) I and III only

(D) II and III only

---

44. If f is a continuous function such that $\int_1^x f(t)\,dt = x\cos^2 x$ for all x, then $f(\frac{\pi}{4}) =$

(A) $\frac{2-\pi}{4}$ (B) $\frac{1}{2}$ (C) $\frac{\pi}{4}$ (D) $\frac{2+\pi}{4}$

---

45. Let f be the function defined by $f(x) = kx^2 - \ln x$ for $x > 0$. If the graph of f has a point of inflection on the x-axis what is the value of k ?

(A) $-2e$ (B) $y = -e^2$ (C) $-\frac{e}{2}$ (D) $2e$

CALCULUS AB
SECTION II, Part A
Time — 30 minutes
Number of problems — 2

A graphing calculator is required for these problems.

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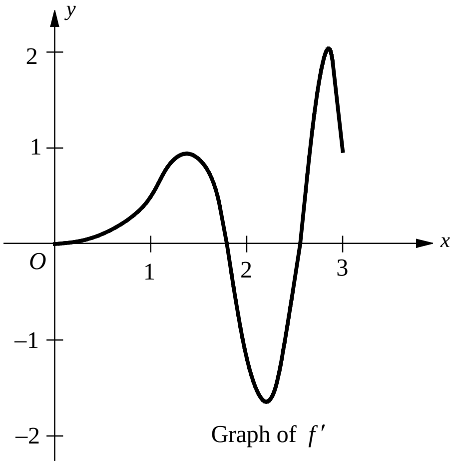
1. The rate of change of the number of honeybees at time t days is modeled by $n(t) = 40\sqrt{t}\sin(\frac{\pi t}{12})$ honeybees per day, for $0 \le t \le 30$. There are 1200 honeybees at time $t = 0$.

(a) Show that the number of honeybees is increasing at time $t = 10$.

(b) At time $t = 10$, is the number of honeybees increasing at an increasing rate, or is the number of honeybees increasing at a decreasing rate? Give a reason for your answer.

(c) According to the model, how many honeybees will there be at time $t = 30$? Round your answer to the nearest whole number.

(d) What is the maximum number of honeybees for $0 \le t \le 30$ rounded to the nearest whole number? Justify your answer.



Graph of f'

2. The figure above shows the graph of f', given by $f'(x) = \ln(x^2 + 1)\sin(x^2)$ on the closed interval $[0,3]$. The function f is twice differentiable with $f(0) = 3$.

(a) Use the graph of f' to determine whether the graph of f concaves up or concaves down on the interval $0 < x < 1$. Justify your answer.

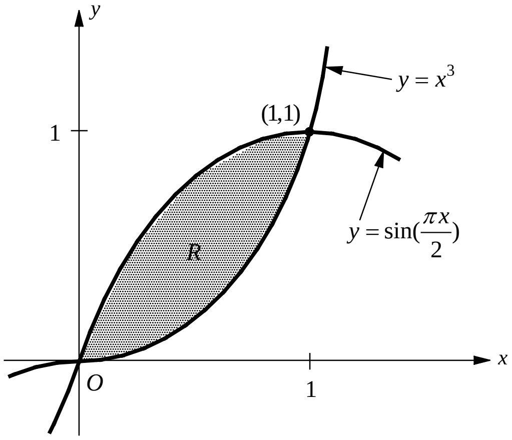
(b) On the closed interval $[0,3]$, find the value of x at which f attains its absolute maximum. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at $x = 2$.

CALCULUS AB
SECTION II, Part B

Time — 60 minutes
Number of problems — 4

No calculator is allowed for these problems.

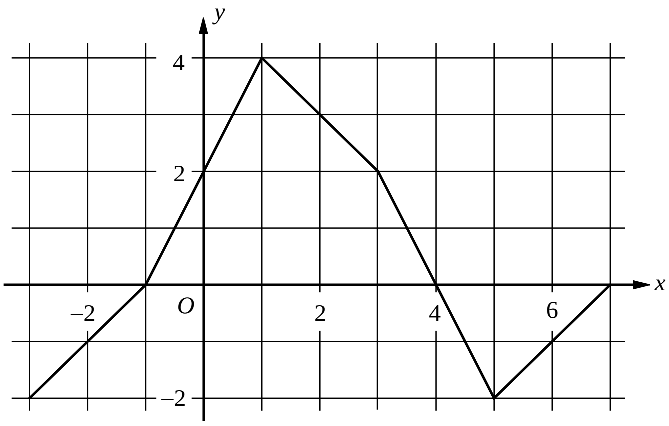


3. Let R be the shaded region in the first quadrant bounded by the graphs of $y = x^3$ and $y = \sin(\frac{\pi x}{2})$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is revolved about the horizontal line $y = -1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.



Graph of f

4. The graph of the function f above consists of five line segments. Let g be the function given by $g(x) = \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(2)$, $g'(2)$ and $g''(2)$.

(b) Find the x-coordinate of each point of inflection of the graph of g on the open interval $-3 < x < 7$. Justify your answer.

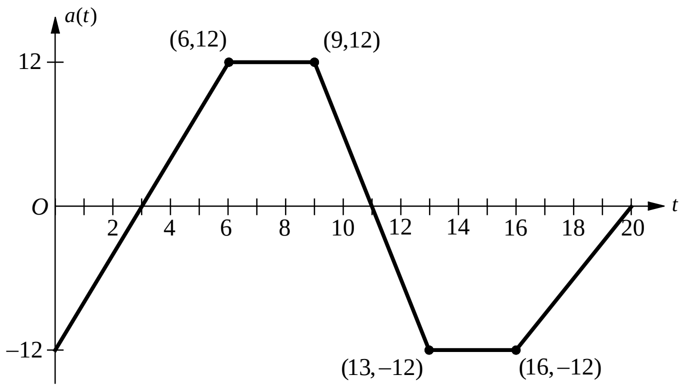
(c) On what interval is g increasing?

(d) Find the average rate of change of g on the interval $0 \le x \le 4$

5. The twice differentiable function f is defined for all real numbers and satisfies the following conditions: $f(0) = -1$, $f'(0) = 3$, and $f''(0) = -2$.

   (a) The function g is given by $g(x) = \sin ax + f(bx)$ for all real numbers, where a and b are constants. Find $g'(0)$ and $g''(0)$ in terms of a and b. Show the work that leads to your answers.

   (b) The function h is given by $h(x) = e^{\sin x} f(x)$ for all real numbers. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.



6. A car is traveling on a straight road with velocity 40 ft/sec at time $t = 0$. For $0 \le t \le 20$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above.

(a) For what values of t is the car's velocity increasing?

(b) At what time in the interval $0 < t < 10$ is the velocity of the car 40 ft/sec? Justify your answer.

(c) On the time interval $0 \le t \le 20$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur?

(d) On the time interval $0 \le t \le 20$, what is the car's absolute minimum velocity, in ft/sec, and at what time does it occur?

AP Calculus AB
Practice Test 2

Answer Sheet

1 Ⓐ Ⓑ Ⓒ Ⓓ
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CALCULUS AB
SECTION I, Part A
Time — 60 minutes
Number of questions — 30

No calculator is allowed for problems on this part of the exam.

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the choices given, decide which is the best answer choice and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

1. An equation of the line normal to the graph of $y = \sec x$ at the point $(\frac{\pi}{4}, \sqrt{2})$ is

(A) $y - \sqrt{2} = \sqrt{2}(x - \frac{\pi}{4})$

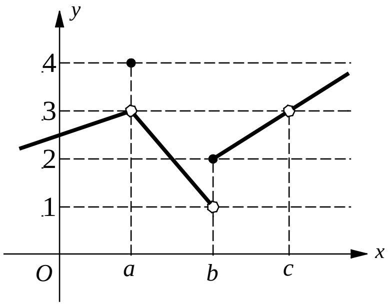
(B) $y - \sqrt{2} = -\frac{1}{\sqrt{2}}(x - \frac{\pi}{4})$

(C) $y - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}(x - \frac{\pi}{4})$

(D) $y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x - \frac{\pi}{4})$

2. The first derivative f' of a function f is given by $f'(x) = \frac{1}{2} - e^{-x}$. On which of the following intervals is f increasing?

(A) $(-\infty, -\ln 2]$ (B) $[-\ln 2, \ln 2]$ (C) $[\ln 2, \infty)$ (D) $[e, \infty)$



3. The graph of the function f is shown in the figure above. Which of the following statements about f is not true?

(A) $\lim_{x \to a} f(x) = 3$

(B) $f(a) = 4$

(C) $\lim_{x \to b} f(x) = 1$

(D) $f(b) = 2$

---

4. If $f(\theta) = \tan^2(2 - \theta)$, then $f'(0) =$

(A) $2 \tan 2$

(B) $-2 \tan 2$

(C) $-2 \tan 2 \sec 2$

(D) $-2 \tan 2 \sec^2 2$

5. If $f(x)=(x^2+1)^x$, then $f'(1)=$

(A) $2+\ln 4$ (B) $2+\ln 2$ (C) $1+\ln 4$ (D) $1+\ln 2$

---

6. If $x=-2$ is the vertical asymptote and $y=1$ is the horizontal asymptote for the graph of the function f, which of the following is the equation of the curve?

(A) $f(x)=\dfrac{x^2-2}{x^2+4}$

(B) $f(x)=\dfrac{x^2-2x}{x^2-4}$

(C) $f(x)=\dfrac{x^2-4x+4}{-x^2+4}$

(D) $f(x)=\dfrac{x^2+2x}{x^2-4}$

---

7. If $g(x)=2x\,f(\sqrt{x})$, then $g'(x)=$

(A) $2\left[\,f'(\sqrt{x})+f(\sqrt{x})\right]$

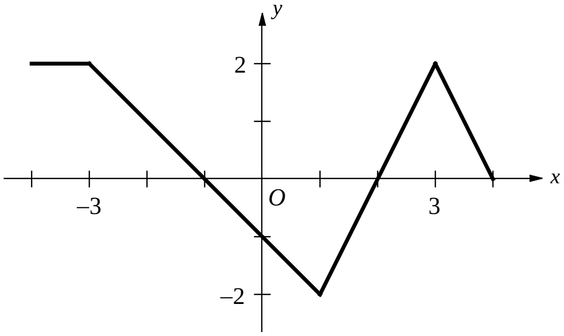
(B) $2\left[x\,f'(\sqrt{x})+f(\sqrt{x})\right]$

(C) $\sqrt{x}\,f'(\sqrt{x})+2\,f(\sqrt{x})$

(D) $2\sqrt{x}\,f'(\sqrt{x})+4\,f(\sqrt{x})$

8. The slope of the tangent to the curve $y^3 + x^2y^2 - ye^{2x} = 6$ at $(0, 2)$ is

(A) $-\frac{9}{4}$ (B) $-\frac{5}{11}$ (C) $\frac{4}{11}$ (D) $\frac{15}{22}$



Graph of f'

9. The graph f' is shown above. Which of the following statements is not true about f ?

(A) f is decreasing for $-3 \le x \le 1$.

(B) f is increasing for $-4 \le x \le -1$ or $2 \le x \le 4$.

(C) f has a local minimum at $x = 2$.

(D) f has a local maximum at $x = -1$.

---

10. Which of the following is the antiderivative of $f(x) = \frac{1}{\sqrt{x}} + \sec^2 x$?

(A) $\frac{-2}{\sqrt{x}} + \tan x + C$

(B) $2\sqrt{x} + \tan x + C$

(C) $-2\sqrt{x} + \sec x \tan x + C$

(D) $2\sqrt{x} + \sec x \tan x + C$

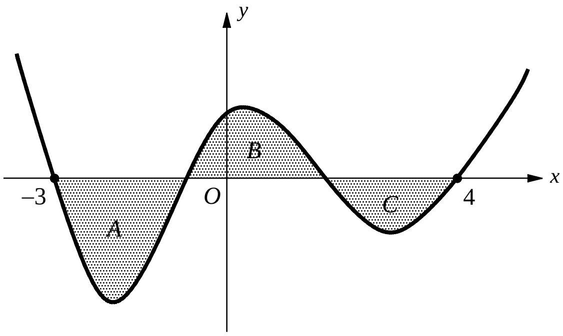
11. If $f''(x) = 10x^{-1/3}$, which of the following could be true?

I. $f(x) = -9x^{5/3} + 5$

II. $f'(x) = 15x^{2/3} - 12$

III. $f'''(x) = -\frac{10}{3}x^{-4/3} + 7$

(A) None (B) I only (C) II only (D) I and II only



12. The shaded regions A, B, and C in the figure above are bounded by the graph of $y = f(x)$ and the x-axis. If the area of region A is 4, region B is 3, and region C is 2, what is the value of $\int_{-3}^{4} \left[f(x) + 2 \right] dx$?

(A) 8 (B) 9 (C) 11 (D) 13

---

13. If $F'(x) = f(x)$ for all real numbers x, and if k is a constant, then $\int_{1}^{2} f(kx)\, dx =$

(A) $\frac{\left[F(2) - F(1)\right]}{k}$

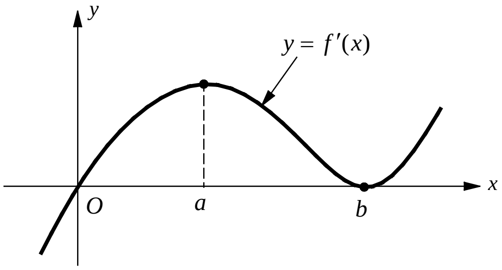
(B) $\frac{\left[F(2k) - F(k)\right]}{k}$

(C) $F(2k) - F(k)$

(D) $k\left[F(2) - F(1)\right]$

14. If $\lim\limits_{x\to 0}\dfrac{\sqrt{3-kx}-\sqrt{3}}{x}=\dfrac{1}{\sqrt{3}}$ what is the value of k ?

(A) -3 (B) -2 (C) $-\sqrt{3}$ (D) $-\sqrt{2}$



15. The graph of f', the derivative of function f, is shown above. If f is a twice differentiable function which of the following statements must be true?

I. $f(a) > f(b)$

II. The graph of f has a point of inflection at $x = b$.

III. The graph of f concaves down on the interval $a < x < b$.

(A) I only (B) II only (C) III only (D) II and III only

---

16. Let f be a differentiable function with $f(\frac{\pi}{4}) = -2$ and $f'(\frac{\pi}{4}) = 3$. If g is the function defined by $g(x) = \cos^2 x \cdot f(x)$, then $g'(\frac{\pi}{4}) =$

(A) $-\dfrac{3\sqrt{2}}{2}$ (B) $-\dfrac{3}{2}$ (C) $\dfrac{7}{2}$ (D) $\dfrac{3\sqrt{2}}{2}$

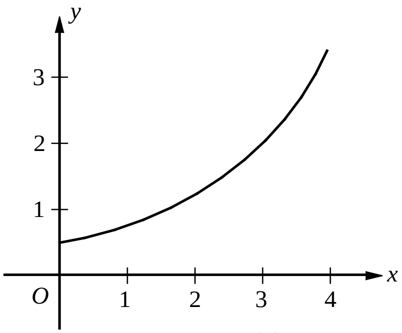
17. The expression $\frac{1}{10}\left[\ln(1.1)+\ln(1.2)+\ln(1.3)+...+\ln(2)\right]$ is a Riemann sum approximation for

(A) $\frac{1}{10}\int_0^1 \ln x\, dx$

(B) $\int_0^1 \ln x\, dx$

(C) $\frac{1}{10}\int_1^2 \ln x\, dx$

(D) $\int_1^2 \ln x\, dx$



Graph of *f*

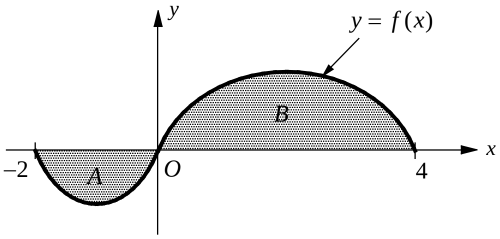
18. The graph of f is shown above for $0 \le x \le 4$. Let L, R, and T be the left Riemann sum, right Riemann sum, and the trapezoidal sum approximation respectively, of $f(x)$ on $[0, 4]$ with 4 subintervals of equal length. Which of the following statements is true?

(A) $L < \int_0^4 f(x)\, dx < T < R$

(B) $L < \int_0^4 f(x)\, dx < R < T$

(C) $R < \int_0^4 f(x)\, dx < L < T$

(D) $T < L < \int_0^4 f(x)\, dx < R$



19. The graph of $y = f(x)$ is shown in the figure above. The shaded region A has area a and the shaded region B has area b. If $g(x) = f(x) + 3$ what is the average value of g on the interval $[-2, 4]$?

(A) $\frac{a+b+3}{6}$ (B) $\frac{-a+b+3}{6}$ (C) $\frac{-a+b}{6}+3$ (D) $\frac{a+b}{6}+3$

---

20. Let f be the function given by $f(x) = e^{\cos x}$. Which of the following statements are true?

I. $\frac{d}{dx}\int_0^{\pi/3} f(x)\,dx = 0$

II. $\int_0^{\pi/3} \frac{d}{dx} f(x)\,dx = \sqrt{e} - e$

III. $\frac{d}{dx}\int_0^{x} f(t)\,dt = e^{\cos x}$

(A) I only (B) I and II only (C) I and III only (D) I , II, and III

---

21. $\int_1^{e^2} (\frac{x-1}{x})\,dx =$

(A) $e^2 - 1$ (B) $e^2 - 2$ (C) $e^3 - e$ (D) $e^2 - 3$

22. Using the substitution $u = \sqrt{x}$, $\int_0^9 \sin(\sqrt{x})\,dx$ is equivalent to

(A) $2\int_0^3 u\sin(u)\,du$

(B) $2\int_0^3 \frac{\sin u}{u}\,du$

(C) $\frac{1}{2}\int_0^3 u\sin(u)\,du$

(D) $\frac{1}{2}\int_0^3 \frac{\sin u}{u}\,du$

---

23. If $\frac{dy}{dx} = 1 + y^2$ and $y(-1) = 0$, then $y =$

(A) $1 - \tan x$ (B) $1 + \sec x$ (C) $\tan(x+1)$ (D) $1 - e^{\tan(x+1)}$

---

x	-1	1	4	6	9
$f(x)$	12	9	5	8	10

24. A function f is continuous on the closed interval $[-1,9]$ and has values that are given in the table above. Using subintervals $[-1,1]$, $[1,4]$, $[4,6]$, and $[6,9]$, what is the trapezoidal approximation of $\int_{-1}^{9} f(x)\,dx$?

(A) 76 (B) 82 (C) 92 (D) 98

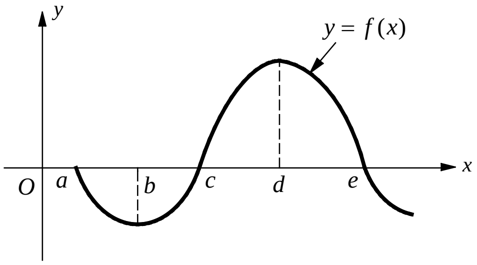
25. $\lim_{x\to\infty}(e^x + x)^{1/x} =$

(A) $-e$ (B) -1 (C) 1 (D) e

---

26. $\int_0^{\pi/4} \frac{1}{e^{\tan x}\cos^2 x}\,dx =$

(A) $-e$ (B) $1-e$ (C) $\frac{1}{e}$ (D) $1-\frac{1}{e}$



27. The graph of the function f is shown in the figure above. If $h(x) = \int_a^x f(t)\,dt$, which of the following is true?

(A) $h(x)$ has a minimum at $x = b$ and has a maximum at $x = d$.

(B) $h(x)$ has a minimum at $x = a$ and has a maximum at $x = e$.

(C) $h(x)$ has a minimum at $x = e$ and has a maximum at $x = c$.

(D) $h(x)$ has a minimum at $x = c$ and has a maximum at $x = e$.

28. Let R be the region in the first quadrant bounded by the graph of $y = a\sqrt{x}$ ($a > 0$), $x = 1$, and $x = 4$. If the area of the region R is 7, what is the value of a?

(A) $\frac{4}{3}$ (B) $\frac{3}{2}$ (C) $\frac{7}{3}$ (D) $\frac{5}{2}$

---

29. If $F(x) = \int_{\tan x}^{0} \frac{dt}{1+t^2}$, then $F'(x) =$

(A) $\sin x$ (B) $\cos x$ (C) -1 (D) $\cos^2 x$

---

30. If the substitution $x = 5\sin\theta$ is made for $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$, where $0 < \theta < \frac{\pi}{2}$, then

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx =$$

(A) $\frac{1}{5}\int \csc\theta\cot\theta\,d\theta$

(B) $-\frac{1}{5}\int \csc^2\theta\,d\theta$

(C) $\frac{1}{25}\int \csc^2\theta\,d\theta$

(D) $\frac{1}{25}\int \sec^2\theta\,d\theta$

CALCULUS AB
SECTION I, Part B
Time ─ 45 minutes
Number of questions ─ 15

A graphing calculator is required for some problems on this part of the exam.

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the choices given, decide which is the best answer choice and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

(1) The exact numerical value of the correct answer does not always appear among the choices given. If this occurs, select the number that best approximates the exact numerical value from the choices given.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(3) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

31. What is the area of the region enclosed by the graphs of $f(x) = \sqrt{x+1}$ and $g(x) = \dfrac{1}{1+x}$ from $x = 0$ to $x = 2$?

(A) 1.362 (B) 1.475 (C)1.699 (D) 1.833

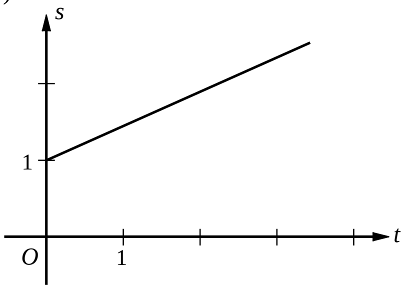
---

32. The maximum acceleration attained on the interval $0 \le x \le 5$ by the particle whose velocity is given by $v(t) = t^3 - 4t^2 + 7t + 3$ is

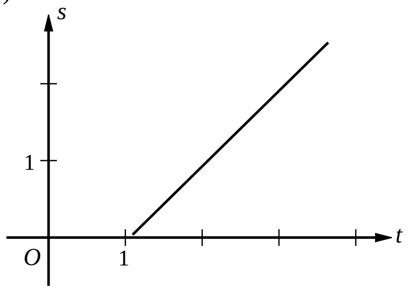
(A) 28 (B) 32 (C) 37 (D) 42

33. A particle moves along the *x*-axis with $a(t) = 1.2$ units /sec^2. When $t = 1$, the particle is at the point $(0,1)$. Which of the following could be the graph of the distance $s(t)$ of the particle as a function of time t.

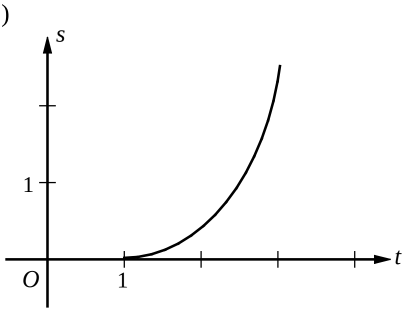
(A)



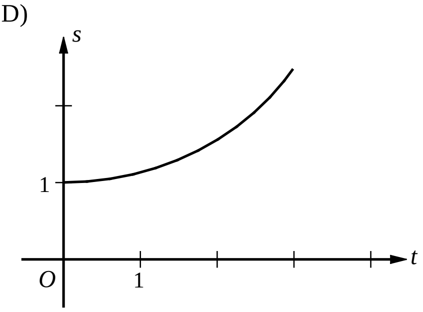
(B)



(C)



(D)



34. Let f be the function given by $f(x) = \cos\left[\ln(x^2+1)\right]$. On the closed interval $[0,8]$, how many values of c satisfies the conclusion of Mean Value Theorem?

(A) 1 (B) 2 (C) 3 (D) 4

35. Let f be a function given by $f(x)=\dfrac{x^2-1}{|x-1|}$. Which of the following statements are true about f ?

I. $\lim\limits_{x\to 1^+} f(x)=2$

II. $\lim\limits_{x\to 1^-} f(x)=-2$

III. $\lim\limits_{x\to 3} f(x)=\lim\limits_{x\to -5} f(x)$

(A) I only (B) I and II only (C) II and III only (D) I , II, and III

---

36. The base of a solid is the region enclosed by the graph of $y=\sin(x^2)$ and the x-axis for $0\le x\le\sqrt{\pi}$. If cross sections of the solid perpendicular to the x-axis are squares, what is the volume of the solid?

(A) 0.670 (B) 0.783 (C) 0.835 (D) 1.032

---

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	5	–1	–3
2	3	–2	–2	1
3	2	4	–1	7

37. The table above gives values of f , f', g , and g' at selected values of x . If $h(x)=g\left[f(x^2)\right]$, what is the value of $h'(1)$?

(A) –14 (B) –8 (C) –3 (D) 6

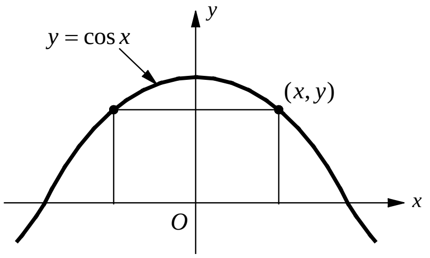
38. The volume V of a sphere is decreasing at a rate of $12\ \text{in}^3/\text{sec}$. What is the rate of decrease of the radius of the sphere, in inches per second, at the instant when the surface area S becomes 36π square inches? ($V = \frac{4}{3}\pi r^3$, and $S = 4\pi r^2$)

(A) $\frac{1}{6\pi}$ (B) $\frac{1}{4\pi}$ (C) $\frac{1}{3\pi}$ (D) $\frac{1}{2\pi}$

---

39. Water is leaking from a tank at the rate of $te^{(-0.1t)}$ gallons per hour. If there are 100 gallons of water in the tank at time $t = 0$, how many gallons of water are in the tank at time $t = 10$?

(A) 58.379 (B) 60.455 (C) 68.702 (D) 73.576



40. The figure above shows a rectangle that has its base on the x-axis and its other two vertices on the curve $y = \cos x$.What is the largest possible area of such a rectangle?

(A) 1.074 (B) 1.122 (C) 1.384 (D) 1.678

41. A particle travels along a straight line with a constant acceleration of 2 ft/sec^2. If the velocity of the particle is 5 ft/sec at time $t = 3$ seconds, how far does the particle travel during the time interval when the velocity increases from 5 ft/sec to 15 ft/sec?

(A) 28 ft (B) 36 ft (C) 42 ft (D) 50 ft

---

42. Let R be the region enclosed by the graph of $y = \frac{4}{x}$ and the line $y = 5 - x$. The volume of the solid obtained by revolving R about the y-axis is given by

(A) $\pi\int_1^4 (5 - y - \frac{4}{y})^2 dy$

(B) $\pi\int_1^4 \left[(\frac{4}{y})^2 - (5 - y)^2\right] dy$

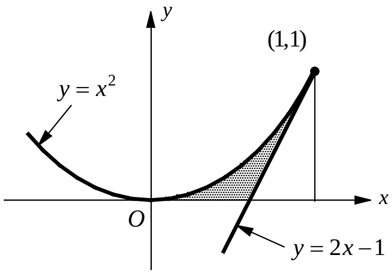
(C) $\pi\int_1^4 \left[(5 - y)^2 - (\frac{4}{y})^2\right] dy$

(D) $2\pi\int_1^4 \left[(5 - x - \frac{4}{x})^2\right] dx$

---

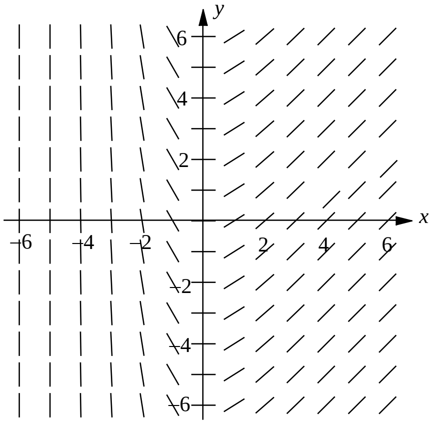
43. Oil is being pumped from an oil well at a rate proportional to the amount of oil left in the well; that is $\frac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t measured in years. There were 2,000,000 gallons of oil in the well at time $t = 0$, and 1,200,000 gallons remaining at time $t = 5$. To the nearest thousand, how many gallons of oil will be left in the well at time $t = 10$?

(A) 570,000 (B) 720,000 (C) 840,000 (D) 920,000



44. The figure above shows a shaded region bounded by the *x*-axis and the graphs of $y = x^2$ and $y = 2x - 1$. If the shaded region is rotated about the *x*-axis, what is the volume of the solid generated?

(A) $\frac{\pi}{30}$ (B) $\frac{\pi}{24}$ (C) $\frac{\pi}{12}$ (D) $\frac{\pi}{8}$



45. The slope field for a certain differential equation is shown above. Which of the following could be a <u>specific solution</u> to that differential equation?

(A) $y = 2e^{-x}$ (B) $y = x + e^{x}$ (C) $y = x + e^{-x}$ (D) $y = x - e^{x}$

CALCULUS AB
SECTION II, Part A
Time — 30 minutes
Number of problems — 2

A graphing calculator is required for these problems.

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1. Let f be the function given by $f(x) = x^3 + 2$, and let g be the function given by $g(x) = mx$, where m is the nonzero constant such that the graph of g is tangent to the graph of f.

   (a) Find the x-coordinate of the point of tangency and the value of m.

   (b) Let R be the region enclosed by the graphs of f and g. Find the area of R.

   (c) Find the volume of the solid generated when R is rotated about the line $y = 3$.

2. The rate at which people enter a movie theater on a given day is modeled by the function S defined by $S(t) = 80 - 12\cos(\frac{t}{5})$. The rate at which people leave the same movie theater is modeled by the function R defined by $R(t) = 12e^{t/10} + 20$. Both $S(t)$ and $R(t)$ are measured in people per hour and these functions are valid for $10 \le t \le 22$. At time $t = 10$, there are no people in the movie theater.

   (a) To the nearest whole number, how many people have entered the movie theater by 8:00 PM ($t = 20$)?

   (b) To the nearest whole number, how many people are in the movie theater at time $t = 20$?

   (c) Let $P(t) = \int_{10}^{t} \left[S(t) - R(t)\right]\, dt$ for $10 \le t \le 22$. Find the value of $P'(20)$ and explain the meaning of $P'(20)$.

   (d) At what time t, for $10 \le t \le 22$, is the number of people in the movie theater a maximum?

CALCULUS AB
SECTION II, Part B
Time — 60 minutes
Number of problems — 4

No calculator is allowed for these problems.

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t (sec)	0	10	20	30	40	50	60	70	80	90
$v(t)$ (ft/sec)	21	23	25	15	6	0	−12	−10	−8	−12

3. A car is traveling on a straight road. The car's velocity v, measured in feet per second, is continuous and differentiable. The table above shows selected values of the velocity function during the time interval $0 \le t \le 90$ seconds.

(a) Find the average acceleration of the car over the time interval $0 \le t \le 90$.

(b) Using correct units explain the meaning of $\int_{40}^{70} |v(t)|\, dt$. Use a trapezoidal approximation with three subintervals of equal length to approximate $\int_{40}^{70} |v(t)|\, dt$.

(c) For $0 < t < 90$, must there be a time t when $v(t) = 10$? Justify your answer.

(d) For $0 < t < 90$, must there be a time t when $a(t) = 0$? Justify your answer.

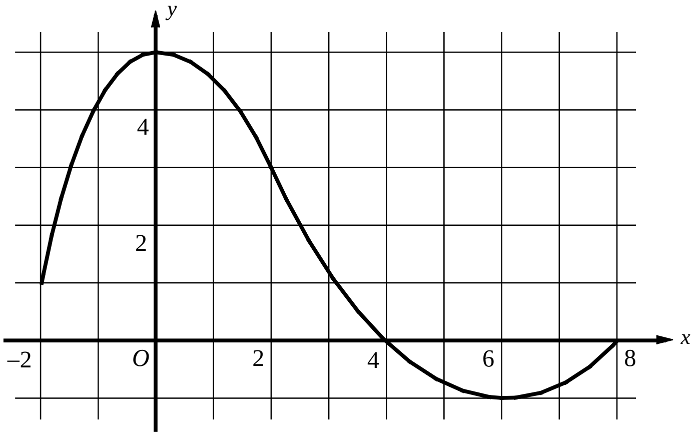
---

4. Consider the curve given by $x^2 + xy + y^2 = 12$.

(a) Find $\dfrac{dy}{dx}$.

(b) Find the two points where the curve crosses the x-axis, and write an equation for the tangent line at each of these two points.

(c) Find the x-coordinate of each point on the curve where the tangent line is horizontal.



Graph of f

5. The graph of a differentiable function f on the closed interval $[-2,8]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 0$ and $x = 6$. Let $h(x) = -3 + \int_0^x f(t)\,dt$ for $-2 \le x \le 8$.

   (a) Find $h(0)$, $h'(0)$, and $h''(0)$.

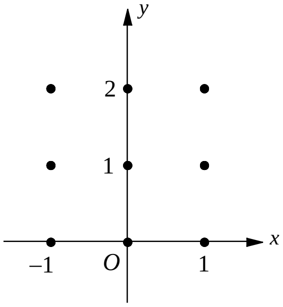
   (b) On what interval is h decreasing? Justify your answer.

   (c) On what intervals does the graph of h concave up? Justify your answer.

   (d) Find a trapezoidal approximation of $\int_{-2}^{8} f(t)\,dt$ using five subintervals of length $\Delta t = 2$.

6. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{-4x^3y^2}{3}$.

(a) On the axis provided, sketch a slope field for the given differential equation at the nine points indicated.



(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(-1) = \dfrac{3}{2}$. Write an equation for the line tangent to the graph of f at $(-1, \dfrac{3}{2})$ and use it to approximate $f(-1.1)$.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $y(-1) = \dfrac{3}{2}$.

AP Calculus BC
Practice Test 1

Answer Sheet

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CALCULUS BC
SECTION I, Part A
Time — 60 minutes
Number of questions — 30

No calculator is allowed for problems on this part of the exam.

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the choices given, decide which is the best answer choice and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

1. If $\lim\limits_{x\to 1} f(x) = 2$ and $\lim\limits_{x\to 1} g(x) = -1$, then $\lim\limits_{x\to 1} \dfrac{x - f(x)}{\left[g(x)\right]^2 + 1} =$

(A) $-\dfrac{1}{5}$ (B) $-\dfrac{1}{2}$ (C) $\dfrac{1}{2}$ (D) nonexistent

---

2. If $f(x) = 2^{\tan x}$, then $f'(\dfrac{\pi}{4}) =$

(A) $\ln 4$ (B) $\ln 8$ (C) $\ln 16$ (D) $\ln 32$

3. If $f(x)=\begin{cases} ax-3, & \text{if } x \le 2 \\ x^2+a, & \text{if } x > 2 \end{cases}$ is continuous on $(-\infty,\infty)$, what is the value of a?

(A) 7 (B) 5 (C) 3 (D) 1

---

4. If four equal subdivisions are used for the internal $[1,2]$, what is the trapezoidal approximation of $\int_1^2 e^{1/x}\,dx$?

(A) $\frac{1}{4}(e+e^{1/1.25}+e^{1/1.5}+e^{1/1.75}+e^{1/2})$

(B) $\frac{1}{4}(e+2e^{1/1.25}+2e^{1/1.5}+2e^{1/1.75}+e^{1/2})$

(C) $\frac{1}{8}(e+2e^{1/1.25}+2e^{1/1.5}+2e^{1/1.75}+e^{1/2})$

(D) $\frac{1}{8}(e+e^{1/1.25}+e^{1/1.5}+e^{1/1.75}+e^{1/2})$

---

5. $\frac{d}{dx}\left[\frac{1}{3}\sec^3 x-\sec x+3\right]$

(A) $\tan^4 x$

(B) $\sec^2 x+\tan^2 x$

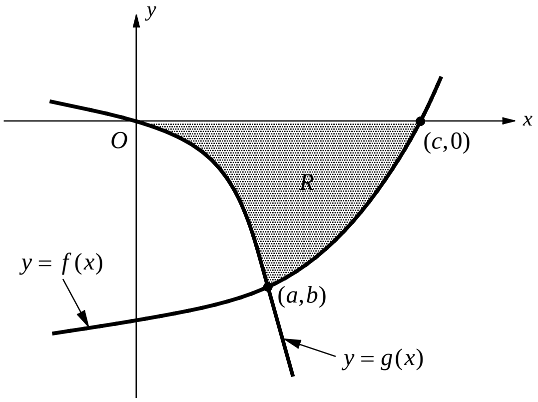
(C) $\sec^2 x-\sec x\tan x$

(D) $\tan^3 x\sec x$

6. Which of the following sequences converge?

I. $\left\{\dfrac{4n}{3n+2}\right\}$ II. $\left\{\dfrac{2^n-9}{e^n}\right\}$ III. $\left\{n\sin(\dfrac{1}{n})\right\}$

(A) I only (B) I and II only (C) II and III only (D) I, II, and III



7. The curves $y = f(x)$ and $y = g(x)$ shown in the figure above intersect at point (a,b). The volume of the solid obtained by revolving R about the x-axis is given by

(A) $\pi\int_0^c [g(x)]^2\,dx - \pi\int_0^c [f(x)]^2\,dx$

(B) $\pi\int_0^a [f(x)]^2\,dx - \pi\int_a^c [g(x)]^2\,dx$

(C) $\pi\int_0^c [f(x)-g(x)]^2\,dx$

(D) $\pi\int_0^a [g(x)]^2\,dx + \pi\int_a^c [f(x)]^2\,dx$

8. If $f'(0) = -1$ and $f(0) = 1$ then $\lim\limits_{h \to 0} \dfrac{f(h) - 1}{h} =$

(A) -1 (B) 0 (C) 1 (D) $-f(x)$

---

9. The length of a curve from $x = a$ to $x = b$ is given by $\int_a^b \sqrt{1 + \sin^2(2x)}\ dx$. Which of the following could be the equation for this curve?

(A) $y = \sin(2x)$ (B) $y = \cos(2x)$ (C) $y = -\dfrac{1}{2}\cos(2x)$ (D) $y = \dfrac{1}{2}\sin(2x)$

---

10. If n is a positive integer, then $\lim\limits_{n \to \infty} \dfrac{3}{n}\left[e^{1+\frac{3}{n}} + e^{1+\frac{6}{n}} + \cdots + e^{1+\frac{3n}{n}} \right]$ can be expressed as

(A) $\int_0^3 e^x\ dx$ (B) $\int_0^3 e^{1+x}\ dx$ (C) $\int_1^4 e^{x/3}\ dx$ (D) $\int_1^4 e^x\ dx$

---

11. $\sum\limits_{n=1}^{\infty} e^{-n+1} \cdot 2^n =$

(A) $\dfrac{e}{2e-1}$ (B) $\dfrac{2e}{e-2}$ (C) $\dfrac{e+2}{e-2}$ (D) $\dfrac{e+2}{2e}$

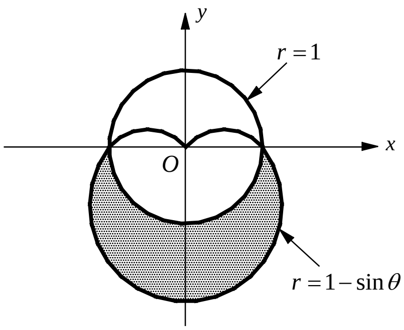
12. $\int \dfrac{1}{x^2+2x+2}\,dx =$

(A) $\arctan(x+1)+C$

(B) $\operatorname{arccot}(x+1)+C$

(C) $-\dfrac{1}{2}(x^2+2x+2)^{-2}+C$

(D) $\ln(x^2+2x+2)+C$



13. Which of the following gives the area of the region inside the polar curve $r = 1 - \sin\theta$ and outside the polar curve $r = 1$, as shown in the figure above?

(A) $\dfrac{1}{2}\int_{\pi}^{2\pi}\left[(1-\sin\theta)^2-1\right]d\theta$

(B) $\dfrac{1}{2}\int_{0}^{2\pi}\left[(1-\sin\theta)^2-1\right]d\theta$

(C) $\dfrac{1}{2}\int_{\pi/2}^{\pi}(1-\sin\theta)^2\,d\theta$

(D) $\int_{0}^{\pi}\left[(1-\sin\theta)^2-1\right]d\theta$

14. A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{3}\left(2 - \frac{P}{60}\right)$, where the initial population $P(0) = 360$ and t is the time in years. What is $\lim_{t \to \infty} P(t)$?

(A) 30 (B) 60 (C) 120 (D) 240

---

15. If $x = e^t$ and $y = (t+1)^2$, then $\frac{d^2y}{dx^2}$ at $t = 1$ is

(A) $\frac{-2}{e}$ (B) $\frac{-2}{e^2}$ (C) $\frac{2}{e}$ (D) $\frac{2}{e^2}$

---

16. A particle moves on the curve $y = \ln(\sqrt{x})$ so that the x-component has velocity $x'(t) = e^t + 1$ for $t > 0$. At time $t = 0$, the particle is at the point $(2, \frac{1}{2}\ln 2)$. At time $t = 1$, the particle is at the point

(A) $\left(e^2, 1\right)$

(B) $\left(e^4, 2\right)$

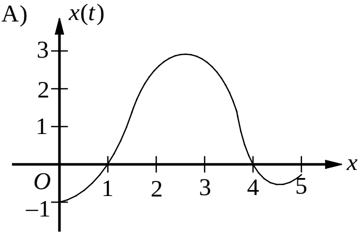
(C) $\left((e+1), \frac{1}{2}\ln(e+1)\right)$

(D) $\left((e+2), \frac{1}{2}\ln(e+2)\right)$

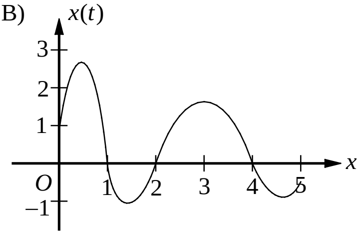
t	0	1	2	3	4	5
$v(t)$	2	0	-2	$-\frac{1}{2}$	0	$\frac{1}{2}$

17. The table above gives selected values of the velocity, $v(t)$, of a particle moving along the x-axis. At time $t = 0$, the particle is at the point $(1, 0)$. Which of the following could be the graph of the position $x(t)$, of the particle for $0 \le t \le 5$?

(A)



(B)



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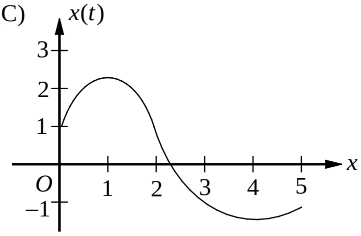

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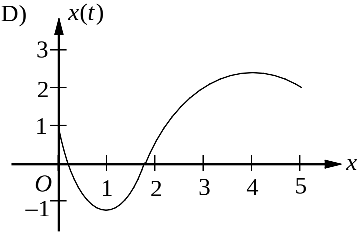

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18. An object moves along a curve in the xy-plane so that its position at any time $t \ge 0$ is given by $(t^2 + 1,\ te^{t/2})$. What is the speed of the object at time $t = 2$?

(A) 3.946 (B) 4.822 (C) 6.749 (D) 8.615

19. What are all values of p for which $\int_1^{\infty} \frac{1}{\sqrt[p]{x}}\, dx$ converges?

(A) $p<1$ (B) $p>1$ (C) $p<0$ (D) $p<-1$

---

20. $\int \frac{\ln x}{x^2}\, dx =$

(A) $x\ln x-\frac{1}{x^3}+C$

(B) $x\ln x+\frac{1}{x}+C$

(C) $-x\ln x-\frac{1}{x}+C$

(D) $-\frac{\ln x}{x}-\frac{1}{x}+C$

---

21. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n-2}{n(n+7)}$ II. $\sum_{n=1}^{\infty} \frac{3^n+4^n}{5^n}$ III. $\sum_{n=1}^{\infty} ne^{-n}$

(A) I only (B) II only (C) I and II only (D) II and III only

22. $\frac{d}{dx}(\arcsin e^{x/2}) =$

(A) $\frac{e^{x/2}}{\sqrt{1-e^{x/2}}}$ (B) $-\frac{e^{x/2}}{\sqrt{1+e^{x}}}$ (C) $\frac{e^{x/2}}{2\sqrt{1-e^{x}}}$ (D) $-\frac{e^{x/2}}{2\sqrt{1-e^{x}}}$

---

23. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n4^n}$ converges?

(A) $-\frac{3}{4} < x < \frac{5}{4}$ (B) $-\frac{3}{4} \le x < \frac{5}{4}$ (C) $-\frac{3}{4} < x \le \frac{5}{4}$ (D) $-\frac{3}{4} \le x \le \frac{5}{4}$

---

24. Let f be a function that is differentiable on the open interval $(-1,8)$. If $f(-1)=7$, $f(5)=7$, and $f(8)=-2$. Which of the following must be true?

I. There exists a number k in the interval $(-1,5)$, such that $f(k)=3$.

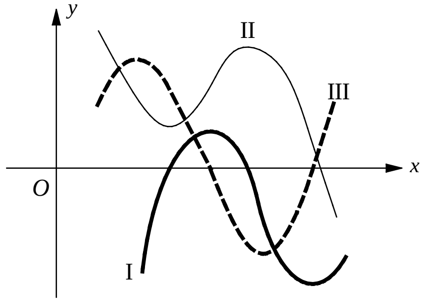
II. f has at least one zero.

III. The graph of f has at least one horizontal tangent.

(A) None
(B) II only
(C) I and II only
(D) II and III only

25. If $\dfrac{dy}{dx} = y^2 \sec^2 x$ and $y(0) = \dfrac{1}{2}$, then $y =$

(A) $\dfrac{1}{2\cos x}$ (B) $\dfrac{1}{1+\cos x}$ (C) $\dfrac{1}{2-\sin x}$ (D) $\dfrac{1}{2-\tan x}$



26. Three graphs labeled I, II, and III are shown above. They are the graphs of f, f', and f''. Which of the following correctly identifies each of the three graphs?

	f	f'	f''
(A)	I	II	III
(B)	II	I	III
(C)	III	I	II
(D)	I	III	II

---

27. $\displaystyle\int_0^\infty xe^{-x^2}\,dx =$

(A) $-\dfrac{1}{2}$ (B) 0 (C) $\dfrac{1}{2}$ (D) ∞

x	$f(x)$	$f'(x)$	$f''(x)$
1	−2	−3	4
2	1	2	−1

28. The table above gives values of f, f', and f'' at selected values of x. If f'' is continuous everywhere, then $\int_1^2 f''(t)\,dt =$

(A) 5 (B) 3 (C) −3 (D) −5

---

29. The position of a particle moving along a line is given by $s(t) = t^3 - 15t^2 + 14$ for $t \geq 0$. For what values of t is the speed of the particle increasing?

(A) $t > 10$ only

(B) $5 < t < 10$ only

(C) $3 < t < 5$ and $t > 10$

(D) $0 < t < 5$ and $t > 10$

---

30. A series expansion of $\dfrac{1-\cos\sqrt{x}}{x}$ is

(A) $\dfrac{1}{3!} - \dfrac{x}{5!} + \dfrac{x^2}{7!} - \cdots + \dfrac{(-1)^{n-1}x^{n-1}}{(2n+1)!} + \cdots$

(B) $\dfrac{1}{2!} - \dfrac{x}{4!} + \dfrac{x^2}{6!} - \cdots + \dfrac{(-1)^{n-1}x^{n-1}}{(2n)!} + \cdots$

(C) $1 - \dfrac{x}{2!} - \dfrac{x^2}{4!} - \cdots + \dfrac{(-1)^{n-1}x^{n-1}}{(n-1)!} + \cdots$

(D) $\dfrac{x}{2!} - \dfrac{x^2}{4!} + \dfrac{x^3}{6!} - \cdots + \dfrac{(-1)^{n-1}x^{n}}{(2n)!} + \cdots$

CALCULUS BC
SECTION I, Part B
Time — 45 minutes
Number of questions — 15

A graphing calculator is required for some problems on this part of the exam.

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the choices given, decide which is the best answer choice and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

(1) The exact numerical value of the correct answer does not always appear among the choices given. If this occurs, select the number that best approximates the exact numerical value from the choices given.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(3) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

31. The rate of consumption of a certain commodity, in thousand units per month, is given by $C(x) = 12e^{0.112x}$, where x represents the number of months. What is the average rate of consumption of the commodity, in thousand units, over the first 6 month period?

(A) 1.085 (B) 1.384 (C) 1.506 (D) 1.916

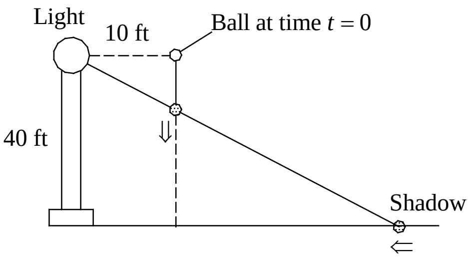
---

32. A particle moving in the *xy*-plane has velocity vector given by $v(t) = \left\langle \frac{1}{\sqrt{1-t^2}}, \frac{-t}{1-t^2} \right\rangle$ for time $t \geq 0$. What is the magnitude of the displacement of the particle between time $t = 0$ to $t = 0.8$?

(A) 0.877 (B) 1.058 (C) 1.099 (D) 1.206

33. Let f be the function given by $f(x) = x^2 + 3x - 1$. If the tangent line to the graph of f at $x = 1$ is used to find an approximate value of f, which of the following is the greatest value of x for which the error resulting from this tangent line approximation is less than 0.3?

(A) 1.3 (B) 1.4 (C) 1.5 (D) 1.6



34. A light shines from the top of a pole 40 feet high. A ball is dropped from the same height from a point 10 feet away from the light, as shown in the figure above. If the position of the ball at time t is given by $y(t) = 40 - 16t^2$, how fast is the shadow moving one second after the ball is released?

(A) −16 ft /sec (B) −32 ft /sec (C) −40 ft /sec (D) −50 ft /sec

---

35. The base of the solid is an elliptical region enclosed by the graph of $\frac{x^2}{4} + y^2 = 1$. If cross sections of the solid perpendicular to the y-axis are isosceles right triangles with the hypotenuse in the base, what is the volume of the solid?

(A) $\frac{16}{3}$ (B) $\frac{20}{3}$ (C) 8 (D) $\frac{26}{3}$

36. If $f(x)=\dfrac{3x^2}{x^4+5}$ and g is an antiderivative of f such that $g(4)=6$, then $g(1)=$

(A) 1.655 (B) 2.704 (C) 3.862 (D) 4.704

---

37. A curve is defined by the polar equation $r=1+3\sin\theta$. When $\theta=\dfrac{5\pi}{6}$, which of the following statements is true of the polar curve?

(A) The curve is closest to the origin.
(B) The curve is getting farther from the origin.
(C) The curve is getting closer to the origin.
(D) The curve has a horizontal tangent.

---

38. The graph of the function represented by the Maclaurin series $x-\dfrac{x^3}{2!}+\dfrac{x^5}{4!}-\dfrac{x^7}{6!}+\cdots+\dfrac{(-1)^n x^{2n+1}}{(2n)!}+\cdots$ intersects the graph of $\ln x$ at $x=$

(A) 0.735 (B) 0.916 (C) 1.347 (D) 1.466

39. The equation of a polar curve is given by $r = 3 + \sin 5\theta$. What is the angle θ that corresponds to the point on the curve with x-coordinate 2?

(A) 0.516 (B) 0.628 (C) 0.705 (D) 0.844

---

40. What is the length of the curve $y = x \ln x$ from $x = 1$ to $x = 2$?

(A) 1.548 (B) 1.713 (C) 1.952 (D) 2.043

---

41. Let h be the function given by $h(x) = \int_0^x 4(x-2)\cos(\frac{x}{2})\, dx$. Which of the following statements about h must be true?

I. h is increasing on $(0, 2)$.
II. $h'(3) > 0$.
III. $h(3) < 0$.

(A) I only (B) II only (C) III only (D) II and III only

42. The number of bacteria in a colony increases at a rate proportional to the number present. If the colony starts with one bacterium and doubles every half-hour, how many bacteria will the colony contain at the end of 12 hours?

(A) 4096 (B) 65,536 (C) 1.049×10^6 (D) 1.678×10^7

---

43. The velocity of a particle moving along the y-axis is given by $v(t)=t^3-5t^2+2t+8$ for $0\le t\le10$. Which of the following expressions gives the change in position of the particle during the time the particle is moving downward?

(A) $\int_2^4 (t^3-5t^2+2t+8)\,dt$

(B) $\int_0^4 (t^3-5t^2+2t+8)\,dt$

(C) $\int_2^4 (3t^2-10t+2)\,dt$

(D) $\int_0^{3.23} (t^3-5t^2+2t+8)\,dt$

44. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = 1 - \frac{xy}{2}$ with the initial condition $f(0) = 1$. What is the approximation for $f(1)$ if Euler's method is used, starting at $x = 0$ with a step size of 0.5?

(A) 1.5 (B) 1.65 (C) 1.762 (D) 1.813

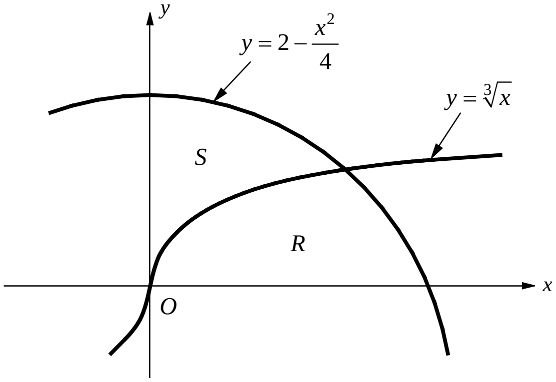
---

45. Let $P(x) = x - 2x^2 + 2x^3 - \frac{4}{3}x^4$ be the fourth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f^{(4)}(0)$?

(A) -32 (B) -16 (C) $-\frac{32}{3}$ (D) $-\frac{16}{3}$

CALCULUS BC
SECTION II, Part A
Time — 30 minutes
Number of problems — 2

A graphing calculator is required for these problems.



1. Let R and S be the region in the first quadrant as shown in the figure. The region R is bounded by the x- axis and the graph of $y = 2 - \frac{x^2}{4}$ and $y = \sqrt[3]{x}$. The region S is bounded by the y-axis and the graph of $y = 2 - \frac{x^2}{4}$ and $y = \sqrt[3]{x}$.

   (a) Find the area of R .

   (b) Find the area of S .

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

---

2. A particle moves along the x-axis with its velocity given by $v(t) = t\ln(t^2)$ for $t \geq 0$. At time $t = 0$, the position of the particle is $x(0) = -1$.

(a) Find the acceleration of the particle at time $t = 0.5$.

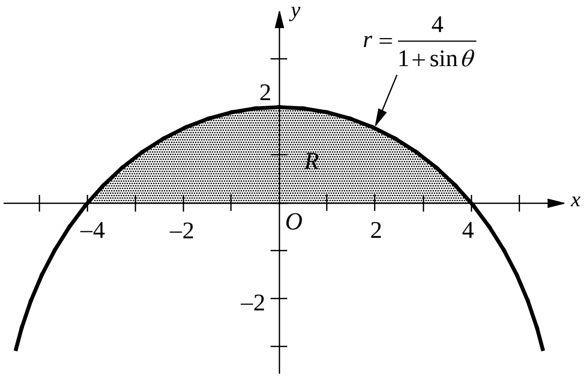
(b) Is the speed of the particle increasing or decreasing at time $t = 0.5$? Give a reason for your answer.

(c) Find the time, $t \geq 0$, at which the particle is farthest to the left. What is the distance between the particle and the origin when it is farthest to the left?

(d) Find the position of the particle at time $t = 0.5$. Is the particle moving toward the origin or away from the origin at time $t = 0.5$? Justify your answer.

CALCULUS BC
SECTION II, Part B
Time — 60 minutes
Number of problems — 4

No calculator is allowed for these problems.



3. The figure above shows the graph of the polar curve $r = \frac{4}{1+\sin\theta}$. Let R be the shaded region bounded by the curve and the x-axis.

(a) Find $\frac{dr}{d\theta}$ at $\theta = \frac{\pi}{6}$. What does the value of $\frac{dr}{d\theta}$ at $\theta = \frac{\pi}{6}$ say about the curve?

(b) Set up, but do not evaluate, an integral expression that represents the area of the polar region R, using the equation $r = \frac{4}{1+\sin\theta}$.

(c) Show that $r = \frac{4}{1+\sin\theta}$ can be written as the equation $y = -\frac{1}{8}x^2 + 2$.

(d) Use the equation $y = -\frac{1}{8}x^2 + 2$ to find the area of the region R.

t (hours)	0	3	6	9	12	15	18	21	24
$P(t)$ (gallons / hour)	700	620	760	1040	1200	1120	960	920	680

4. The rate of fuel consumption in a factory, in gallons per hour, recorded during a 24-hour period is given by a twice differentiable function P of time t. The table of selected values of $P(t)$, for the time interval $0 \le t \le 24$, is shown above.

(a) Use the data from the table to find an approximation for $P'(7.5)$. Indicate the units of measure.

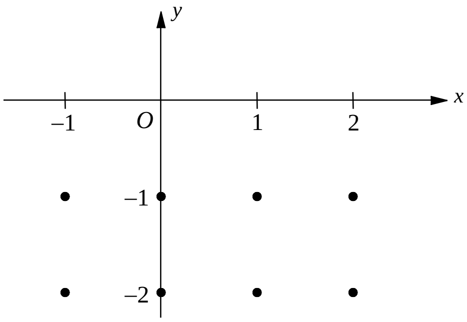
(b) The rate of fuel consumption is increasing the fastest at time $t = 7.5$ minutes. What is the value of $P''(7.5)$?

(c) Approximate the average value of the rate of fuel consumption on the interval $12 \le t \le 24$ using a left Riemann sum with the four subintervals indicated by the data in the table above.

(d) For $12 \le t \le 24$ hours, $P(t)$ is strictly a decreasing function of time t. Is the data in the table consistent with the assertion that $P''(t) < 0$ for every x in the interval $12 < t < 24$? Explain your answer.

5. Consider the differential equation $\frac{dy}{dx} = -x - y$.

(a) On the axis provided, sketch a slope field for the given differential equation at the eight points indicated, and sketch the solution curve that passes through the point $(0,-1)$.



(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = -1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

(c) The solution curve that passes through the point $(0,-1)$ has a local maximum at $x = \ln 2$. What is the y-coordinate of this local maximum?

(d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (b) is less than or greater than $f(-0.4)$. Explain your reasoning.

---

6. The Maclaurin series for $\tan^{-1} x$ is $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$.

The continuous function f is defined by $f(x) = \frac{\tan^{-1} x}{x}$ for $x \neq 0$ and $f(0) = 1$.

(a) Write the first three nonzero terms and the general term for the Maclaurin series of $f'(x)$.

(b) Use the result from part (a) to find the sum of the infinite series

$$-\frac{2}{3}\cdot\frac{1}{\sqrt{3}} + \frac{4}{5}\cdot\frac{1}{3\sqrt{3}} + \frac{6}{7}\cdot\frac{1}{3^2\sqrt{3}} + \cdots + \frac{(-1)^n(2n)}{2n+1}\cdot\frac{1}{3^{n-1}\sqrt{3}} + \cdots.$$

(c) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Maclaurin series representing $g(x)$.

(d) Show that $1 - \frac{1}{3^2} + \frac{1}{5^2}$ approximates $g(1)$ with an error less than $\frac{1}{40}$.

AP Calculus BC
Practice Test 2

Answer Sheet

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CALCULUS BC
SECTION I, Part A
Time — 60 minutes
Number of questions — 30

No calculator is allowed for problems on this part of the exam.

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the choices given, decide which is the best answer choice and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

1. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{1+x-e^x} =$

(A) -1 (B) 0 (C) 1 (D) nonexistent

---

2. If $f(x)=e^{-x}$ and $g(x)=f\left(f(x)\right)$, then $g'(0)=$

(A) $\dfrac{2}{e}$ (B) $-\dfrac{1}{e}$ (C) $\dfrac{1}{e}$ (D) $\dfrac{1}{e^2}$

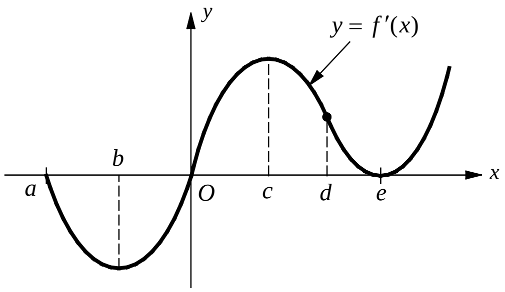
3. If $3\sin x\cos y = 1$, then $\dfrac{dy}{dx} =$

(A) $-\cot x\cot y$

(B) $\cot x\cot y$

(C) $\tan x\tan y$

(D) $-\tan x\tan y$



4. The graph of f', the derivative of the function f, is shown in the figure above. For what values of x does the graph of f concave up ?

(A) $b < x < d$

(B) $a < x < 0$ or $x > d$

(C) $b < x < c$ or $x > e$

(D) $a < x < b$ or $c < x < e$

---

5. If $h(x) = \arctan x + \arctan(\frac{1}{x})$, then $h'(x) =$

(A) $\dfrac{2}{1+x^2}$ (B) $\dfrac{2x}{1+x^2}$ (C) 0 (D) 1

6. Let $f(x)=\int_{-1}^{2x^4-x} e^t\ dt$. At what value of x is $f(x)$ a minimum?

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt[3]{2}}$ (D) $\frac{1}{\sqrt{2}}$

---

7. What is $\lim_{h\to 0}\frac{\tan(\frac{\pi}{4}+h)-1}{h}$?

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\sqrt{2}$ (D) 2

---

8. Given $f(x)=\begin{cases} x^3 & \text{for } x<0, \\ \sin 2x & \text{for } x\geq 0, \end{cases}$, then $\int_{-1}^{\pi/2} f(x)\ dx =$

(A) $\frac{1}{4}$ (B) $\frac{3}{8}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

---

9. Which of the following integrals gives the length of the graph of $y=1+4x^{3/2}$ between $x=0$ to $x=3$?

(A) $\int_0^3 \sqrt{1+36x}\ dx$

(B) $\int_0^3 \sqrt{1+36x^2}\ dx$

(C) $\int_0^3 \sqrt{1+36x^3}\ dx$

(D) $\int_0^3 \sqrt{1+48x}\ dx$

10. If the function f given by $f(x)=\sqrt{x}$ has an average value of 2 on the closed interval $[0,k]$, then $k=$

(A) 3 (B) 4 (C) 6 (D) 9

---

11. If the substitution $x=3\sec\theta$ is made for $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$, where $0\le\theta<\frac{\pi}{2}$, then $\int \frac{\sqrt{x^2-9}}{x^3}\,dx=$

(A) $\frac{1}{3}\int \sin^2\theta\, d\theta$

(B) $-\frac{1}{3}\int \cos^2\theta\, d\theta$

(C) $\frac{1}{3}\int \tan^2\theta\, d\theta$

(D) $-\frac{1}{3}\int \cot^2\theta\, d\theta$

---

x	–3	–1	2	4	5
$f(x)$	2	4	1	–3	3
$g(x)$	–1	–2	0	7	4

12. The functions f and g are differentiable for all real numbers. The table above gives values of f and g for selected values of x. If $\int_{-3}^{5} f(x)g'(x)\,dx=9$, then $\int_{-3}^{5} f'(x)g(x)\,dx=$

(A) –2 (B) 5 (C) 12 (D) 17

13. A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{90}\right)$, where the initial population $P(0) = 110$ and t is the time in years. For what values of P is the population growing the fastest?

(A) 30 (B) 45 (C) 55 (D) 90

---

14. If $\frac{dy}{dx} = \frac{e^x}{3y^2}$ and $y(0) = 2$, then $y =$

(A) $\sqrt[3]{e^x + 7}$ (B) $\sqrt{e^x + 3}$ (C) $2e^x$ (D) $\sqrt[3]{3e^x + 5}$

---

15. Which of the following series can be used to determine the series $\sum_{n=1}^{\infty} \frac{n^2 + 2n}{\sqrt{9 + n^7}}$ converges, using limit comparison test?

(A) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ (B) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (C) $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ (D) $\sum_{n=1}^{\infty} \frac{1}{n^5}$

---

16. $\int \cos x \ \ln(\sin x) \, dx =$

(A) $-\cos x \ \ln(\sin x) - \cos x + C$

(B) $-\cos x \ \ln(\sin x) + \sin x + C$

(C) $\sin x \ \ln(\sin x) - \sin x + C$

(D) $\sin x \ \ln(\sin x) + \sin x + C$

17. $\lim\limits_{h\to 0} \dfrac{\int_{1}^{1+h} (\tan^{-1} x)\, dx}{h} =$

(A) -1 (B) 0 (C) 1 (D) $\dfrac{\pi}{4}$

---

18. For what value(s) of t does the curve defined by the parametric equations $x = t^3 - 3t^2 + 2$ and $y = t^4 - 7t$ have a vertical tangent?

(A) 0 and 2 only (B) 0 and 1 only (C) 1 and 2 only (D) 2 only

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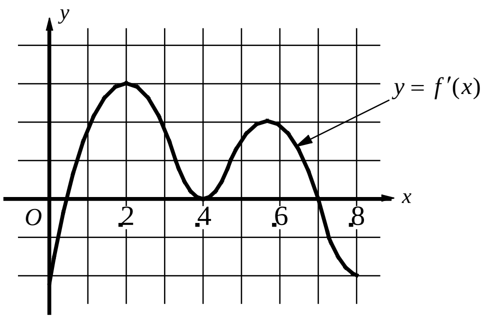


19. The graph of f', the derivative of f, is shown in the figure above. Which of the following statements is <u>not</u> true about f ?

(A) f has two relative maxima for $0 \le x \le 8$.

(B) f is decreasing for $x \ge 7$.

(C) f is increasing for $2 \le x \le 4$.

(D) f concaves up for $0 \le x \le 2$.

20. Which of the following series converge?

I. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\cdots$

II. $\frac{2}{5}-\frac{3}{6}+\frac{4}{7}-\frac{5}{8}+\cdots$

III. $\sum_{n=1}^{\infty}\frac{n!}{1\cdot 3\cdot 5\cdot\ \cdot\ \cdot(2n+1)}$

(A) I only (B) II only (C) III only (D) I and III only

---

21. A particle moves along the x-axis so that at any time $t \geq 0$ its velocity is given by $v(t)=e^{\sin t}+\sin(e^{t})$. Which of the following statements must be true?

I. The particle is moving to the right at time $t=1$.

II. The particle's acceleration is negative at time $t=1$.

III. The particle's speed is decreasing at time $t=1$.

(A) I only (B) I and II only (C) II and III only (D) I, II, and III

---

22. Which of the following gives the area of the region inside the polar curve $r=2\sin\theta$ and outside the polar curve $r=\sin\theta$?

(A) $\frac{3}{2}\int_{0}^{\frac{\pi}{2}}\sin^{2}\theta\,d\theta$

(B) $2\int_{0}^{\frac{\pi}{2}}\sin^{2}\theta\,d\theta$

(C) $3\int_{0}^{\frac{\pi}{2}}\sin^{2}\theta\,d\theta$

(D) $4\int_{0}^{\frac{\pi}{2}}\sin^{2}\theta\,d\theta$

23. The fourth-degree Taylor polynomial of xe^{-x} about $x=0$ is

(A) $-1+x-x^2+\frac{x^3}{2}-\frac{x^4}{6}$

(B) $1-x+x^2-\frac{x^3}{2}+\frac{x^4}{6}$

(C) $-x+x^2-\frac{x^3}{2}+\frac{x^4}{6}$

(D) $x-x^2+\frac{x^3}{2}-\frac{x^4}{6}$

---

24. If $\int_1^\infty \frac{\ln x}{x^2}\,dx=1$, then which of the following must be true?

I. $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges.

II. $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ diverges.

III. $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}=1$

(A) I only (B) II only (C) III only (D) I and III only

---

25. If f is continuous on $[0,3]$ and $\int_0^3 f(x)\,dx=5$, then $\int_0^3 f(3-x)\,dx=$

(A) –5 (B) 5 (C) –2 (D) 2

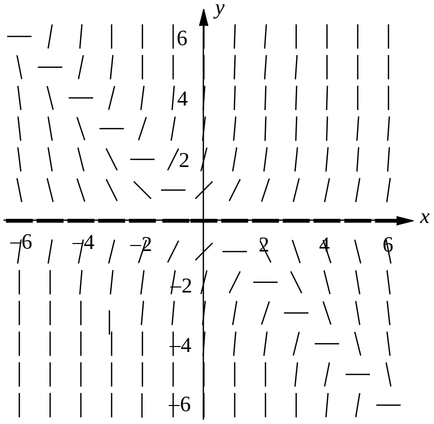
26. The length of the path described by the parametric equations $x = \dfrac{t}{1+t}$, $y = \ln(1+t)$, for $0 \le t \le 1$, is given by

(A) $\int_0^1 \sqrt{\dfrac{t^2}{(1+t)^2} + [\ln(1+t)]^2}\, dt$

(B) $\int_0^1 \sqrt{\dfrac{1}{(1+t)^2} + \dfrac{1}{(1+t)}}\, dt$

(C) $\int_0^1 \sqrt{\dfrac{1}{(1+t)^4} + \dfrac{1}{(1+t)^2}}\, dt$

(D) $\int_0^1 \sqrt{\dfrac{t}{(1+t)^4} + \dfrac{1}{(1+t)}}\, dt$



27. Shown above is a slope field for which of the following differential equations?

(A) $\dfrac{dy}{dx} = y(x+y)$ (B) $\dfrac{dy}{dx} = x(x-y)$ (C) $\dfrac{dy}{dx} = -x-y$ (D) $\dfrac{dy}{dx} = x+y$

28. What are all values of x for which the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 3^n}$ converges?

(A) $-3 < x < 3$ (B) $-3 \le x < 3$ (C) $-3 < x \le 3$ (D) $-3 \le x \le 3$

---

29. Let f be the function defined by $f(x) = x^4 - 2x + 5$. If $g(x) = f^{-1}(x)$ and $g(4) = 1$, what is the value of $g'(4)$?

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$

---

30. A function f has a Maclaurin series given by $1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \cdots + \frac{(-1)^{n-1} x^{4(n-1)}}{(2n-1)!} + \cdots$. Which of the following is an expression for $f(x)$?

(A) $x \sin x$

(B) $x^2 \cos x$

(C) $\frac{\sin(x^2)}{x^2}$

(D) $x^2 e^{-x} - x^3$

CALCULUS BC
SECTION I, Part B
Time — 45 minutes
Number of questions — 15

A graphing calculator is required for some problems on this part of the exam.

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the choices given, decide which is the best answer choice and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

(1) The exact numerical value of the correct answer does not always appear among the choices given. If this occurs, select the number that best approximates the exact numerical value from the choices given.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(3) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

31. If three equal subintervals on $[1,4]$ are used, what is the trapezoidal approximation of $\int_1^4 \ln(x^2)\,dx$?

(A) 3.306 (B) 4.265 (C) 4.969 (D) 5.339

---

32. In some chemical reaction, the rate at which the amount of substance changes with time is proportional to the amount present. If 100 liters of chemical substance reduces to 70 liters during the first 10 hours of the reaction, how many liters of chemical will remain after another 12 hours?

(A) 37.595 (B) 40.406 (C) 45.626 (D) 50.388

33. Two particles start at the origin and move along the x-axis. For $0 \le t \le 12$, their respective position functions are given by $x_1 = -\frac{2}{3}\cos(\frac{3}{2}t)$ and $x_2 = \ln(3t)$. For how many values of t do the particles have the same velocity?

(A) 1 (B) 3 (C) 5 (D) 6

---

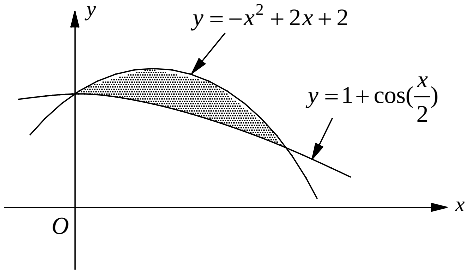
34. A cup noodle heated to a temperature of 220°F is placed in a 72°F room at time $t = 0$ minutes. The temperature of the cup noodle is changing at a rate of $-80e^{-0.6t}$ degrees Fahrenheit per minut38 To the nearest degree, what is the temperature of the cup noodle at $t = 3$ minutes?

(A) 92°F (B) 98°F (C) 109°F (D) 124°F

---

35. The x-coordinate of the point on the curve $y = e^x$ nearest to $(1,1)$ is

(A) 0.365 (B) 0.438 (C) 0.506 (D) 0.584



36. The figure above shows the shaded region enclosed by the graphs of $y = -x^2 + 2x + 2$ and $y = 1 + \cos(\frac{x}{2})$. What is the volume of the solid when the shaded region is revolved about the x-axis?

(A) 16.082 (B) 19.765 (C) 24.445 (D) 28.216

37. The rate of change of the altitude of a hot air balloon is given by $h(t) = 4\sin(e^{x/3}) + 1$ for $0 \le t \le 6$. Which of the following expressions gives the change in altitude of the balloon during the time the altitude is decreasing?

(A) $\int_{0}^{3.694} h'(t)\,dt$

(B) $\int_{1.355}^{4.650} h'(t)\,dt$

(C) $\int_{1.355}^{4.650} h(t)\,dt$

(D) $\int_{3.666}^{5.390} h(t)\,dt$

---

t(sec)	0	1	2	3	4	5	6
$a(t)$ (ft /sec^2)	7	4	5	3	6	8	6

38. The data for the acceleration $a(t)$ of a car from 0 to 6 seconds is given in the table above. If the velocity at $t = 0$ is -7 feet per second, the approximate value of the velocity at $t = 6$, computed using a midpoint Riemann sum with three subintervals of equal length, is

(A) 23 ft /sec (B) 27 ft /sec (C) 32 ft /sec (D) 37 ft /sec

---

39. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} \frac{1+n}{\sqrt[3]{1+n^6}}$ II. $\sum_{n=1}^{\infty} (-1)^n 3^{1/n}$ III. $\sum_{n=1}^{\infty} \frac{(-2)^n n}{3^{n-1}}$

(A) I only (B) II only (C) III only (D) I and II only

40. Which of the following is the approximation for the value of $\cos 1$ obtained by using the sixth-degree Taylor polynomial about $x = 0$ for $\cos x$?

(A) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}$

(B) $1+\frac{1}{2}+\frac{1}{24}+\frac{1}{720}$

(C) $1-\frac{1}{2}+\frac{1}{24}-\frac{1}{720}$

(D) $1-\frac{1}{6}+\frac{1}{120}-\frac{1}{720}$

---

41. Let f be the function defined by $f(x) = e^x - x$. What is the value of k for which the instantaneous rate of change of f at $x = k$ is the same as the average value of f over the interval $-2 \le x \le 3$?

(A) 1.086 (B) 1.502 (C) 2.071 (D) 3.286

---

x	1	1.1	1.2	1.3	1.4	1.5	1.6
$f'(x)$	–1.8	–1.6	–1.5	–1.2	0.8	1.2	1.5

42. The table above gives selected values for the derivative of a function f on the interval $1 \le x \le 1.6$. If $f(1) = -1$ and Euler's method with a step size of 0.3 is used to approximate $f(1.6)$, what is the resulting approximation?

(A) –2.4 (B) –1.9 (C) –.82 (D) 0.91

43. If $f(x) = \sum_{n=1}^{\infty} (\sin^2 x)^n$, then $f(2) =$

(A) 0.508 (B) 1 (C) 2.635 (D) 4.774

---

44. The position of a particle moving in a *xy*-plane is given by the parametric equations $x(t) = t^3 - 3t^2$ and $y(t) = t - \ln t^2$. For what values of t is the particle at rest?

(A) -1 only (B) 0 only (C) 2 only (D) -1 and 2 only

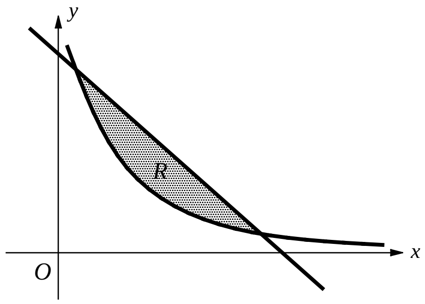
---

45. Let f be the function given by $f(x) = 4\sin(\frac{x}{2})$ for all x . The function g satisfies $g(x) = k\, f''(ax)$ for all x . If the Taylor series for g about $x = 0$ is given by $g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$, what is the value of k ?

(A) $-\frac{1}{4}$ (B) $-\frac{1}{2}$ (C) -1 (D) $\frac{1}{3}$

CALCULUS BC
SECTION II, Part A
Time — 30 minutes
Number of problems — 2

A graphing calculator is required for these problems.



1. Let R be the shaded region bounded by the graph of $y = 2 - \ln x$ and the line $y = 4 - x$, as shown above.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$

   (c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the line $x = -1$.

2. An object moving along a curve in the *xy*-plane is at position $(x(t),\ y(t))$ at time t, where $\frac{dx}{dt} = \sqrt{t^2+2}$ and $\frac{dy}{dt} = e^{\sin t}$ for $0 \le t \le 2\pi$. At time $t = 1$, the object is at position $(1, -2)$.

   (a) Write an equation for the line tangent to the curve at position $(1, -2)$.

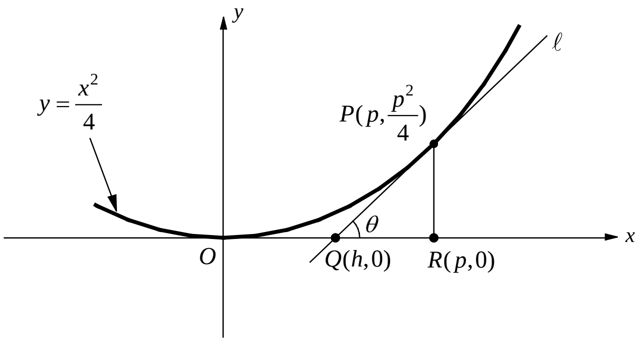
   (b) Find the *y*-coordinate of the position of the object at time $t = \pi$.

   (c) Find the speed of the object at time $t = \pi$.

   (d) Find the total distance traveled by the object over the time interval $0 \le t \le 2\pi$.

CALCULUS BC
SECTION II, Part B
Time — 60 minutes
Number of problems — 4

No calculator is allowed for these problems.



3. In the figure above, line ℓ is tangent to the graph of $y=\frac{x^2}{4}$ at point P, with coordinates $(p,\frac{p^2}{4})$, where $p>0$. Point R has coordinates $(p,0)$ and line ℓ crosses the x-axis at point Q, with coordinates $(h,0)$.

   (a) For all $p>0$, find h in terms of p.

   (b) Suppose p is increasing at a constant rate of 4 units per second. When $p=2$ what is the rate of change of angle θ with respect to time?

   (c) Suppose p is increasing at a constant rate of 4 units per second. When $p=2$, what is the rate of change of the area of ΔPQR with respect to time?

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4. Consider the differential equation $f'(x) = \dfrac{3x\sqrt{f(x)}}{2}$.

   (a) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $y(1) = 4$. Write an equation for the line tangent to the graph of f at $(1, 4)$, and use it to approximate $f(1.1)$.

   (b) Find $f''(1)$ with the initial condition $y(1) = 4$.

   (c) Find the particular solution $y = f(x)$ to the differential equation $\dfrac{dy}{dx} = \dfrac{3x\sqrt{y}}{2}$ with the initial condition $y(1) = 4$.

5. A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{2}\left(1 - \frac{P}{32}\right)$.

(a) If $P(0) = 10$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 40$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 40$, for what value of P is the population growing the fastest?

(c) Find the slope of the graph of P at the point of inflection.

(d) Let f be a function with $f(0) = \frac{1}{2}$ such that all points (x, y) on the graph of f satisfy the differential equation $\frac{dy}{dx} = \frac{y}{2}\left(1 - \frac{x}{32}\right)$. Find $y = f(x)$.

6. The function f has a Taylor series about $x = 3$ that converges to $f(x)$ for all x in the interval of convergence. The n th derivative of f at $x = 3$ is given by $f^{(n)}(3) = \frac{(n+1)!}{2^n}$ for $n \geq 1$, and $f(3) = 1$.

   (a) Write the first four terms and the general term of the Taylor series for f about $x = 3$.

   (b) Find the radius of convergence of the Taylor series for f about $x = 3$. Show the work that leads to your answer.

   (c) Let h be a function satisfying $h(3) = 2$ and $h'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for h about $x = 3$.

   (d) Does the Taylor series for h, as defined in part (c), converge at $x = 1$? Justify your answer.

Chapter 1 Limits and Continuity

1.1 Rate of Change

1. C 2. B 3. (a) -1.3 ft /sec (b) 0.9 ft /sec

1.2 The Limit of a Function and One Sided Limits

1. C 2. D 3. D 4. B 5. 1 6. 2 7. 4 8. DNE 9. 3 10. $\frac{\pi}{4}$
11. 10

1.3 Calculating Limits Using the Limit Laws

1. D 2. C 3. B 4. A 5. D 6. D 7. A 8. 4 9. $\frac{1}{\sqrt{2x+1}}$
10. $5/2$ 11. 2

1.4 Properties of Continuity and Intermediate Value Theorem

1. D 2. C 3. B 4. D 5. $a=\pi$, $b=\frac{3\pi}{4}$ 6. $\frac{1}{2}$ 7. 6

1.5 Limits and Asymptotes

1. B 2. C 3. C 4. A 5. D 6. C 7. (a) $x=2$ (b) $y=0$
8. (a) $x=-2$ (b) $y=0$

Chapter 2 Differentiation

2.1 Definition of Derivatives and the Power Rule

1. A 2. B 3. D 4. D 5. C 6. C 7. B 8. $m=-2/3$, $k=-8/3$

9. $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left[f(x)+x^3h-xh^3-f(h)\right]-f(x)}{h}=\lim_{h\to 0}\frac{x^3h-xh^3-f(h)}{h}$$
$$=\lim_{h\to 0}\frac{h(x^3-xh^2)}{h}-\lim_{h\to 0}\frac{f(h)}{h}=\lim_{h\to 0}\frac{\cancel{h}(x^3-xh^2)}{\cancel{h}}-1=x^3-1$$

10. (a) 1 (b) 2 (c) No (d) $a=1/8$, $b=4$

2.2 The Product and Quotient Rules and Higher Derivatives

1. A 2. C 3. B 4. B 5. C 6. D 7. D 8. -4 9. $1/8$

2.3 The Chain Rule and the Composite Functions

1. D 2. D 3. C 4. A 5. 1 6. 13 7. $\frac{13}{16}$ 8. 20 9. $-\frac{5}{16}$

10. (a) $f(g(x))=2x$, $\frac{d}{dx}\left[f(g(x))\right]=\frac{d}{dx}\left[2x\right] \Rightarrow f'(g(x))\cdot g'(x)=2 \Rightarrow g'(x)=\frac{2}{f'(g(x))}$

(b) $f'(x)=1+\left[f(x)\right]^2 \Rightarrow f'(g(x))=1+\left[f(g(x))\right]^2=1+\left[2x\right]^2=1+4x^2$

Therefore, $g'(x)=\frac{2}{f'(g(x))}=\frac{2}{1+4x^2}$.

2.4 Derivatives of Trigonometric Functions

1. B 2. C 3. D 4. A 5. C 6. B 7. B 8. $-4\sqrt{2}+3$ 9. $a=-1$, $b=\pi$

2.5 Derivatives of Exponential and Logarithmic Functions

1. C 2. C 3. A 4. B 5. D 6. D 7. C 8. B 9. 10 10. x

2.6 Tangent Lines and Normal Lines

1. C 2. C 3. B 4. C 5. D 6. A 7. (a) 30 (b) $y=\frac{1}{2}x+\frac{15}{2}$ (c) 7.5

2.7 Implicit Differentiation

1. A 2. C 3. B 4. D 5. D 6. B 7. C 8. (a) $\frac{dy}{dx}=\frac{3x^2-y}{x-2y}$

(b) At $(1,-1)$ $y+1=\frac{4}{3}(x-1)$, at $(1,2)$ $y-2=-\frac{1}{3}(x-1)$ (c) $x=0.822$ and $x=-0.709$

9. (a) $\frac{dy}{dx}=\frac{y-2x}{2y-x}$ (b) At $(2,-1)$ $y+1=\frac{5}{4}(x-2)$, at $(2,3)$ $y-3=-\frac{1}{4}(x-2)$ (c) $x=\pm\frac{2\sqrt{21}}{3}$

2.8 Derivatives of an Inverse Function

1. C 2. B 3. A 4. D 5. D 6. (a) $y-2=\frac{1}{4}(x+1)$ (b) $h(1)=3$, $h'(1)=-4$ (c) $-\frac{1}{4}$

2.9 Derivatives of Inverse Trigonometric Functions

1. D 2. C 3. A 4. B 5. (a) $x^{\tan^{-1}x}(\frac{\tan^{-1}x}{x}+\frac{\ln x}{1+x^2})$ (b) $y-1=\frac{\pi}{4}(x-1)$

2.10 Approximating a Derivative

1. C 2. (a) $19.2^\circ F/\text{mon}$ (b) $23.5^\circ F/\text{mon}$ (c) $26.472^\circ F/\text{mon}$

Chapter 3 Applications of Differentiation

3.1 Related Rates

1. B 2. D 3. C 4. C 5. A 6. D 7. (a) 1 (b) $\frac{9\sqrt{5}}{5}$ units/sec (c) $-\frac{1}{5}$

8. (a) $-\frac{1}{48\pi}$ ft/min (b) $r=\sqrt{50y-y^2}$ (c) $-\frac{7}{1152\pi}$ ft/min

9. (a) $\frac{3}{2}$ ft/sec (b) -7 ft^2/sec (c) $-\frac{1}{8}$ rad/sec 10. (a) $\frac{dy}{dx}=-\frac{3y}{4y+3x}$ (b) $(0,-\frac{\sqrt{2}}{2})$, $(0,\frac{\sqrt{2}}{2})$ (c) $\frac{3}{2}$

3.2 Position, Velocity, and Acceleration

1. C 2. B 3. D 4. C 5. A 6. B 7. D 8. (a) $v(t)=4(t-2)^2(t-5)$
$a(t)=12(t-2)(t-4)$ (b) $t=4$ (c) $t>5$ (d) $2<t<4$ (e) $2<t<4$ or $t>5$

3.3 Roll's Theorem and the Mean Value Theorem

1. D 2. C 3. C 4. B 5. B 6. (a) Yes. Since $v(50) = -1.2 < -1 < -0.4 = v(45)$, the Intermediate Value Theorem guarantees a t in $(45, 50)$ so that $v(t) = -1$

(b) Since $v(5) = v(20)$, the MVT guarantees a t in $(5, 20)$ so that $a(t) = v'(t) = 0$. The smallest instances that the acceleration of the car could equal zero is 1.

3.4 The First Derivative Test and the Extreme Values of Functions

1. D 2. A 3. B 4. D 5. B 6. A 7. B 8. C 9. C 10.D

11. A 12. D 13. (a) f attains a relative minimum at $x = 1$, because f' changes from negative to positive at $x = 1$. (b) f attains a relative maximum at $x = -2$, because f' changes from positive to negative $x = -2$. (c) The absolute maximum occurs at $x = 7$.

3.5 The Second Derivative Test

1. B 2. D 3. C 4. D 5. A 6. D 7. A 8. C 9. B 10. C

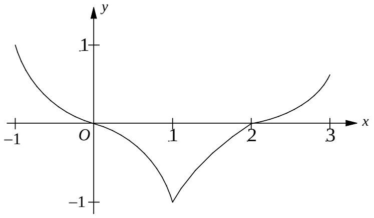
11. B 12. (a) $x = -1,\ 1,$ and 4 (b) $x = 6$ (c) $y - 2 = -2(x + 2)$

13. (a) $y + 1 = 2(x - 1)$ (b) We don't have sufficient information as to whether f'' changes sign at $x = 1$.
(c) $y = 3$ (d) g has a local minimum at $x = 1$.

3.6 Curves of f, f', f'' and Curve Sketching

1. C 2. C 3. B 4. A 5. D 6. C 7. C 8. (a) $x = 0,\ 2,$ and 5
(b) $0 < x < 2$ and $5 < x < 8$ (c) $0 < x < 1$ and $3 < x < 7$ (d) $x = 2$ (e) $x = 1,\ 3,$ and 7

9. (a) f has a relative minimum at $x = 1$. (b)
(c) h has a relative maximum at $x = 0$.
h has a relative minimum at $x = 2$.
(d) The graph of h has a POI at $x = 1$.



3.7 Optimization Problems

1. D 2. A 3. C 4. B

5. (a) $A = \frac{1}{4}k\sqrt{64 - k^2}$ (b) $k = 4\sqrt{2}$ 6. (a) $y = -2zx + z^2 + 3$ (b) $z = 1$

3.8 Tangent Line Approximation and Differentials

1. C 2. D 3. C 4. A 5.B 6. C 7. D 8. (a) $y = -\frac{1}{2}x + 1$ (b) 0.95

(c) $f^{-1} = \sin^{-1}\left[\ln\left|\frac{2}{x} - 1\right|\right]$

9. (a) $f'(0) \approx \frac{f(1) - f(-2)}{1 - (-2)} = -\frac{2}{3}$ (b) $y = \frac{9}{5}x - \frac{27}{5}$ (c) $y = 2x - 5$ (d) If the graph is CU, the tangent line approximation is smaller than the real value. Therefore, $f(5) \geq \frac{9}{5}(5) - \frac{27}{5} = \frac{18}{5}$.

(e) The secant line connecting $(1, -3)$ and $(6, 7)$ lies above the graph for $1 \leq x \leq 6$.
Therefore, $f(5) \leq 2(5) - 5 = 5$.

10. (a) $y = \frac{5}{12}x + \frac{7}{4}$ (b) $h''(3) = \frac{1}{4}\left[\frac{2f''(2) - f'(2)}{8}\right]$. $h''(3)$ is negative, since $f''(2) < 0$ and $f'(2) > 0$.

(c) If the curve is CD, tangent line lies above the curve and the secant line lies below the curve.
Therefore, $h(2) \leq \frac{5}{12}(2) + \frac{7}{4} = \frac{31}{12}$. Equation of secant line is $y = x$. Thus, $h(2) \geq 2$.

Chapter 4 Integration

4.1 Antiderivatives and Indefinite Integrals

1. B 2. D 3. C 4. C 5. -7

4.2 Riemann Sums

1. C 2. D 3. C 4. A 5. A 6. B

4.3 Definite Integral and Area Under a Curve

1. A 2. C 3. B 4. D 5. D 6. A 7. C 8. C 9. B 10. D

11. (a) $A = \dfrac{1}{\sqrt{k}}\int_1^2 \sqrt{x-1}\,dx$ (b) $k = \dfrac{1}{9}$

(c) The tangent line is $y = \dfrac{1}{2\sqrt{k}}x$. Thus the tangent line passes through $(0,0)$.

12. (a) 1 (b) 2π (c) $-\dfrac{5}{2}$ (d) $2\pi + \dfrac{11}{2}$

4.4 Properties of Definite Integral

1. C 2. D 3. C 4. B 5. (a) 10 (b) 3 (c) -18 (d) -10 (e) 7

6. (a) $3n$ (b) $2n-1$ (c) $k = \dfrac{3}{2}$

4.5 Trapezoidal Rule

1. A 2. C 3. D 4. C 5. B 6. 0.527 7. 21

4.6 Fundamental Theorem of Calculus Part 1

1. C 2. C 3. B 4. C 5. B 6. A 7. (a) $-1 \le x \le 3$ (b) 2 (c) $x = 3$

8. (a) 0 (b) $x = 2$ and 6 (c) $\dfrac{1}{2}$ (d) There are two values of c.

4.7 Fundamental Theorem of Calculus Part 2

1. B 2. C 3. A 4. C 5. (a) 15 (b) 12 (c) 19 (d) 17

6. (a) $f(-3) = \dfrac{13}{2}$, $f(7) = \dfrac{15}{2} + 2\pi$ (b) $y = x+1$ (c) $1 < x < 7$ (d) $-1 < x < 4$

7. (a) $g(0) = 2$, $g'(0) = 2$, $g''(0) = 1$ (b) $-2 < x < 1$ and $7 < x < 9$ (c) $-2 < x < 4$ and $8 < x < 9$

8. (a) $h(8) = 2$, $h'(6) = -2$, $h''(4) = -2$ (b) h has a relative minimum at $x = 1$.
h has a relative maximum at $x = 5$. (c) $y - 11 = 4(x - 35)$

4.8 Integration by Substitution

1. D 2. D 3. C 4. A 5. B 6. B 7. C 8. 5

4.9 Integration of Exponential and Logarithmic Functions

1. B 2. C 3. D 4. B 5. A 6. A 7. D
8. (a) $C = 45{,}000$, $k = -1.861$ (b) 174,069

Chapter 5 Applications of Integration

5.1 Area of a Region Between Two Curves

1. C 2. B 3. C 4. D Q 5. (a) $\frac{e}{2}+\frac{1}{2e}-1$ (b) $\frac{e}{2}-\frac{1}{2e}-1$

5.2 Volumes by Disk and Washers

1. C 2. D 3. B 4. C 5. B 6. A 7. D 8. A

9. (a) $y=-x+1$ (b) 2.670 (c) $V=\pi\int_0^{2.313}\left(\left[(2-(x^3-2x^2-x+\cos x)\right]^2-\left[2-(-x+1)\right]^2\right)dx$

10. (a) $\ln 4+\frac{1}{e}$ (b) 2.225π (c) 8.348π 11. (a) $V=\pi\left[3-\frac{3}{k^3+1}\right]$ (b) 1

5.3 Volumes of Solids with Known Cross Sections

1. B 2. D 3. A 4. C 5. A 6. C 7. D 8. (a) 4 (b) 24π (c) 3

9. (a) Volume $=\int_0^{12}\pi\left[\frac{D(x)}{2}\right]^2 dx$ (b) Volume $\approx 19.386\text{ m}^3$

(c) Yes. Since $D(2)=D(8)=1.5$, MVT guarantees that there is at least one x in $(2,8)$ such that $D'(x)=0$.

10. (a) $\int_0^{1.5}[f(x)-g(x)]\,dx$ (b) $\int_{1.5}^{4}[g(x)-f(x)]\,dx$ (c) 3.776 (d) 16.584

5.4 The Total Change Theorem (Application of FTC)

1. B 2. C 3. C 4. D 5. A 6. (a) 13.925 (b) 116.075

(c) $f(t)=50+8t-\int_0^t \frac{1}{2}t^{2/3}\,dt$ (d) $t=64$ 7. (a) $P'(3)=-2.041$. The rate at which granules of plastic is changing is decreasing by 2.041 tons per hour per hour at time $t=3$ hours.

(b) $t=4.550$ (c) 11.532 tons

5.5 Distance Traveled by a Particle Along a Line

1. B 2. D 3. D 4. C 5. A 6. B 7. B 8. A 9. C

10. (a) The particle is moving to the left at time $t=2$. (b) $a(2)=-8\sin(3)+\cos(3)$. No. The velocity of the particle is decreasing at time $t=2$, since $a(2)<0$.

(c) Yes (d) 3.991 (e) 4.921 (f) 1.992

11. (a) -14 (b) $2/3<t<4$ (c) -4 (d) -5 (e) -10 (f) 17.037

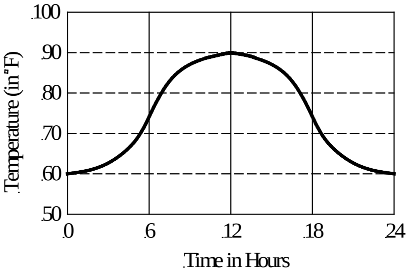
12. (a) 0 (b) 1 (c) $2<t<6$ (d) $0<t<3$ (e) On the interval $2<t<3$, $v\ (=s')<0$ and $a\ (=s'')<0$. Since v and a have the same sign on $2<t<3$, the speed of particle is increasing.

(f) $a\ (=s'')>0$, on the interval $3<t<8$ since on this interval the curve of s is concave upward.

5.6 Average Value of a Function

1. C 2. D 3. C 4. B 5. B 6. A 7. (a) -1 (b) 0 (c) -1

8. (a)



(b) 78° F

(c) $7.298\le t\le 16.702$

(d) \$7.32

5.7 Length of a Curve (Distance Traveled Along a Curve) $\boxed{BC}$

1. B 2. C 3. A 4. D 5. C 6. D

7. (a) $P(x, y) = (\cot^2\theta, \cot\theta)$ (b) 3.168 (c) 1.442 units /min

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Chapter 6 Techniques of Integration

6.1 Basic Integration Rules

1. B 2. D 3. C 4. A 5. D 6. $\pi\left[\ln(\sqrt{2}+1)-\sqrt{2}/2\right]$

6.2 Trigonometric Integrals

1. B 2. C 3. B 4. D 5. A 6. $\frac{1}{3}$

6.3 Trigonometric Substitutions

1. D 2. B 3. C 4. D 5. C 6. (a) 15.904 (b) $27\int_0^{\pi/2}\cos^4\theta\, d\theta$

6.4 L'Hospital's Rule

1. B 2. D 3. A 4. D 5. C 6. C 7. B 8. 3 9. $-\frac{1}{2}$

6.5 Integration by Partial Fractions $\boxed{BC}$

1. C 2. C 3. A 4. D 5. (a) $\int \frac{-dx}{x(x-1)}$ (b) $\ln\left|\frac{\cos\theta}{\cos\theta-1}\right|+C$

6.6 Integration by Parts $\boxed{BC}$

1. C 2. B 3. B 4. C 5. C 6. D 7. A 8. D 9. A 10. $\frac{\pi}{2}-1$

6.7 Improper Integrals $\boxed{BC}$

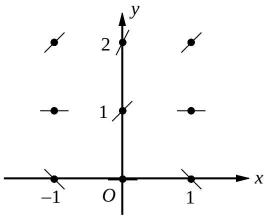
1. D 2. B 3. C 4. A 5. B 6. C

7. (a) $\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{\sqrt{x^2+1}}\,dx = \lim_{b\to\infty}\left[\sqrt{x^2+1}\right]_1^b = \lim_{b\to\infty}\left[\sqrt{b^2+1}-\sqrt{1+1}\right]=\infty$ (b) 1

Chapter 7 Further Applications of Integration

7.1 Slope Field

1. B 2. C 3.



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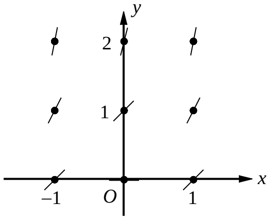

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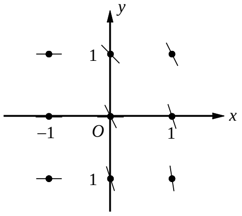
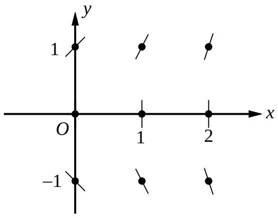


7.2 Separable Differential Equations

1. D 2. B 3. C 4. A 5. C

6. (a)



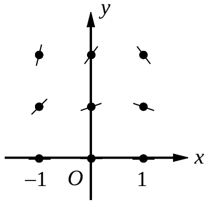
(b) $y = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$, $f(1.2) \approx 1.963$

(c) $y = \sqrt{x^2 + 2x}$

(d) $f(1.2) = 1.959$

7. (a) $y = \frac{3}{e^2}x + 2$ (b) $f''(0) = \frac{2e^2 - 9}{e^4}$ (c) $y = \ln\left|x^2 + 3x + e^2\right|$.

8. (a)



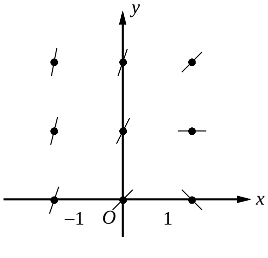
(b) $\frac{d^2y}{dx^2} = \frac{1}{3}\left[-2y^2 + (2y - 4xy)\frac{y^2(1-2x)}{3}\right]$

(c) $\left.\frac{dy}{dx}\right|_{(\frac{1}{2},4)} = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(\frac{1}{2},4)} = -\frac{32}{3} < 0$

Therefore, f has a relative maximum at $x = 1/2$.

(d) $y = \frac{3}{x^2 - x + 1}$

9. (a)



(b) $\frac{d^2y}{dx^2} = -2x + y - 1$ If the curve is CD, $y'' < 0$.

$-2x + y - 1 < 0 \Rightarrow y < 2x + 1$

Therefore, solution curves will be concave down on the half-plane below the line $y = 2x + 1$.

(c) $\left.\frac{dy}{dx}\right|_{(0,-1)} = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,-1)} < 0$. Therefore, f has a relative maximum at $(0,-1)$.

(d) $m = 2$, $b = 1$

7.3 Exponential Growth and Decay

1. D 2. B 3. C 4. A 5. (a) $V = 16e^{-0.00866t}$ (b) 0.03464 oz/sec (c) $t = 240$ seconds

7.4 Logistic Equations $\boxed{BC}$

1. D 2. C 3. B 4. A 5. D

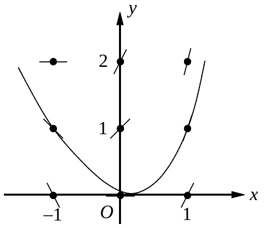
6. (a) $y = e^{2t - \frac{t^2}{4} - 3}$ (b) 0 (c) $\lim_{t\to\infty} g(t) = 5 \Rightarrow \lim_{t\to\infty} g'(t) = 0$ (d) POI at $y = 5/2$, Slope $= 5/4$

7.5 Euler's Method $\boxed{BC}$

1. C 2. B 3. B 4. A 5. D 6. (a) $\frac{d^2y}{dx^2}=1-\frac{1}{2}x+y$

(b) $\left.\frac{dy}{dx}\right|_{(0,-\frac{1}{2})}=0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,-\frac{1}{2})}>0$ Therefore, f has a relative minimum at $x=0$. (c) $k=5$

7. (a)



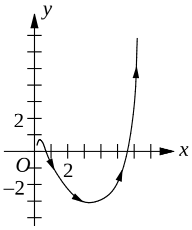
(b) 1.65
(c) −2
(d) g has a local minimum at $(1,-2)$.

Chapter 8 Parametric Equations, Vectors, and Polar Coordinates

8.1 Slopes and Tangents for the Parametric Curves

1. D 2. B 3. C 4. B 5. D 6. C 7. A

8. (a)



(b) $y(t)$ attains its minimum value at $t=3.426$.
$x(3.426)=3.706$, $y(3.426)=-3.288$

(c) $y+\pi=-\frac{1}{2}(x-\pi)$

9. (a) $x(t)=\sqrt{t^2+9}-2$ (b) $\frac{dy}{dt}=\left[3(\sqrt{t^2+9}-2)^2-8(\sqrt{t^2+9}-2)\right]\left[\frac{t}{\sqrt{t^2+9}}\right]$

(c) $x(4)=3$ and $y(4)=-5$ (d) $y+5=3(x-3)$

8.2 Arc Length in Parametric Form

1. B 2. C 3. D 4. A

5. (a) $y-2=0.321(x-3)$ (b) 3.166 (c) $\left(x(3),y(3)\right)=(3.927,1.877)$ (d) 0.935

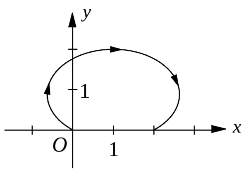
8.3 Vector Valued Functions

1. C 2. D 3. B 4. A 5. D 6. C

7. (a) $t=0.950$ (b) $a(2)=\left(-e^2\sin(e^2),\ \frac{-4}{e^2}\right)$ (c) 3.544 (d) 2.954

8. (a) $x(5)=13.245$ (b) $y-4=5.091(x-1)$ (c) 3.057 (d) $(-5.6,-3.544)$

9. (a)



(b) $x(1)=1$ $y(1)=2$
(c) $v(t)=(1-\pi\cos(\pi t),\pi\sin(\pi t))$
(d) $\int_0^2\sqrt{(1-\pi\cos(\pi t))^2+(\pi\sin(\pi t))^2}\,dt=6.443$

10. (a) $t=\frac{3}{7}$ (b) 10.9 (c) $t=1.920$ (d) 12.384 (e) 10.354 (f) $\theta=1.475$

11. (a) (6.946,1.5) (b) –0.105 (c) 4.793 (d) 10.072

8.4 Polar Coordinates and Slopes of Curves

1. C 2. A 3. B 4. D 5. (a) $\frac{dr}{d\theta}=\frac{(1-2\sin(2\theta))}{2\sqrt{\theta+\cos(2\theta)}}$ (b) $\theta=0.910$

(c) r is decreasing on this interval. The curve is getting closer to the origin. (d) $\theta=5\pi/12$

8.5 Areas in Polar Coordinates

1. C 2. B 3. A 4. D 5. B 6. D 7. (a) $P(2,\frac{\pi}{4})$, $\theta=\frac{\pi}{4}$

(b) $\frac{1}{2}\int_{0}^{\pi/4}\left[(2+\cos 2\theta)^2-2^2\right]d\theta$ (c) $\frac{1}{2}\int_{\pi/4}^{\pi/2}\left[2^2-(2+\cos 2\theta)^2\right]d\theta$ (d) $\frac{17}{16}\pi-1$ (e) $-\sqrt{3}$

8. (a) $\frac{1}{2}\int_{\pi/2}^{\pi}(2+2\cos\theta)^2\,d\theta$ (b) $\frac{dx}{d\theta}=-2\sin\theta-4\cos\theta\sin\theta$, $\frac{dy}{d\theta}=2\cos\theta+2\cos 2\theta$ (c) $y=x+2$

Chapter 9 Infinite Sequences and Series

9.1 Sequences and Series

1. C 2. D 3. D 4. B 5. A 6. D 7. C 8. 2

9.2 The Integral Test and *p*-Series

1. C 2. D 3. B 4. D 5. A 6. (a) converges (b) diverges

9.3 The Comparison Tests

1. D 2. C 3. D 4. B 5. (a) converges (b) diverges

9.4 Alternating Series Tests

1. D 2. C 3. C 4. B 5. D 6. A 7. B 8. C

9. $\left|f(1)-(1-\frac{3}{2!}+\frac{9}{4!})\right|\le\frac{27}{6!}=0.0375<\frac{1}{20}$

9.5 The Ratio and Root Tests

1. C 2. D 3. (a) diverges (b) converges (c) diverges

9.6 Convergence of Power Series

1. B 2. A 3. D 4. C 5. C 6. $R=1$, Interval of convergence is $(-1,1)$.

9.7 Representations of Functions as Power Series

1. B 2. C 3. D 4. (a) $g(x)=1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots$

(b) $g(x^2)=1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots$ (c) $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+\frac{(-1)^{n-1}x^{2n-1}}{2n-1}+\cdots$

(d) $h(1)=\tan^{-1}(1)$

5. (a) $f'(0)=0$, $f''(0)=-3$, f has a local maximum at $x=0$ because $f'(0)=0$ and $f''(0)<0$.

(b) $\left| f(1)-(1-\frac{3}{2!}+\frac{5}{4!}) \right| \le \frac{7}{6!}=\frac{7}{720}<\frac{1}{100}$ (c) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$

9.8 Taylor Polynomial and Lagrange Error Bound

1. D 2. C 3. A 4. B 5. C 6. B

7. (a) $f(2)=3$, $f'''(2)=-72$ (b) $P_3(x)=-2+10(x-2)-36(x-2)^2+12(x-2)^3$, $f'(2.1)=-1.348$

(c) $3(x-2)-(x-2)^2+\frac{5}{3}(x-2)^3-3(x-2)^4$ (d) No.

8. (a) $P(x)=1+2x-2x^2-\frac{4}{3}x^3$ (b) $\frac{(-2)^{19}}{19!}$ (c) $32\cdot\frac{0.0016}{24}=\frac{4}{1875}<\frac{1}{100}$ (d) $h(x)=x+x^2-\frac{2}{3}x^3$

9.9 Taylor Series and Maclaurin Series

1. B 2. A 3. D 4. C 5. C 6. B 7. D 8. A 9. C 10. B

11. (a) $f(x)=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots+\frac{(-1)^n x^n}{n!}+\cdots$ (b) $-\frac{x}{2!}+\frac{x^2}{3!}-\frac{x^3}{4!}+\frac{x^4}{5!}-\cdots+\frac{(-1)^n x^n}{(n+1)!}+\cdots$

(c) $g'(-1)=-\frac{1}{2!}-\frac{2}{3!}-\frac{3}{4!}-\cdots=\sum_{n=1}^{\infty}\frac{-n}{(n+1)!}$, $g'(x)=\frac{-1+xe^{-x}+e^{-x}}{x^2} \Rightarrow g'(-1)=-1$.

Thus, $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}=1$.

12. (a) $f'''(0)=-\frac{1}{4}$, $f^{(15)}(0)=-\frac{1}{16}$ (b) $(-1,1]$ (c) $y'(0)=0$, $y''(0)=\frac{2}{3}>0$

Since $y'(0)=0$ and $y''(0)>0$, y has a relative minimum at $x=0$.

Calculus AB Practice Test 1

SECTION I, Part A

1. B 2. A 3. C 4. B 5. C 6. D 7. B 8. A 9. A 10. B

11. D 12. C 13. B 14. B 15. A 16. A 17. D 18. D 19. C 20. B

21. D 22. C 23. D 24. C 25. C 26. D 27. A 28. B 29. A 30. B

SECTION I, Part B

31. C 32. A 33. D 34. C 35. C 36. D 37. D 38. C 39. B 40. A

41. D 42. D 43. C 44. A 45. C

SECTION II, Part A

1. (a) Since $n(10)=40\sqrt{10}\sin(\frac{10\pi}{12})=63>0$, the number of honeybees is increasing at $t=10$.

(b) The number of honeybees is increasing at a decreasing rate, because $n(10)>0$ and $n'(10)<0$.

(c) 1440 (d) 1927

2. (a) On the interval $0<x<1$, f concaves up because f' is increasing.

(b) f has an absolute maximum at $x=\sqrt{\pi}$. (c) $y-3.586=-1.218(x-2)$

SECTION II, Part B

3. (a) $\frac{2}{\pi}-\frac{1}{4}$ (b) $V=\pi\int_0^1\left[\left(\sin(\frac{\pi x}{2})+1\right)^2-(x^3+1)^2\right]dx$ (c) $V=\int_0^1 3\left(\sin(\frac{\pi x}{2})-x^3\right)^2 dx$

4. (a) $g(2)=\frac{11}{2}$, $g'(2)=3$, $g''(2)=-1$ (b) $x=1$, $x=5$ (c) $-1\le x\le 4$ (d) $\frac{5}{2}$

5. (a) $g'(0)=a+3b$, $g''(0)=-2b^2$ (b) $h'(x)=e^{\sin x}\cdot f'(x)+e^{\sin x}\cos x\cdot f(x)$, $y=2x-1$

6. (a) $3\le t\le 11$ (b) At time $t=6$, because $v(6)-v(0)=\int_0^6 a(t)\,dt=0$.
(c) 88 ft /sec, at time $t=11$. (d) 16 ft /sec, at time $t=20$.

Calculus AB Practice Test 2

SECTION I, Part A

1. B 2. C 3. C 4. D 5. A 6. B 7. C 8. C 9. A 10. B
11. C 12. C 13. B 14. B 15. D 16. C 17. D 18. A 19. C 20. D
21. D 22. A 23. C 24. B 25. D 26. D 27. D 28. B 29. C 30. C

SECTION I, Part B

31. C 32. D 33. C 34. B 35. D 36. A 37. A 38. C 39. D 40. B
41. D 42. C 43. B 44. A 45. C

SECTION II, Part A

1. (a) $x=1$, $m=3$ (b) 6.75 (c) 52.071π or 163.587
2. (a) 900 (b) 139 (c) $P'(20)=-20.825$. The number of people in the movie theater is decreasing at the rate of 21 people / hour at time $t=20$. (d) $t=17.776$

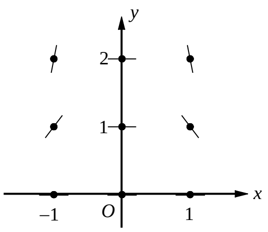
SECTION II, Part B

3. (a) $-\frac{11}{30}$ ft / sec^2 (b) The distance that the car travels from $t=40$ to $t=70$. $\int_{40}^{70}|v(t)|\,dt=200$ ft
(c) Yes. Since $v(30)=15$ and $v(40)=6$, IVT guarantees a t in $(30,40)$ so that $v(t)=10$.
(d) Yes. Since $v(60)=v(90)=-12$, MVT guarantees a t in $(60,90)$ so that $a(t)=v'(t)=0$.

4. (a) $\frac{dy}{dx}=\frac{-2x-y}{x+2y}$ (b) $(2\sqrt{3},0)$, $y-0=-2(x-2\sqrt{3})$. $(-2\sqrt{3},0)$, $y-0=-2(x+2\sqrt{3})$. (c) $x=\pm 2$

5. (a) $h(0)=-3$, $h'(0)=5$, $h''(0)=0$ (b) $4\le x\le 8$ (c) $-2\le x\le 0$ and $6\le x\le 8$ (d) 15

6. (a)



(b) $y=3x+\frac{9}{2}$, $f(-1.1)\approx 1.2$

(c) $y=\frac{3}{x^4+1}$

Calculus BC Practice Test 1

SECTION I, Part A

1. B 2. C 3. A 4. C 5. D 6. D 7. D 8. A 9. C 10. D

11. B 12. A 13. A 14. C 15. B 16. D 17. C 18. C 19. A 20. D

21. D 22. C 23. B 24. D 25. D 26. B 27. C 28. A 29. D 30. B

SECTION I, Part B

31. D 32. B 33. C 34. D 35. A 36. D 37. C 38. C 39. C 40. B

41. D 42. D 43. A 44. D 45. A

SECTION II, Part A

1. (a) 2.299 (b) 1.472 (c) 2.139

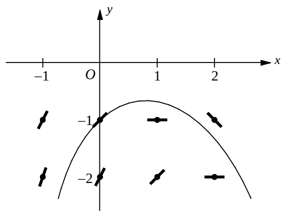
2. (a) 0.614 (b) $v(0.5) = -0.693 < 0$ and $a(0.5) = 0.614 > 0$. The speed of the particle is decreasing because at time $t = 0.5$, $v(t)$ and $a(t)$ have opposite signs.
   (c) At $t = 1$, the particle is farthest to the left. The distance between the particle and the origin is 1.5.
   (d) $x(0.5) = -1.298$ At time $t = 0.5$, the particle is to the left of the origin and moving to the left since $v(0.5) < 0$. Therefore, the particle is moving away from the origin at time $t = 0.5$.

SECTION II, Part B

3. (a) $\left.\dfrac{dr}{d\theta}\right|_{\theta=\frac{\pi}{6}} = -\dfrac{8\sqrt{3}}{9}$. The curve is getting closer to the origin when $\theta = \dfrac{\pi}{6}$ because r is positive and $\dfrac{dr}{d\theta}$ is negative at $\theta = \dfrac{\pi}{6}$. (b) $A = \dfrac{1}{2}\displaystyle\int_0^{\pi} r^2\, d\theta = \dfrac{1}{2}\displaystyle\int_0^{\pi} \left(\dfrac{4}{1+\sin\theta}\right)^2 d\theta = \dfrac{32}{3}$
   (c) Make substitution. $y = r\sin\theta$ and $x^2 + y^2 = r^2$. (d) $A = \displaystyle\int_{-4}^{4} \left(-\dfrac{1}{8}x^2 + 2\right) dx = \dfrac{32}{3}$

4. (a) $P'(7.5) \approx 93.333$ gallons / hr^2 (b) $P''(7.5) = 0$ (c) 1050 gallons / hour (d) No.

5. (a)



   (b) $f(-0.4) \approx -1.48$
   (c) $-\ln 2$
   (d) $\dfrac{d^2y}{dx^2} = -1 + x + y$ is negative because $x(= -0.4)$ and $y(\approx -1.48)$ are both negative. Thus the graph concaves down and the approximation is greater than $f(-0.4)$.

6. (a) $-\dfrac{2x}{3} + \dfrac{4x^3}{5} - \dfrac{6x^5}{7} + \cdots + \dfrac{(-1)^n(2n)x^{2n-1}}{2n+1} + \cdots$ (b) $\dfrac{3\sqrt{3}}{4} - \dfrac{\pi}{2}$
   (c) $x - \dfrac{x^3}{3^2} + \dfrac{x^5}{5^2} - \dfrac{x^7}{7^2} + \cdots + \dfrac{(-1)^{n-1}x^{2n-1}}{(2n-1)^2} + \cdots$ (d) $\left|g(1) - \left(1 - \dfrac{1}{3^2} + \dfrac{1}{5^2}\right)\right| \le \dfrac{1}{7^2} < \dfrac{1}{40}$

Calculus BC Practice Test 2

SECTION I, Part A

1. A 2. C 3. B 4. C 5. C 6. B 7. D 8. D 9. A 10. D

11. A 12. B 13. B 14. A 15. B 16. C 17. D 18. A 19. A 20. D

21. D 22. C 23. D 24. A 25. B 26. C 27. A 28. D 29. B 30. C

SECTION I, Part B

31. C 32. C 33. D 34. C 35. A 36. C 37. D 38. A 39. D 40. C

41. B 42. B 43. D 44. C 45. A

SECTION II, Part A

1. (a) 1.949 (b) 63.773 (c) $V = \pi \int_{0.853}^{3.841} \left[(4 - y + 1)^2 - (e^{2-y} + 1)^2 \right] dy$

2. (a) $y + 2 = 1.339(x - 1)$ (b) 2.577 (c) 3.587 (d) 25.105

SECTION II, Part B

3. (a) $h = \dfrac{p}{2}$ (b) $\dfrac{d\theta}{dt} = 1$ (c) $\left.\dfrac{dA}{dt}\right|_{p=2} = 3$

4. (a) $y = 3x + 1$, $f(1.1) \approx 4.3$ (b) $f''(1) = \dfrac{33}{8}$ (c) $y = (\dfrac{3}{8}x^2 + \dfrac{13}{8})^2$.

5. (a) When $P(0) = 10$, $\lim\limits_{t \to \infty} P(t) = 32$ and when $P(0) = 40$, $\lim\limits_{t \to \infty} P(t) = 32$.

   (b) $P = 16$ (c) 4 (d) $y = \dfrac{1}{2} e^{\frac{x}{2} - \frac{x^2}{128}}$

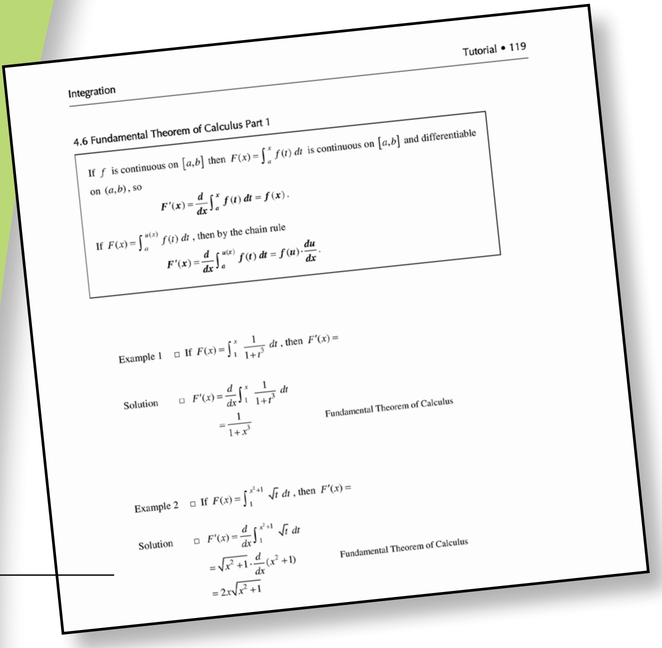
6. (a) $f(x) = 1 + (x - 3) + \dfrac{3}{2^2}(x - 3)^2 + \dfrac{4}{2^3}(x - 3)^3 + \cdots + \dfrac{n + 1}{2^n}(x - 3)^n + \cdots$

   (b) $R = 2$ (c) $h(x) = 2 + (x - 3) + \dfrac{1}{2}(x - 3)^2 + \dfrac{1}{2^2}(x - 3)^3 + \cdots + \dfrac{1}{2^n}(x - 3)^{n+1} + \cdots$

   (d) Since the geometric series only converges for $|r| = \left| \dfrac{x - 3}{2} \right| < 1$, the Taylor series does not converge at $x = 1$.

Understand the Basic Concepts of Calculus and Learn Problem-Solving Techniques

Major definitions, theorems, and important formulas are enclosed by a box in the beginning of each section, followed by examples and solutions.



Integration

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4.6 Fundamental Theorem of Calculus Part 1

If f is continuous on $[a,b]$ then $F(x)=\int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b), so

$$\boldsymbol{F'(x)=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)}.$$

If $F(x)=\int_a^{u(x)} f(t)\,dt$, then by the chain rule

$$\boldsymbol{F'(x)=\frac{d}{dx}\int_a^{u(x)} f(t)\,dt=f(u)\cdot\frac{du}{dx}}.$$

Example 1 □ If $F(x)=\int_1^x \frac{1}{1+t^3}\,dt$, then $F'(x)=$

Solution □ $F'(x)=\frac{d}{dx}\int_1^x \frac{1}{1+t^3}\,dt$ Fundamental Theorem of Calculus

$=\frac{1}{1+x^3}$

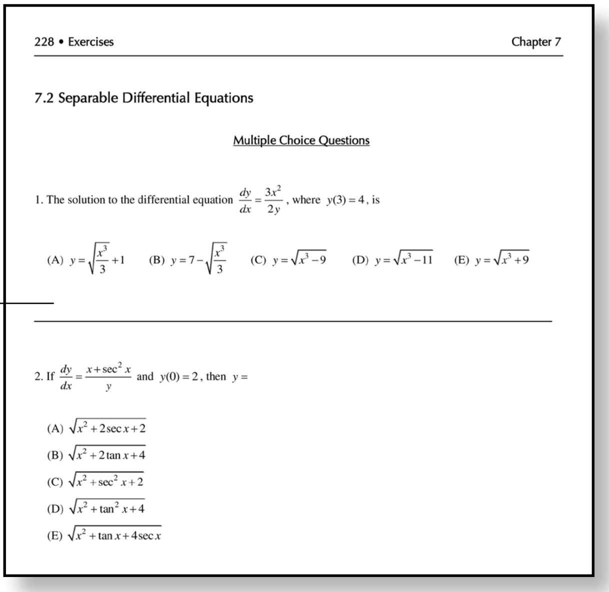
Example 2 □ If $F(x)=\int_1^{x^2+1} \sqrt{t}\,dt$, then $F'(x)=$

Solution □ $F'(x)=\frac{d}{dx}\int_1^{x^2+1} \sqrt{t}\,dt$ Fundamental Theorem of Calculus

$=\sqrt{x^2+1}\cdot\frac{d}{dx}(x^2+1)$

$=2x\sqrt{x^2+1}$

A set of exercises is included at the end of each chapter. These multiple choice and free response questions are grouped by section in order to help students master discrete concepts for the AP Calculus Test.



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Chapter 7

7.2 Separable Differential Equations

Multiple Choice Questions

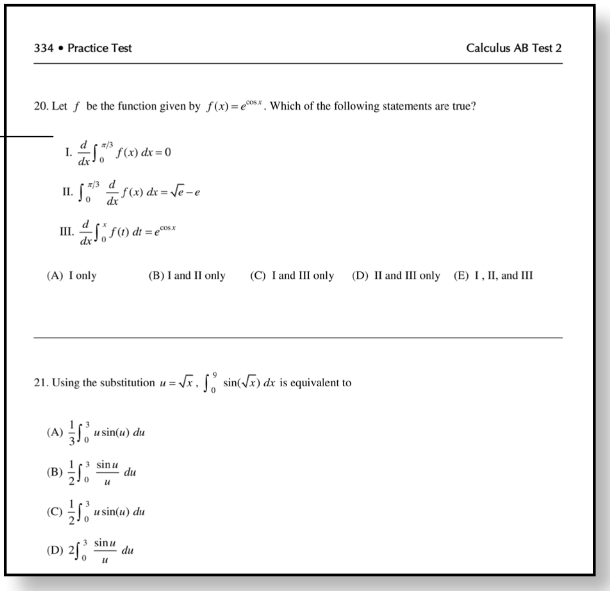
1. The solution to the differential equation $\frac{dy}{dx}=\frac{3x^2}{2y}$, where $y(3)=4$, is

(A) $y=\sqrt{\frac{x^3}{3}}+1$ (B) $y=7-\sqrt{\frac{x^3}{3}}$ (C) $y=\sqrt{x^3-9}$ (D) $y=\sqrt{x^3-11}$ (E) $y=\sqrt{x^3+9}$

2. If $\frac{dy}{dx}=\frac{x+\sec^2 x}{y}$ and $y(0)=2$, then $y=$

(A) $\sqrt{x^2+2\sec x+2}$
(B) $\sqrt{x^2+2\tan x+4}$
(C) $\sqrt{x^2+\sec^2 x+2}$
(D) $\sqrt{x^2+\tan^2 x+4}$
(E) $\sqrt{x^2+\tan x+4\sec x}$

There are 2 AB practice tests and 2 BC practice tests, each with 45 multiple choice questions and 6 free response questions. Many prep books use some of the same questions in their AB and BC tests, but our AB and BC practice tests never share questions.



334 • Practice Test

Calculus AB Test 2

20. Let f be the function given by $f(x)=e^{\cos x}$. Which of the following statements are true?

I. $\frac{d}{dx}\int_0^{\pi/3} f(x)\,dx=0$
II. $\int_0^{\pi/3} \frac{d}{dx} f(x)\,dx=\sqrt{e}-e$
III. $\frac{d}{dx}\int_0^{x} f(t)\,dt=e^{\cos x}$

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I , II, and III

21. Using the substitution $u=\sqrt{x}$, $\int_0^9 \sin(\sqrt{x})\,dx$ is equivalent to

(A) $\frac{1}{3}\int_0^3 u\sin(u)\,du$
(B) $\frac{1}{2}\int_0^3 \frac{\sin u}{u}\,du$
(C) $\frac{1}{2}\int_0^3 u\sin(u)\,du$
(D) $2\int_0^3 \frac{\sin u}{u}\,du$

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