

Sequences and Series



Then

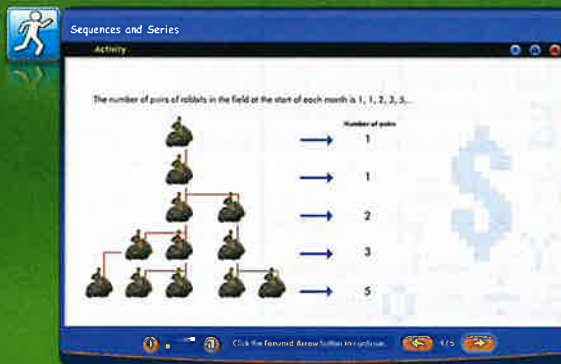
- You simplified and evaluated algebraic expressions.

Now

- You will:
 - Use arithmetic and geometric sequences and series.
 - Use special sequences and iterate functions.
 - Expand powers by using the Binomial Theorem.
 - Prove statements by using mathematical induction.

Why? ▲

- CONSERVATION AND NATURE** Mathematics occurs in aspects of nature in astonishing ways. The Fibonacci sequence manifests itself in seeds, flowers, pine cones, fruits, and vegetables. Sequences and series can further help us conserve our natural resources by making water filtration systems more efficient.



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Animation



Vocabulary



eGlossary



Personal Tutor



Virtual Manipulatives



Graphing Calculator



Audio



Foldables



Self-Check Practice



Worksheets



Get Ready for the Chapter

Diagnose Readiness | You have two options for checking prerequisite skills.

1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Solve each equation.

- $-6 = 7x + 78$
- $768 = 3x^4$
- $23 - 5x = 8$
- $2x^3 + 4 = -50$
- PLANTS** Lauri has 48 plants for her two gardens. She plants 12 in the small garden. In the other garden she wants 4 plants in each row. How many rows will she have?

Graph each function.

- $\{(1, 3), (2, 5), (3, 7), (4, 9), (5, 11)\}$
- $\{(1, -15), (2, -12), (3, -9), (4, -6), (5, -3)\}$
- $\left\{(1, 27), (2, 9), (3, 3), (4, 1), \left(5, \frac{1}{3}\right)\right\}$
- $\left\{(1, 1), (2, 2), \left(3, \frac{5}{2}\right), \left(4, \frac{11}{4}\right), \left(5, \frac{23}{8}\right)\right\}$
- DAY CARE** A child care center has expenses of \$125 per day. They charge \$50 per child per day. The function $P(c) = 50c - 125$ gives the amount of money the center makes when there are c children there. How much will they make if there are 8 children?

Evaluate each expression for the given value(s) of the variable(s).

- $\frac{a}{3}(b + c)$ if $a = 9$, $b = -2$, and $c = -8$
- $r + (n - 2)t$ if $r = 15$, $n = 5$, and $t = -1$
- $x \cdot y^z + 1$ if $x = -2$, $y = \frac{1}{3}$, and $z = 5$
- $\frac{a(1 - bc)^2}{1 - b}$ if $a = -3$, $b = -4$, and $c = 1$

QuickReview

Example 1

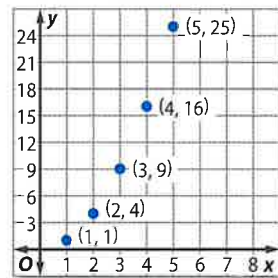
Solve $25 = 3x^3 + 400$.

$$\begin{aligned} 25 &= 3x^3 + 400 && \text{Original equation} \\ -375 &= 3x^3 && \text{Subtract 400 from each side.} \\ -125 &= x^3 && \text{Divide each side by 3.} \\ \sqrt[3]{-125} &= \sqrt[3]{x^3} && \text{Take the cube root of each side.} \\ -5 &= x && \text{Simplify.} \end{aligned}$$

Example 2

Graph the function $\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$. State the domain and range.

The domain of a function is the set of all possible x -values. So, the domain of the function is $\{1, 2, 3, 4, 5\}$. The range of a function is the set of all possible y -values. So, the range of this function is $\{1, 4, 9, 16, 25\}$.



Example 3

Evaluate $2 \cdot 3^{x+y}$ if $x = -2$ and $y = -3$.

$$\begin{aligned} 2 \cdot 3^{x+y} &= 2 \cdot 3^{-2+(-3)} && \text{Substitute.} \\ &= 2 \cdot 3^{-5} && \text{Simplify.} \\ &= \frac{2}{3^5} && \text{Rewrite with positive exponent.} \\ &= \frac{2}{243} && \text{Evaluate the power.} \end{aligned}$$

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.



Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 10. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

FOLDABLES Study Organizer



Sequences and Series Make this Foldable to help you organize your Chapter 10 notes about sequences and series. Begin with one $8\frac{1}{2}$ " by 11" sheet of paper.

- 1** Fold in half, matching the short sides.



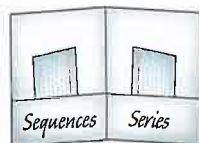
- 2** Unfold and fold the long side up 2 inches to form a pocket.



- 3** Staple or glue the outer edges to complete the pocket.



- 4** Label each side as shown. Use index cards to record notes and examples.



New Vocabulary



English		Español
sequence	p. 659	sucesión
finite sequence	p. 659	sucesión finita
infinite sequence	p. 659	sucesión infinita
arithmetic sequence	p. 659	sucesión aritmética
common difference	p. 659	diferencia común
geometric sequence	p. 661	sucesión geométrica
common ratio	p. 661	razón común
arithmetic means	p. 667	media aritmética
series	p. 668	serie
arithmetic series	p. 668	serie aritmética
partial sum	p. 668	suma parcial
geometric means	p. 675	media geométrica
geometric series	p. 676	serie geométrica
convergent series	p. 683	serie convergente
divergent series	p. 683	serie divergente
recursive sequence	p. 692	sucesión recursiva
iteration	p. 694	iteración
mathematical induction	p. 705	inducción matemática
induction hypothesis	p. 705	hipótesis inductiva

Review Vocabulary



coefficient **coeficiente** the numerical factor of a monomial

$$15x^3$$

coefficient
coeficiente

formula **fórmula** a mathematical sentence that expresses the relationship between certain quantities

function **función** a relation in which each element of the domain is paired with exactly one element in the range

Sequences as Functions



Then

You analyzed linear and exponential functions.

Now

- 1 Relate arithmetic sequences to linear functions.
- 2 Relate geometric sequences to exponential functions.

Why?

During their routine, a high school marching band marches in rows. There is one performer in the first row, three performers in the next row, and five in the third row. This pattern continues for the rest of the rows.



New Vocabulary

- sequence
- term
- finite sequence
- infinite sequence
- arithmetic sequence
- common difference
- geometric sequence
- common ratio



Common Core State Standards

Content Standards
 F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Mathematical Practices

- 2 Reason abstractly and quantitatively.
- 7 Look for and make use of structure.

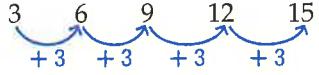
1 Arithmetic Sequences A **sequence** is a set of numbers in a particular order or pattern. Each number in a sequence is called a **term**. A sequence may be a **finite sequence** containing a limited number of terms, such as $\{-2, 0, 2, 4, 6\}$, or an **infinite sequence** that continues without end, such as $\{0, 1, 2, 3, \dots\}$. The first term of a sequence is denoted a_1 , the second term is denoted a_2 , and so on.

KeyConcept Sequences as Functions

Words	A sequence is a function in which the domain consists of natural numbers, and the range consists of real numbers.	
Symbols	Domain: 1 2 3 ... n	the position of a term
	Range: a_1 a_2 a_3 ... a_n	the terms of the sequence
Examples	Finite Sequence $\{3, 6, 9, 12, 15\}$	Infinite Sequence $\{3, 6, 9, 12, 15, \dots\}$
	Domain: $\{1, 2, 3, 4, 5\}$ Range: $\{3, 6, 9, 12, 15\}$	Domain: $\{\text{all natural numbers}\}$ Range: $\{y \mid y \text{ is a multiple of } 3, y \geq 3\}$

In an **arithmetic sequence**, each term is determined by adding a constant value to the previous term. This constant value is called the **common difference**.

Consider the sequence 3, 6, 9, 12, 15. This sequence is arithmetic because the terms share a common difference. Each term is 3 more than the previous term.



Example 1 Identify Arithmetic Sequences

Determine whether each sequence is arithmetic.

a. 5, -6, -17, -28, ...

The common difference is -11 .
 The sequence is arithmetic.

b. -4, 12, 28, 42, ...

There is no common difference.
 This is not an arithmetic sequence.

Guided Practice

1A. 7, 12, 16, 20, ...

1B. -6, 3, 12, 21, ...



You can use the common difference to find terms of an arithmetic sequence.



Example 2 Graph an Arithmetic Sequence

Consider the arithmetic sequence 18, 14, 10, ...

a. Find the next four terms of the sequence.

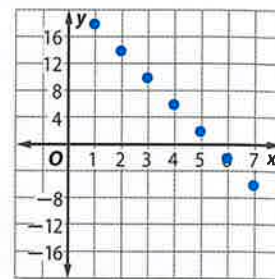
Step 1 To determine the common difference, subtract any term from the term directly after it. The common difference is $10 - 14$ or -4 .

Step 2 To find the next term, add -4 to the last term.

Continue to add -4 to find the following terms.

$$\begin{array}{cccccc}
 10 & & 6 & & 2 & & -2 & & -6 \\
 & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\
 & +(-4) & +(-4) & +(-4) & +(-4) & +(-4) & +(-4) & +(-4) & +(-4)
 \end{array}$$

The next four terms are 6, 2, -2 , and -6 .



b. Graph the first seven terms of the sequence.

The domain contains the terms $\{1, 2, 3, 4, 5, 6, 7\}$ and the range contains the terms $\{18, 14, 10, 6, 2, -2, -6\}$. So, graph the corresponding ordered pairs.

GuidedPractice

2. Find the next four terms of the arithmetic sequence 18, 11, 4, Then graph the first seven terms.

Notice that the graph of the terms of the arithmetic sequence lie on a line. An arithmetic sequence is a linear function in which the term number n is the independent variable, the term a_n is the dependent variable, and the common difference is the slope.



Real-World Example 3 Find a Term

MARCHING BANDS Refer to the beginning of the lesson. Suppose the director wants to determine how many performers will be in the 14th row during the routine.

Understand Because the difference between any two consecutive rows is 2, the common difference for the sequence is 2.

Plan Use point-slope form to write an equation for the sequence. Let $m = 2$ and $(x_1, y_1) = (3, 5)$. Then solve for $x = 14$.

Solve	$ \begin{aligned} (y - y_1) &= m(x - x_1) \\ (y - 5) &= 2(x - 3) \\ y - 5 &= 2x - 6 \\ y &= 2x - 1 \\ y &= 2(14) - 1 \\ y &= 28 - 1 \text{ or } 27 \end{aligned} $	<p>Point-slope form $m = 2$ and $(x_1, y_1) = (3, 5)$ Multiply. Add 5 to each side. Replace x with 14. Simplify.</p>
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Check You can find the terms of the sequence by adding 2, starting with row 1, until you reach row 14.

GuidedPractice

3. **MONEY** Geraldo's employer offers him a pay rate of \$9 per hour with a \$0.15 raise every three months. How much will Geraldo earn per hour after 3 years?



Real-WorldLink

Each year, about 100 bands compete in the Bands of America Grand National Championships.

Source: Bands of America



2 Geometric Sequences Another type of sequence is a geometric sequence. In a **geometric sequence**, each term is determined by multiplying a nonzero constant by the previous term. This constant value is called the **common ratio**. Consider the sequence $\frac{1}{16}, \frac{1}{4}, 1, 4, 16$. This sequence is geometric because the terms share a common ratio. Each term is 4 times as much as the previous term.

$$\frac{1}{16} \quad \frac{1}{4} \quad 1 \quad 4 \quad 16$$

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WatchOut!

Ratios If you find the ratio of a term to the previous term, set up the remaining ratios the same way.

Example 4 Identify Geometric Sequences

Determine whether each sequence is geometric.

a. $-2, 6, -18, 54, \dots$

Find the ratios of the consecutive terms.

$$\frac{6}{-2} = -3 \quad \frac{-18}{6} = -3 \quad \frac{54}{-18} = -3$$

The ratios are the same, so the sequence is geometric.

b. $8, 16, 24, 32, \dots$

$$\frac{16}{8} = 2 \quad \frac{24}{16} = 1.5 \quad \frac{32}{24} = 1.\bar{3}$$

The ratios are not the same, so the sequence is not geometric.

Guided Practice

4A. $-8, 2, -0.5, 0.125, \dots$

4B. $1, 3, 7, 15, \dots$

You can use the common ratio to determine more terms of a geometric sequence.

Example 5 Graph a Geometric Sequence

Consider the geometric sequence $32, 8, 2, \dots$.

a. Find the next three terms of the sequence.

Step 1 Find the value of the common ratio: $\frac{2}{8}$ or $\frac{1}{4}$.

Step 2 To find the next term, multiply the previous term by $\frac{1}{4}$.

Continue multiplying by $\frac{1}{4}$ to find the following terms.

$$2 \quad \frac{1}{2} \quad \frac{1}{8} \quad \frac{1}{32}$$

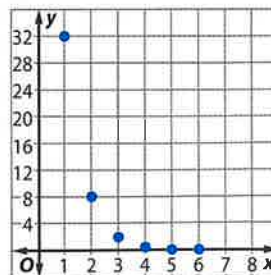
$\underbrace{\hspace{1.5cm}}_{\times \frac{1}{4}}$
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The next three terms are $\frac{1}{2}, \frac{1}{8},$ and $\frac{1}{32}$.

b. Graph the first six terms of the sequence.

Domain: $\{1, 2, 3, 4, 5, 6\}$

Range: $\left\{32, 8, 2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}\right\}$



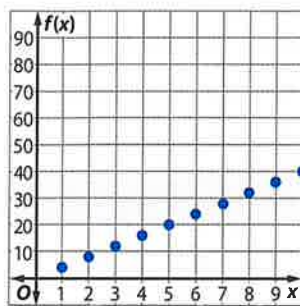
Guided Practice

5. Find the next two terms of $7, 21, 63, \dots$. Then graph the first five terms.

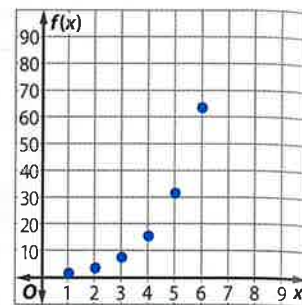
Review Vocabulary

exponential function a function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$

Examine the graph in Example 5. While the graph of an arithmetic sequence is linear, the graph of a geometric sequence is exponential and can be represented by $f(x) = r^x$, where r is the common ratio, $r > 0$, and $r \neq 1$.

Arithmetic

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	4	8	12	16	20	24	28	32	36	40

Geometric

x	1	2	3	4	5	6
$f(x)$	2	4	8	16	32	64

Arithmetic and geometric sequences are functions in which the domain, defined by the term number n , contains the set of or subset of positive integers. The characteristics of arithmetic and geometric sequences can be used to classify sequences.

Example 6 Classify Sequences

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

- a. 16, 24, 36, 54, ...

Check for a common difference.

$$54 - 36 = 18 \quad 36 - 24 = 12 \quad \times$$

Check for a common ratio.

$$\frac{54}{36} = \frac{3}{2} \quad \frac{36}{24} = \frac{3}{2} \quad \frac{24}{16} = \frac{3}{2} \quad \checkmark$$

Because there is a common ratio, the sequence is geometric.

- b. 1, 4, 9, 16, ...

Check for a common difference.

$$16 - 9 = 7 \quad 9 - 4 = 5 \quad \times$$

Check for a common ratio.

$$\frac{16}{9} = 1.\bar{7} \quad \frac{9}{4} = 2.25 \quad \times$$

Because there is no common difference or ratio, the sequence is neither arithmetic nor geometric.

- c. 23, 17, 11, 5, ...

Check for a common difference.

$$5 - 11 = -6 \quad 11 - 17 = -6 \quad 17 - 23 = -6 \quad \checkmark$$

Because there is a common difference, the sequence is arithmetic.

Guided Practice

6A. $\frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, \dots$

6B. $2, -\frac{3}{2}, \frac{9}{8}, -\frac{27}{32}, \dots$

6C. $-4, 4, 5, -5, \dots$





Example 1

Determine whether each sequence is arithmetic. Write *yes* or *no*.

- 1. 8, -2, -12, -22
- 2. -19, -12, -5, 2, 9
- 3. 1, 2, 4, 8, 16
- 4. 0.6, 0.9, 1.2, 1.8, ...

Example 2

Find the next four terms of each arithmetic sequence. Then graph the sequence.

- 5. 6, 18, 30, ...
- 6. 15, 6, -3, ...
- 7. -19, -11, -3, ...
- 8. -26, -33, -40, ...

Example 3

9. FINANCIAL LITERACY Kelly is saving her money to buy a car. She has \$250, and she plans to save \$75 per week from her job as a waitress.

- a. How much will Kelly have saved after 8 weeks?
- b. If the car costs \$2000, how long will it take her to save enough money at this rate?

Example 4

Determine whether each sequence is geometric. Write *yes* or *no*.

- 10. -8, -5, -1, 4, ...
- 11. 4, 12, 36, 108, ...
- 12. 27, 9, 3, 1, ...
- 13. 7, 14, 21, 28, ...

Example 5

Find the next three terms of each geometric sequence. Then graph the sequence.

- 14. 8, 12, 18, 27, ...
- 15. 8, 16, 32, 64, ...
- 16. 250, 50, 10, 2, ...
- 17. 9, -3, 1, $-\frac{1}{3}$, ...

Example 6

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

- 18. 5, 1, 7, 3, 9, ...
- 19. 200, -100, 50, -25, ...
- 20. 12, 16, 20, 24, ...

Practice and Problem Solving

Extra Practice is on page R10.

Example 1

Determine whether each sequence is arithmetic. Write *yes* or *no*.

- 21. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$
- 22. -9, -3, 0, 3, 9
- 23. 14, -5, -19, ...
- 24. $\frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{11}{9}, \dots$

Example 2

Find the next four terms of each arithmetic sequence. Then graph the sequence.

- 25. -4, -1, 2, 5, ...
- 26. 10, 2, -6, -14, ...
- 27. -5, -11, -17, -23, ...
- 28. -19, -2, 15, ...
- 29. $\frac{1}{5}, \frac{4}{5}, \frac{7}{5}, \dots$
- 30. $\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}$

Example 3

31. THEATER There are 28 seats in the front row of a theater. Each successive row contains two more seats than the previous row. If there are 24 rows, how many seats are in the last row of the theater?

32. CCSS SENSE-MAKING Mario began an exercise program to get back in shape. He plans to row 5 minutes on his rowing machine the first day and increase his rowing time by one minute and thirty seconds each day.

- a. How long will he row on the 18th day?
- b. On what day will Mario first row an hour or more?
- c. Is it reasonable for this pattern to continue indefinitely? Explain.



Example 4 Determine whether each sequence is geometric. Write *yes* or *no*.

33. 21, 14, 7, ... 34. 124, 186, 248, ... 35. -27, 18, -12, ...
36. 162, 108, 72, ... 37. $\frac{1}{2}, -\frac{1}{4}, 1, -\frac{1}{2}, \dots$ 38. -4, -2, 0, 2, ...

Example 5 Find the next three terms of the sequence. Then graph the sequence.

39. 0.125, -0.5, 2, ... 40. 18, 12, 8, ... 41. 64, 48, 36, ...
42. 81, 108, 144, ... 43. $\frac{1}{3}, 1, 3, 9, \dots$ 44. 1, 0.1, 0.01, 0.001, ...

Example 6 Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning.

45. 3, 12, 27, 48, ... 46. 1, -2, -5, -8, ... 47. 12, 36, 108, 324, ...
48. $-\frac{2}{5}, -\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, \dots$ 49. $\frac{5}{2}, 3, \frac{7}{2}, 4, \dots$ 50. 6, 9, 14, 21, ...

51. **READING** Sareeta took an 800-page book on vacation. If she was already on page 112 and is going to be on vacation for 8 days, what is the minimum number of pages she needs to read per day to finish the book by the end of her vacation?
52. **DEPRECIATION** Tammy's car is expected to depreciate at a rate of 15% per year. If her car is currently valued at \$24,000, to the nearest dollar, how much will it be worth in 6 years?
53. **CCSS REGULARITY** When a piece of paper is folded onto itself, it doubles in thickness. If a piece of paper that is 0.1 mm thick could be folded 37 times, how thick would it be?

H.O.T. Problems Use Higher-Order Thinking Skills

54. **REASONING** Explain why the sequence 8, 10, 13, 17, 22 is not arithmetic.
55. **OPEN ENDED** Describe a real-life situation that can be represented by an arithmetic sequence with a common difference of 8.
56. **CHALLENGE** The sum of three consecutive terms of an arithmetic sequence is 6. The product of the terms is -42. Find the terms.
57. **ERROR ANALYSIS** Brody and Gen are determining whether the sequence 8, 8, 8, ... is *arithmetic*, *geometric*, *neither*, or *both*. Is either of them correct? Explain your reasoning.

Brody

The sequence has a common difference of 0. The sequence is arithmetic.

Gen

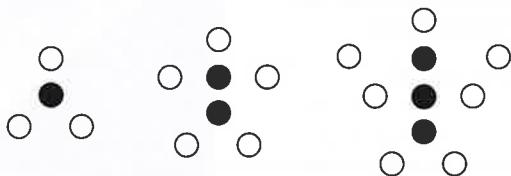
The sequence has a common ratio of 1. The sequence is geometric.

58. **OPEN ENDED** Find a geometric sequence, an arithmetic sequence, and a sequence that is neither geometric nor arithmetic that begins 3, 9,
59. **REASONING** If a geometric sequence has a ratio r such that $|r| < 1$, what happens to the terms as n increases? What would happen to the terms if $|r| \geq 1$?
60. **WRITING IN MATH** Describe what happens to the terms of a geometric sequence when the common ratio is doubled. What happens when it is halved? Explain your reasoning.



Standardized Test Practice

61. **SHORT RESPONSE** Mrs. Aguilar's rectangular bedroom measures 13 feet by 11 feet. She wants to purchase carpet for the bedroom that costs \$2.95 per square foot, including tax. How much will it cost to carpet her bedroom?
62. The pattern of filled circles and white circles below can be described by a relationship between two variables.



Which rule relates w , the number of white circles, to f , the number of dark circles?

- A $w = 3f$ C $w = 2f + 1$
 B $f = \frac{1}{2}w - 1$ D $f = \frac{1}{3}w$

63. **SAT/ACT** Donna wanted to determine the average of her six test scores. She added the scores correctly to get T , but divided by 7 instead of 6. Her average was 12 less than the actual average. Which equation could be used to determine the value of T ?

- F $6T + 12 = 7T$ J $\frac{T}{6} = \frac{T-12}{7}$
 G $\frac{T}{7} = \frac{T-12}{6}$ K $\frac{T}{6} = 12 - \frac{T}{7}$
 H $\frac{T}{7} + 12 = \frac{T}{6}$

64. Find the next term in the geometric sequence

$8, 6, \frac{9}{2}, \frac{27}{8}, \dots$

- A $\frac{11}{8}$ C $\frac{9}{4}$
 B $\frac{27}{16}$ D $\frac{81}{32}$

Spiral Review

Solve each system of equations. (Lesson 9-7)

65. $y = 5$

$y^2 = x^2 + 9$

66. $y - x = 1$

$x^2 + y^2 = 25$

67. $3x = 8y^2$

$8y^2 - 2x^2 = 16$

Write each equation in standard form. State whether the graph of the equation is a parabola, circle, ellipse, or hyperbola. Then graph the equation. (Lesson 9-6)

68. $6x^2 + 6y^2 = 162$

69. $4y^2 - x^2 + 4 = 0$

70. $x^2 + y^2 + 6y + 13 = 40$

Graph each function. (Lesson 8-4)

71. $f(x) = \frac{6}{(x-2)(x+3)}$

72. $f(x) = \frac{-3}{(x-2)^2}$

73. $f(x) = \frac{x^2 - 36}{x + 6}$

74. **HEALTH** A certain medication is eliminated from the bloodstream at a steady rate. It decays according to the equation $y = ae^{-0.1625t}$, where t is in hours. Find the half-life of this substance. (Lesson 7-8)

Skills Review

Write an equation of each line.

75. passes through $(6, 4)$, $m = 0.5$

76. passes through $(2, \frac{1}{2})$, $m = -\frac{3}{4}$

77. passes through $(0, -6)$, $m = 3$

78. passes through $(0, 4)$, $m = \frac{1}{4}$

79. passes through $(1, 3)$ and $(8, -\frac{1}{2})$

80. passes through $(-5, 1)$ and $(5, 16)$



Then

- You determined whether a sequence was arithmetic.

Now

- Use arithmetic sequences.
- Find sums of arithmetic series.

Why?

- In the 18th century, a teacher asked his class of elementary students to find the sum of the counting numbers 1 through 100. A pupil named Karl Gauss correctly answered within seconds, astonishing the teacher. Gauss went on to become a great mathematician.

He solved this problem by using an arithmetic series.



New Vocabulary

arithmetic means
series
arithmetic series
partial sum
sigma notation



Common Core State Standards

Content Standards

A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Mathematical Practices

8 Look for and express regularity in repeated reasoning.

1 Arithmetic Sequences In Lesson 10-1, you used the point-slope form to find a specific term of an arithmetic sequence. It is possible to develop an equation for any term of an arithmetic sequence using the same process.

Consider the arithmetic sequence $a_1, a_2, a_3, \dots, a_n$ in which the common difference is d .

$$\begin{array}{ll} (y - y_1) = m(x - x_1) & \text{Point-slope form} \\ (a_n - a_1) = d(n - 1) & (x, y) = (n, a_n), (x_1, y_1) = (1, a_1), \text{ and } m = d \\ a_n = a_1 + d(n - 1) & \text{Add } a_1 \text{ to each side.} \end{array}$$

You can use this equation to find any term in an arithmetic sequence when you know the first term and the common difference.

KeyConcept n th Term of an Arithmetic Sequence

The n th term a_n of an arithmetic sequence in which the first term is a_1 and the common difference is d is given by the following formula, where n is any natural number.

$$a_n = a_1 + (n - 1)d$$

You will prove this formula in Exercise 80.

Example 1 Find the n th Term

Find the 12th term of the arithmetic sequence 9, 16, 23, 30, ...

Step 1 Find the common difference.

$$16 - 9 = 7 \quad 23 - 16 = 7 \quad 30 - 23 = 7$$

So, $d = 7$.

Step 2 Find the 12th term.

$$\begin{array}{ll} a_n = a_1 + (n - 1)d & \text{\textit{n}th term of an arithmetic sequence} \\ a_{12} = 9 + (12 - 1)(7) & a_1 = 9, d = 7, \text{ and } n = 12 \\ = 9 + 77 \text{ or } 86 & \text{Simplify.} \end{array}$$

Guided Practice

Find the indicated term of each arithmetic sequence.

1A. $a_1 = -4, d = 6, n = 9$

1B. a_{20} for $a_1 = 15, d = -8$



If you are given some terms of an arithmetic sequence, you can write an equation for the n th term of the sequence.



Example 2 Write Equations for the n th Term

Write an equation for the n th term of each arithmetic sequence.

a. 5, -13, -31, ...

$d = -13 - 5$ or -18 ; 5 is the first term.

$$a_n = a_1 + (n - 1)d \quad \text{\textit{n}th term of an arithmetic sequence}$$

$$a_n = 5 + (n - 1)(-18) \quad a_1 = 5 \text{ and } d = -18$$

$$a_n = 5 + (-18n + 18) \quad \text{Distributive Property}$$

$$a_n = -18n + 23 \quad \text{Simplify.}$$

b. $a_5 = 19$, $d = 6$

First, find a_1 .

$$a_n = a_1 + (n - 1)d \quad \text{\textit{n}th term of an arithmetic sequence}$$

$$19 = a_1 + (5 - 1)(6) \quad a_5 = 19, n = 5, \text{ and } d = 6$$

$$19 = a_1 + 24 \quad \text{Multiply.}$$

$$-5 = a_1 \quad \text{Subtract 24 from each side.}$$

Then write the equation.

$$a_n = a_1 + (n - 1)d \quad \text{\textit{n}th term of an arithmetic sequence}$$

$$a_n = -5 + (n - 1)(6) \quad a_1 = -5 \text{ and } d = 6$$

$$a_n = -5 + (6n - 6) \quad \text{Distributive Property}$$

$$a_n = 6n - 11 \quad \text{Simplify.}$$

StudyTip

Checking Solutions Check your solution by using it to determine the first three terms of the sequence.

GuidedPractice

2A. 12, 3, -6, ...

2B. $a_6 = 12$, $d = 8$

Sometimes you are given two terms of a sequence, but they are not consecutive terms of the sequence. The terms between any two nonconsecutive terms of an arithmetic sequence, called **arithmetic means**, can be used to find missing terms of a sequence.



Example 3 Find Arithmetic Means

Find the arithmetic means in the sequence $-8, \underline{?}, \underline{?}, \underline{?}, \underline{?}, 22, \dots$

Step 1 Since there are four terms between the first and last terms given, there are $4 + 2$ or 6 total terms, so $n = 6$.

Step 2 Find d .

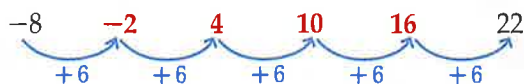
$$a_n = a_1 + (n - 1)d \quad \text{\textit{n}th term of an arithmetic sequence}$$

$$22 = -8 + (6 - 1)d \quad a_1 = -8, a_6 = 22, \text{ and } n = 6$$

$$30 = 5d \quad \text{Distributive Property}$$

$$6 = d \quad \text{Divide each side by 5.}$$

Step 3 Use d to find the four arithmetic means.



The arithmetic means are -2, 4, 10, and 16.

GuidedPractice

3. Find the five arithmetic means between -18 and 36.

ReadingMath

arithmetic mean the average of two or more numbers

arithmetic means the terms between any two nonconsecutive terms of an arithmetic sequence



2 Arithmetic Series A **series** is formed when the terms of a sequence are added. An **arithmetic series** is the sum of the terms of an arithmetic sequence. The sum of the first n terms is called the **partial sum** and is denoted S_n .

KeyConcept Partial Sum of an Arithmetic Series		
Formula	Given	The sum S_n of the first n terms is:
General	a_1 and a_n	$S_n = n\left(\frac{a_1 + a_n}{2}\right)$
Alternate	a_1 and d	$S_n = \frac{n}{2} [2a_1 + (n - 1)d]$

Sometimes a_1 , a_n , or n must be determined before the sum of an arithmetic series can be found. When this occurs, use the formula for the n th term.

Example 4 Use the Sum Formulas

Find the sum of $12 + 19 + 26 + \dots + 180$.

Step 1 $a_1 = 12$, $a_n = 180$, and $d = 19 - 12$ or 7 .

We need to find n before we can use one of the formulas.

$$a_n = a_1 + (n - 1)d \quad \text{\textit{nth term of an arithmetic sequence}}$$

$$180 = 12 + (n - 1)(7) \quad \text{\textit{a}_n = 180, a_1 = 12, and d = 7}$$

$$168 = 7n - 7 \quad \text{\textit{Simplify.}}$$

$$25 = n \quad \text{\textit{Solve for n.}}$$

Step 2 Use either formula to find S_n .

$$S_n = \frac{n}{2} [2a_1 + (n - 1)d] \quad \text{\textit{Sum formula}}$$

$$S_{25} = \frac{25}{2} [2(12) + (25 - 1)(7)] \quad \text{\textit{n = 25, a}_1 = 12, and d = 7}$$

$$S_{25} = 12.5(192) \text{ or } 2400 \quad \text{\textit{Simplify.}}$$

GuidedPractice

Find the sum of each arithmetic series.

4A. $2 + 4 + 6 + \dots + 100$

4B. $n = 16$, $a_n = 240$, and $d = 8$.

You can use a sum formula to find terms of a series.

Example 5 Find the First Three Terms

Find the first three terms of the arithmetic series in which $a_1 = 7$, $a_n = 79$, and $S_n = 430$.

Step 1 Find n .

$$S_n = n\left(\frac{a_1 + a_n}{2}\right) \quad \text{\textit{Sum formula}}$$

$$430 = n\left(\frac{7 + 79}{2}\right) \quad \text{\textit{S}_n = 430, a}_1 = 7, \text{ and } a_n = 79$$

$$430 = n(43) \quad \text{\textit{Simplify.}}$$

$$10 = n \quad \text{\textit{Divide each side by 43.}}$$

WatchOut!

Common Difference

Don't confuse the sign of the common difference in an arithmetic sequence. Check that the rule actually produces the terms of a sequence.



Step 2 Find d .

$$\begin{aligned} a_n &= a_1 + (n - 1)d && n\text{th term of an arithmetic sequence} \\ 79 &= 7 + (10 - 1)d && a_n = 79, a_1 = 7, \text{ and } n = 10 \\ 72 &= 9d && \text{Subtract 7 from each side.} \\ 8 &= d && \text{Divide each side by 9.} \end{aligned}$$

Step 3 Use d to determine a_2 and a_3 .

$$\begin{aligned} a_2 &= 7 + 8 \text{ or } 15 && a_3 = 15 + 8 \text{ or } 23 \\ \text{The first three terms are } &7, 15, \text{ and } 23. \end{aligned}$$

Guided Practice

Find the first three terms of each arithmetic series.

5A. $S_n = 120, n = 8, a_n = 36$

5B. $a_1 = -24, a_n = 288, S_n = 5280$

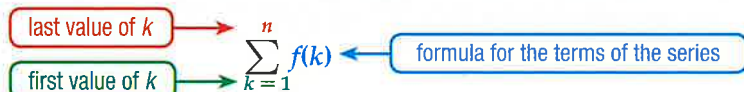
The sum of a series can be written in shorthand by using **sigma notation**.

Reading Math

Sigma Notation The name comes from the Greek letter sigma, which is used in the notation.

Key Concept Sigma Notation

Symbols



Example

$$\begin{aligned} \sum_{k=1}^{12} (4k + 2) &= [4(1) + 2] + [4(2) + 2] + [4(3) + 2] + \cdots + [4(12) + 2] \\ &= 6 + 10 + 14 + \cdots + 50 \end{aligned}$$

Test-Taking Tip

CCSS Perseverance
Sometimes it is necessary to break a problem into parts, solve each part, then combine the solutions of the parts.

Standardized Test Example 6 Use Sigma Notation

Find $\sum_{k=4}^{18} (6k - 1)$.

A 846

B 910

C 975

D 1008

Read the Test Item

You need to find the sum of the series. Find a_1 , a_n , and n .

Solve the Test Item

There are $18 - 4 + 1$ or 15 terms, so $n = 15$.

$$a_1 = 6(4) - 1 \text{ or } 23 \quad a_n = 6(18) - 1 \text{ or } 107$$

Find the sum.

$$S_n = n \left(\frac{a_1 + a_n}{2} \right) \quad \text{Sum formula}$$

$$S_{15} = 15 \left(\frac{23 + 107}{2} \right) \quad n = 15, a_1 = 23, \text{ and } a_n = 107$$

$$S_{15} = 15(65) \text{ or } 975 \quad \text{The correct answer is C.}$$

Guided Practice

6. Find $\sum_{m=9}^{21} (5m + 6)$.

F 972

G 1053

H 1281

J 1701



Example 1 Find the indicated term of each arithmetic sequence.

1. $a_1 = 14, d = 9, n = 11$

2. a_{18} for 12, 25, 38, ...

Example 2 Write an equation for the n th term of each arithmetic sequence.

3. 13, 19, 25, ...

4. $a_5 = -12, d = -4$

Example 3 Find the arithmetic means in each sequence.

5. 6, ?, ?, ?, 42

6. -4, ?, ?, ?, 8

Example 4 Find the sum of each arithmetic series.

7. the first 50 natural numbers

8. $4 + 8 + 12 + \dots + 200$

9. $a_1 = 12, a_n = 188, d = 4$

10. $a_n = 145, d = 5, n = 21$

Example 5 Find the first three terms of each arithmetic series.

11. $a_1 = 8, a_n = 100, S_n = 1296$

12. $n = 18, a_n = 112, S_n = 1098$

Example 6 13. **MULTIPLE CHOICE** Find $\sum_{k=1}^{12} (3k + 9)$.

A 45

C 342

B 78

D 410

Practice and Problem Solving

Extra Practice is on page R10.

Example 1 Find the indicated term of each arithmetic sequence.

14. $a_1 = -18, d = 12, n = 16$

15. $a_1 = -12, n = 66, d = 4$

16. $a_1 = 9, n = 24, d = -6$

17. a_{15} for -5, -12, -19, ...

18. a_{10} for -1, 1, 3, ...

19. a_{24} for 8.25, 8.5, 8.75, ...

Example 2 Write an equation for the n th term of each arithmetic sequence.

20. 24, 35, 46, ...

21. 31, 17, 3, ...

22. $a_9 = 45, d = -3$

23. $a_7 = 21, d = 5$

24. $a_4 = 12, d = 0.25$

25. $a_5 = 1.5, d = 4.5$

26. 9, 2, -5, ...

27. $a_6 = 22, d = 9$

28. $a_8 = -8, d = -2$

29. $a_{15} = 7, d = \frac{2}{3}$

30. -12, -17, -22, ...

31. $a_3 = -\frac{4}{5}, d = \frac{1}{2}$

32. **CCSS STRUCTURE** José averaged 123 total pins per game in his bowling league this season. He is taking bowling lessons and hopes to bring his average up by 8 pins each new season.

a. Write an equation to represent the n th term of the sequence.

b. If the pattern continues, during what season will José average 187 per game?

c. Is it reasonable for this pattern to continue indefinitely? Explain.

Example 3 Find the arithmetic means in each sequence.

33. 24, ?, ?, ?, ?, -1

34. -6, ?, ?, ?, ?, 49

35. -28, ?, ?, ?, ?, 7

36. 84, ?, ?, ?, ?, 39

37. -12, ?, ?, ?, ?, ?, -66

38. 182, ?, ?, ?, ?, ?, 104



Example 4

Find the sum of each arithmetic series.

39. the first 100 even natural numbers 40. the first 200 odd natural numbers
 41. the first 100 odd natural numbers 42. the first 300 even natural numbers
 43. $-18 + (-15) + (-12) + \dots + 66$ 44. $-24 + (-18) + (-12) + \dots + 72$
 45. $a_1 = -16, d = 6, n = 24$ 46. $n = 19, a_n = 154, d = 8$
 47. **CONTESTS** The prizes in a weekly radio contest began at \$150 and increased by \$50 for each week that the contest lasted. If the contest lasted for eleven weeks, how much was awarded in total?

Example 5

Find the first three terms of each arithmetic series.

48. $n = 32, a_n = -86, S_n = 224$ 49. $a_1 = 48, a_n = 180, S_n = 1368$
 50. $a_1 = 3, a_n = 66, S_n = 759$ 51. $n = 28, a_n = 228, S_n = 2982$
 52. $a_1 = -72, a_n = 453, S_n = 6858$ 53. $n = 30, a_n = 362, S_n = 4770$
 54. $a_1 = 19, n = 44, S_n = 9350$ 55. $a_1 = -33, n = 36, S_n = 6372$
 56. **PRIZES** A radio station is offering a total of \$8500 in prizes over ten hours. Each hour, the prize will increase by \$100. Find the amounts of the first and last prize.

Example 6

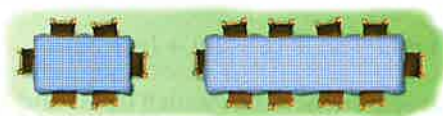
Find the sum of each arithmetic series.

57. $\sum_{k=1}^{16} (4k - 2)$ 58. $\sum_{k=4}^{13} (4k + 1)$ 59. $\sum_{k=5}^{16} (2k + 6)$ 60. $\sum_{k=0}^{12} (-3k + 2)$

61. **FINANCIAL LITERACY** Daniela borrowed some money from her parents. She agreed to pay \$50 at the end of the first month and \$25 more each additional month for 12 months. How much does she pay in total after the 12 months?
 62. **GRAVITY** When an object is in free fall and air resistance is ignored, it falls 16 feet in the first second, an additional 48 feet during the next second, and 80 feet during the third second. How many total feet will the object fall in 10 seconds?

Use the given information to write an equation that represents the n th term in each arithmetic sequence

63. The 100th term of the sequence is 245. The common difference is 13.
 64. The eleventh term of the sequence is 78. The common difference is -9 .
 65. The sixth term of the sequence is -34 . The 23rd term is 119.
 66. The 25th term of the sequence is 121. The 80th term is 506.
 67. **CCSS MODELING** The rectangular tables in a reception hall are often placed end-to-end to form one long table. The diagrams below show the number of people who can sit at each of the table arrangements.



- a. Make drawings to find the next three numbers as tables are added one at a time to the arrangement.
 b. Write an equation representing the n th number in this pattern.
 c. Is it possible to have seating for exactly 100 people with such an arrangement? Explain.



68. **PERFORMANCE** A certain company pays its employees according to their performance. Belinda is paid a flat rate of \$200 per week plus \$24 for every unit she completes. If she earned \$512 in one week, how many units did she complete?
69. **SALARY** Terry currently earns \$28,000 per year. If Terry expects a \$4000 increase in salary every year, after how many years will he have a salary of \$100,000 per year?
70. **SPORTS** While training for cross country, Silvia plans to run 3 miles per day for the first week, and then increase the distance by a half mile each of the following weeks.
- Write an equation to represent the n th term of the sequence.
 - If the pattern continues, during which week will she be running 10 miles per day?
 - Is it reasonable for this pattern to continue indefinitely? Explain.
71. **MULTIPLE REPRESENTATIONS** Consider $\sum_{k=1}^x (2k + 2)$.
- Tabular** Make a table of the partial sums of the series for $1 \leq k \leq 10$.
 - Graphical** Graph $(k, \text{partial sum})$.
 - Graphical** Graph $f(x) = x^2 + 3x$ on the same grid.
 - Verbal** What do you notice about the two graphs?
 - Analytical** What conclusions can you make about the relationship between quadratic functions and the sum of arithmetic series?
 - Algebraic** Find the arithmetic series that relates to $g(x) = x^2 + 8x$.

Find the value of x .

72. $\sum_{k=3}^x (6k - 5) = 928$

73. $\sum_{k=5}^x (8k + 2) = 1032$

H.O.T. Problems Use Higher-Order Thinking Skills

74. **CCSS CRITIQUE** Eric and Juana are determining the formula for the n th term for the sequence $-11, -2, 7, 16, \dots$. Is either of them correct? Explain your reasoning.

Eric

$$\begin{aligned} d &= 16 - 7 \text{ or } 9, a_1 = -11 \\ a_n &= -11 + (n - 1)9 \\ &= 9n - 20 \end{aligned}$$

Juana

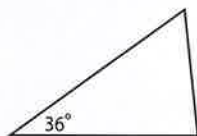
$$\begin{aligned} d &= 16 - 7 \text{ or } 9, a_1 = -11 \\ a_n &= 9n - 11 \end{aligned}$$

75. **REASONING** If a is the third term in an arithmetic sequence, b is the fifth term, and c is the eleventh term, express c in terms of a and b .
76. **CHALLENGE** There are three arithmetic means between a and b in an arithmetic sequence. The average of the arithmetic means is 16. What is the average of a and b ?
77. **CHALLENGE** Find S_n for $(x + y) + (x + 2y) + (x + 3y) + \dots$
78. **OPEN ENDED** Write an arithmetic series with 8 terms and a sum of 324.
79. **WRITING IN MATH** Compare and contrast arithmetic sequences and series.
80. **PROOF** Prove the formula for the n th term of an arithmetic sequence.
81. **PROOF** Derive a sum formula that does not include a_1 .
82. **PROOF** Derive the Alternate Sum Formula using the General Sum Formula.



Standardized Test Practice

83. **SAT/ACT** The measures of the angles of a triangle form an arithmetic sequence. If the measure of the smallest angle is 36° , what is the measure of the largest angle?



- A 54°
 B 75°
 C 84°
 D 90°
 E 97°

84. The area of a triangle is $\frac{1}{2}q^2 - 8$ and the height is $q + 4$. Which expression best describes the triangle's base?

- F $(q + 1)$ H $(q - 3)$
 G $(q + 2)$ J $(q - 4)$.

85. The expression $1 + \sqrt{2} + \sqrt[3]{3}$ is equivalent to

- A $\sum_{k=1}^3 k^{\frac{1}{k}}$ C $\sum_{k=1}^3 k^{-k}$
 B $\sum_{k=1}^3 k^k$ D $\sum_{k=1}^3 \sqrt{k}$

86. **SHORT RESPONSE** Trevor can type a 200-word essay in 6 hours. Minya can type the same essay in $4\frac{1}{2}$ hours. If they work together, how many hours will it take them to type the essay?

Spiral Review

Determine whether each sequence is arithmetic. Write *yes* or *no*. (Lesson 10-1)

87. $-6, 4, 14, 24, \dots$

88. $2, \frac{7}{5}, \frac{4}{5}, \frac{1}{5}, \dots$

89. $10, 8, 5, 1, \dots$

Solve each system of inequalities by graphing. (Lesson 9-7)

90. $x + 2y > 1$

$x^2 + y^2 \leq 25$

91. $x + y \leq 2$

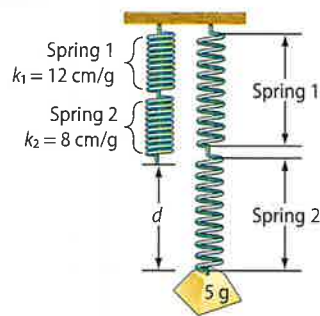
$4x^2 - y^2 \geq 4$

92. $x^2 + y^2 \geq 4$

$4y^2 + 9x^2 \leq 36$

93. **PHYSICS** The distance a spring stretches is related to the mass attached to the spring. This is represented by $d = km$, where d is the distance, m is the mass, and k is the spring constant. When two springs with spring constants k_1 and k_2 are attached in a series, the resulting spring constant k is found by the equation $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$. (Lesson 8-6)

- a. If one spring with constant of 12 centimeters per gram is attached in a series with another spring with constant of 8 centimeters per gram, find the resultant spring constant.
- b. If a 5-gram object is hung from the series of springs, how far will the springs stretch? Is this answer reasonable in this context?



Graph each function. State the domain and range. (Lesson 7-1)

94. $f(x) = \frac{2}{3}(2^x)$

95. $f(x) = 4^x + 3$

96. $f(x) = 2\left(\frac{1}{3}\right)^x - 1$

Skills Review

Solve each equation. Round to the nearest ten-thousandth.

97. $5^x = 52$

98. $4^{3p} = 10$

99. $3^{n+2} = 14.5$

100. $16^{d-4} = 3^{3-d}$

Then

- You determined whether a sequence was geometric.

Now

- Use geometric sequences.
- Find sums of geometric series.

Why?

- Julian sees a new band at a concert. He e-mails a link for the band's Web site to five of his friends. They each forward the link to five of their friends. The link is forwarded again following the same pattern. How many people will receive the link on the eighth round of e-mails?

**New Vocabulary**

geometric means
geometric series

**Common Core State Standards****Content Standards**

A.SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

Mathematical Practices

- Look for and express regularity in repeated reasoning.

1 Geometric Sequences As with arithmetic sequences, there is a formula for the n th term of a geometric sequence. This formula can be used to determine any term of the sequence.

KeyConcept n th Term of a Geometric Sequence

The n th term a_n of a geometric sequence in which the first term is a_1 and the common ratio is r is given by the following formula, where n is any natural number.

$$a_n = a_1 r^{n-1}$$

You will prove this formula in Exercise 68.

Real-World Example 1 Find the n th Term

MUSIC If the pattern continues, how many e-mails will be sent in the eighth round?

Understand We need to determine the number of forwarded e-mails on the eighth round. Five e-mails were sent on the first round. Each of the five recipients sent five e-mails on the second round, and so on.

Plan This is a geometric sequence, and the common ratio is 5. Use the formula for the n th term of a geometric sequence.

Solve $a_n = a_1 r^{n-1}$ n th term of a geometric sequence

$$a_8 = 5(5)^{8-1} \quad a_1 = 5, r = 5, \text{ and } n = 8$$

$$a_8 = 5(78,125) \text{ or } 390,625 \quad 5^7 = 78,125$$

Check Write out the first eight terms by multiplying by the common ratio.

5, 25, 125, 625, 3125, 15,625, 78,125, 390,625

There will be 390,625 e-mails sent on the 8th round.

Guided Practice

- E-MAILS** Shira receives a joke in an e-mail that asks her to forward it to four of her friends. She forwards it, then each of her friends forwards it to four of their friends and so on. If the pattern continues, how many people will receive the e-mail on the ninth round of forwarding?





Math HistoryLink

Archytas (428–347 B.C.) Geometric sequences, or geometric progressions, were first studied by the Greek mathematician Archytas. His studies of these sequences came from his interest in music and octaves.

If you are given some of the terms of a geometric sequence, you can determine an equation for finding the n th term of the sequence.

Example 2 Write an Equation for the n th Term



Write an equation for the n th term of each geometric sequence.

a. 0.5, 2, 8, 32, ...

$r = 8 \div 2$ or 4; 0.5 is the first term.

$$a_n = a_1 r^{n-1} \quad \text{\textit{n}th term of a geometric sequence}$$

$$a_n = 0.5(4)^{n-1} \quad a_1 = 0.5 \text{ and } r = 4$$

b. $a_4 = 5$ and $r = 6$

Step 1 Find a_1 .

$$a_n = a_1 r^{n-1} \quad \text{\textit{n}th term of a geometric sequence}$$

$$5 = a_1(6^{4-1}) \quad a_n = 5, r = 6, \text{ and } n = 4$$

$$5 = a_1(216) \quad \text{Evaluate the power.}$$

$$\frac{5}{216} = a_1 \quad \text{Divide each side by 216.}$$

Step 2 Write the equation.

$$a_n = a_1 r^{n-1} \quad \text{\textit{n}th term of a geometric sequence}$$

$$a_n = \frac{5}{216}(6)^{n-1} \quad a_1 = \frac{5}{216} \text{ and } r = 6$$

GuidedPractice

Write an equation for the n th term of each geometric sequence.

2A. $-0.25, 2, -16, 128, \dots$

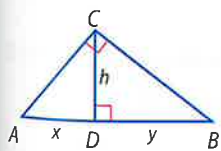
2B. $a_3 = 16, r = 4$

Like arithmetic means, **geometric means** are the terms between two nonconsecutive terms of a geometric sequence. The common ratio r can be used to find the geometric means.

ReadingMath

Geometric Means

A geometric mean can also be represented geometrically. In the figure below, h is the geometric mean between x and y .



Example 3 Find Geometric Means



Find three geometric means between 2 and 1250.

Step 1 Since there are three terms between the first and last term, there are $3 + 2$ or 5 total terms, so $n = 5$.

Step 2 Find r .

$$a_n = a_1 r^{n-1} \quad \text{\textit{n}th term of a geometric sequence}$$

$$1250 = 2r^{5-1} \quad a_n = 1250, a_1 = 2, \text{ and } n = 5$$

$$625 = r^4 \quad \text{Divide each side by 2.}$$

$$\pm 5 = r \quad \text{Take the 4th root of each side.}$$

Step 3 Use r to find the four arithmetic means.

$$\begin{array}{ccccccc} 2 & \xrightarrow{10} & 50 & \xrightarrow{250} & 1250 & \text{or} & 2 & \xrightarrow{-10} & 50 & \xrightarrow{-250} & 1250 \\ \times 5 & & \times 5 & & \times 5 & & \times -5 & & \times -5 & & \times -5 \end{array}$$

The geometric means are 10, 50, and 250 or $-10, 50,$ and -250 .

GuidedPractice

3. Find four geometric means between 0.5 and 512.



2 Geometric Series A **geometric series** is the sum of the terms of a geometric sequence. The sum of the first n terms of a series is denoted S_n . You can use either of the following formulas to find the partial sum S_n of the first n terms of a geometric series.

KeyConcept Partial Sum of a Geometric Series

Given	The sum S_n of the first n terms is:
a_1 and n	$S_n = \frac{a_1 - a_1 r^n}{1 - r}, r \neq 1$
a_1 and a_n	$S_n = \frac{a_1 - a_n r}{1 - r}, r \neq 1$

Real-World Example 4 Find the Sum of a Geometric Series

MUSIC Refer to the beginning of the lesson. If the pattern continues, what is the total number of e-mails sent in the eight rounds?

Five e-mails are sent in the first round and there are 8 rounds of e-mails. So, $a_1 = 5$, $r = 5$ and $n = 8$.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \quad \text{Sum formula}$$

$$S_8 = \frac{5 - 5 \cdot 5^8}{1 - 5} \quad a_1 = 5, r = 5, \text{ and } n = 8$$

$$S_8 = \frac{-1,953,120}{-4} \quad \text{Simplify the numerator and denominator.}$$

$$S_8 = 488,280 \quad \text{Divide.}$$

There will be 488,280 e-mails sent after 8 rounds.

GuidedPractice

Find the sum of each geometric series.

4A. $a_1 = 2, n = 10, r = 3$

4B. $a_1 = 2000, a_n = 125, r = \frac{1}{2}$

As with arithmetic series, sigma notation can also be used to represent geometric series.

WatchOut!

Sigma Notation Notice in Example 5 that you are being asked to evaluate the sum from the 3rd term to the 10th term.

Example 5 Sum in Sigma Notation

Find $\sum_{k=3}^{10} 4(2)^k - 1$.

Find a_1, r , and n . In the first term, $k = 3$ and $a_1 = 4 \cdot 2^3 - 1$ or 16. The base of the exponential function is r , so $r = 2$. There are $10 - 3 + 1$ or 8 terms, so $n = 8$.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \quad \text{Sum formula}$$

$$= \frac{16 - 16(2)^8}{1 - 2} \quad a_1 = 16, r = 2, \text{ and } n = 8$$

$$= 4080 \quad \text{Use a calculator.}$$

GuidedPractice

Find each sum.

5A. $\sum_{k=4}^{12} \frac{1}{4} \cdot 3^{k-1}$

5B. $\sum_{k=2}^9 \frac{2}{3} \cdot 4^{k-1}$



You can use the formula for the sum of a geometric series to help find a particular term of the series.



Example 6 Find the First Term of a Series

Find a_1 in a geometric series for which $S_n = 13,116$, $n = 7$, and $r = 3$.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \quad \text{Sum formula}$$

$$13,116 = \frac{a_1 - a_1(3^7)}{1 - 3} \quad S_n = 13,116, r = 3, \text{ and } n = 7$$

$$13,116 = \frac{a_1(1 - 3^7)}{1 - 3} \quad \text{Distributive Property}$$

$$13,116 = \frac{-2186a_1}{-2} \quad \text{Subtract.}$$

$$13,116 = 1093a_1 \quad \text{Simplify.}$$

$$12 = a_1 \quad \text{Divide each side by 1093.}$$

Guided Practice

6. Find a_1 in a geometric series for which $S_n = -26,240$, $n = 8$, and $r = -3$.

Check Your Understanding

= Step-by-Step Solutions begin on page R14.



Example 1

1. **CCSS REGULARITY** Dean is making a family tree for his grandfather. He was able to trace many generations. If Dean could trace his family back 10 generations, starting with his parents how many ancestors would there be?

Example 2

Write an equation for the n th term of each geometric sequence.

2. 2, 4, 8, ...

3. 18, 6, 2, ...

4. -4, 16, -64, ...

5. $a_2 = 4$, $r = 3$

6. $a_6 = \frac{1}{8}$, $r = \frac{3}{4}$

7. $a_2 = -96$, $r = -8$

Example 3

Find the geometric means of each sequence.

8. 0.25, $\frac{?}{?}$, $\frac{?}{?}$, 64

9. 0.20, $\frac{?}{?}$, $\frac{?}{?}$, 125

Example 4

10. **GAMES** Miranda arranges some rows of dominoes so that after she knocks over the first one, each domino knocks over two more dominoes when it falls. If there are ten rows, how many dominoes does Miranda use?

Example 5

Find the sum of each geometric series.

11. $\sum_{k=1}^6 3(4)^{k-1}$

12. $\sum_{k=1}^8 4\left(\frac{1}{2}\right)^{k-1}$

Example 6

Find a_1 for each geometric series described.

13. $S_n = 85\frac{5}{16}$, $r = 4$, $n = 6$

14. $S_n = 91\frac{1}{12}$, $r = 3$, $n = 7$

15. $S_n = 1020$, $a_n = 4$, $r = \frac{1}{2}$

16. $S_n = 121\frac{1}{3}$, $a_n = \frac{1}{3}$, $r = \frac{1}{3}$



- Example 1** 17. **WEATHER** Heavy rain in Brieanne's town caused the river to rise. The river rose three inches the first day, and each day after rose twice as much as the previous day. How much did the river rise in five days?

Find a_n for each geometric sequence.

18. $a_1 = 2400, r = \frac{1}{4}, n = 7$ 19. $a_1 = 800, r = \frac{1}{2}, n = 6$
 20. $a_1 = \frac{2}{9}, r = 3, n = 7$ 21. $a_1 = -4, r = -2, n = 8$
 22. **BIOLOGY** A certain bacteria grows at a rate of 3 cells every 2 minutes. If there were 260 cells initially, how many are there after 21 minutes?

- Example 2** Write an equation for the n th term of each geometric sequence.

23. $-3, 6, -12, \dots$ 24. $288, -96, 32, \dots$ 25. $-1, 1, -1, \dots$
 26. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \dots$ 27. $8, 2, \frac{1}{2}, \dots$ 28. $12, -16, \frac{64}{3}, \dots$
 29. $a_3 = 28, r = 2$ 30. $a_4 = -8, r = 0.5$ 31. $a_6 = 0.5, r = 6$
 32. $a_3 = 8, r = \frac{1}{2}$ 33. $a_4 = 24, r = \frac{1}{3}$ 34. $a_4 = 80, r = 4$

- Example 3** Find the geometric means of each sequence.

35. $810, \underline{\quad}, \underline{\quad}, \underline{\quad}, 10$ 36. $640, \underline{\quad}, \underline{\quad}, \underline{\quad}, 2.5$
 37. $\frac{7}{2}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \frac{56}{81}$ 38. $\frac{729}{64}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \frac{324}{9}$
 39. Find two geometric means between 3 and 375.
 40. Find two geometric means between 16 and -2 .

- Example 4** 41. **CCSS PERSEVERANCE** A certain water filtration system can remove 70% of the contaminants each time a sample of water is passed through it. If the same water is passed through the system four times, what percent of the original contaminants will be removed from the water sample?

Find the sum of each geometric series.

42. $a_1 = 36, r = \frac{1}{3}, n = 8$ 43. $a_1 = 16, r = \frac{1}{2}, n = 9$
 44. $a_1 = 240, r = \frac{3}{4}, n = 7$ 45. $a_1 = 360, r = \frac{4}{3}, n = 8$
 46. **VACUUMS** A vacuum claims to pick up 80% of the dirt every time it is run over the carpet. Assuming this is true, what percent of the original amount of dirt is picked up after the seventh time the vacuum is run over the carpet?

- Example 5** Find the sum of each geometric series.

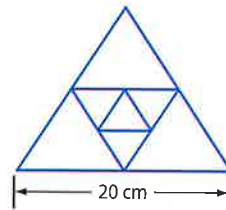
47. $\sum_{k=1}^7 4(-3)^{k-1}$ 48. $\sum_{k=1}^8 (-3)(-2)^{k-1}$ 49. $\sum_{k=1}^9 (-1)(4)^{k-1}$ 50. $\sum_{k=1}^{10} 5(-1)^{k-1}$

- Example 6** Find a_1 for each geometric series described.

51. $S_n = -2912, r = 3, n = 6$ 52. $S_n = -10,922, r = 4, n = 7$
 53. $S_n = 1330, a_n = 486, r = \frac{3}{2}$ 54. $S_n = 4118, a_n = 128, r = \frac{2}{3}$
 55. $a_n = 1024, r = 8, n = 5$ 56. $a_n = 1875, r = 5, n = 7$



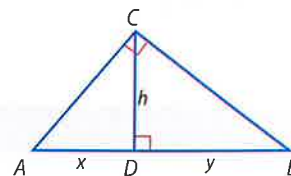
57. **SCIENCE** One minute after it is released, a gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will it rise in 5 minutes?
58. **CHEMISTRY** Radon has a half-life of about 4 days. This means that about every 4 days, half of the mass of radon decays into another element. How many grams of radon remain from an initial 60 grams after 4 weeks?
59. **CCSS REASONING** A virus goes through a computer, infecting the files. If one file was infected initially and the total number of files infected doubles every minute, how many files will be infected in 20 minutes?
60. **GEOMETRY** In the figure, the sides of each equilateral triangle are twice the size of the sides of its inscribed triangle. If the pattern continues, find the sum of the perimeters of the first eight triangles.



61. **PENDULUMS** The first swing of a pendulum travels 30 centimeters. If each subsequent swing travels 95% as far as the previous swing, find the total distance traveled by the pendulum after the 30th swing.
62. **PHONE CHAINS** A school established a phone chain in which every staff member calls two other staff members to notify them when the school closes due to weather. The first round of calls begins with the superintendent calling both principals. If there are 94 total staff members and employees at the school, how many rounds of calls are there?
63. **TELEVISIONS** High Tech Electronics advertises a weekly installment plan for the purchase of a popular brand of high definition television. The buyer pays \$5 at the end of the first week, \$5.50 at the end of the second week, \$6.05 at the end of the third week, and so on for one year. (Assume that 1 year = 52 weeks.)
- What will the payments be at the end of the 10th, 20th, and 40th weeks?
 - Find the total cost of the TV.
 - Why is the cost found in part **b** not entirely accurate?

H.O.T. Problems Use Higher-Order Thinking Skills

64. **PROOF** Derive the General Sum Formula using the Alternate Sum Formula.
65. **PROOF** Derive a sum formula that does not include a_1 .
66. **OPEN ENDED** Write a geometric series for which $r = \frac{3}{4}$ and $n = 6$.
67. **REASONING** Explain how $\sum_{k=1}^{10} 3(2)^k - 1$ needs to be altered to refer to the same series if $k = 1$ changes to $k = 0$. Explain your reasoning.
68. **PROOF** Prove the formula for the n th term of a geometric sequence.
69. **CHALLENGE** The fifth term of a geometric sequence is $\frac{1}{27}$ th of the eighth term. If the ninth term is 702, what is the eighth term?
70. **CHALLENGE** Use the fact that h is the geometric mean between x and y in the figure at the right to find h^4 in terms of x and y .
71. **OPEN ENDED** Write a geometric series with 6 terms and a sum of 252.
72. **WRITING IN MATH** How can you classify a sequence? Explain your reasoning.



Standardized Test Practice

73. Which of the following is closest to $\sqrt[3]{7.32}$?

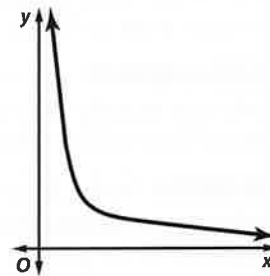
- A 1.8
- B 1.9
- C 2.0
- D 2.1

74. The first term of a geometric series is 5, and the common ratio is -2 . How many terms are in the series if its sum is -6825 ?

- F 5
- G 9
- H 10
- J 12

75. **SHORT RESPONSE** Danette has a savings account. She withdraws half of the contents every year. After 4 years, she has \$2000 left. How much did she have in the savings account originally?

76. **SAT/ ACT** The curve below could be part of the graph of which function?



- A $y = \sqrt{x}$
- B $y = x^2 - 5x + 4$
- C $y = -x + 20$
- D $y = \log x$
- E $xy = 4$

Spiral Review

77. **MONEY** Elena bought a high-definition LCD television at the electronics store. She paid \$200 immediately and \$75 each month for a year and a half. How much did Elena pay in total for the TV? (Lesson 10-2)

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. (Lesson 10-1)

78. $\frac{1}{10}, \frac{3}{5}, \frac{7}{20}, \frac{17}{20}, \dots$

79. $-\frac{7}{25}, -\frac{13}{50}, -\frac{6}{25}, -\frac{11}{50}, \dots$

80. $-\frac{22}{3}, -\frac{68}{9}, -\frac{208}{27}, -\frac{632}{81}, \dots$

Find the center and radius of each circle. Then graph the circle. (Lesson 9-3)

81. $(x - 3)^2 + (y - 1)^2 = 25$

82. $(x + 3)^2 + (y + 7)^2 = 81$

83. $(x - 3)^2 + (y + 7)^2 = 50$

84. Suppose y varies jointly as x and z . Find y when $x = 9$ and $z = -5$, if $y = -90$ when $z = 15$ and $x = -6$. (Lesson 8-5)

85. **SHOPPING** A certain store found that the number of customers who will attend a sale can be modeled by $N = 125\sqrt[3]{100Pt}$, where N is the number of customers expected, P is the percent of the sale discount, and t is the number of hours the sale will last. Find the number of customers the store should expect for a sale that is 50% off and will last four hours. (Lesson 6-4)

Skills Review

Evaluate each expression if $a = -2$, $b = \frac{1}{3}$, and $c = -12$.

86. $\frac{3ab}{c}$

87. $\frac{a - c}{a + c}$

88. $\frac{a^3 - c}{b^2}$

89. $\frac{c + 3}{ab}$

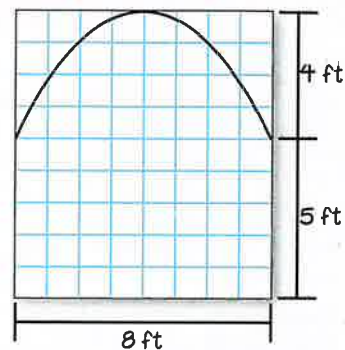


Algebra Lab

Area Under a Curve



The Morgans are renovating the outside of their house so that there is an archway above the front entrance. Mr. Morgan made a scale drawing of the archway in which each line on the grid paper represents one foot of the actual archway. Mrs. Morgan modeled the shape of the top with the quadratic equation $y = -0.25x^2 + 3x$.



Activity

Find the area of the opening under the archway.

Method 1

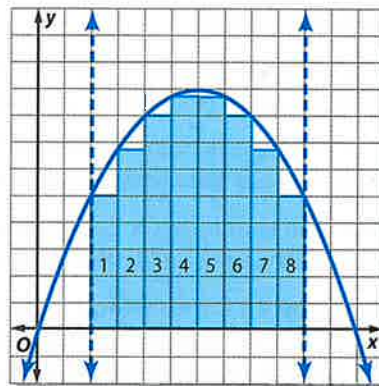
Step 1 Make a table of values for $y = -0.25x^2 + 3x$. Then graph the equation.

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	2.75	5	6.75	8	8.75	9	8.75	8	6.75	5	2.75	0

Step 2 Divide the figure into regions.

To estimate the area inside the archway, you can divide the archway into rectangles as shown in blue.

Because the left and right sides of the archway are 5 feet high and $y = 5$ when $x = 2$ and when $x = 10$, the opening of the entrance extends from $x = 2$ to $x = 10$.



Step 3 Find the area of the regions.

Rectangle	1	2	3	4	5	6	7	8
Width (ft)	1	1	1	1	1	1	1	1
Height (ft)	5	6.75	8	8.75	8.75	8	6.75	5
Area (ft ²)	5	6.75	8	8.75	8.75	8	6.75	5

The approximate area of the archway is the sum of the areas of the rectangles.

$$5 + 6.75 + 8 + 8.75 + 8.75 + 8 + 6.75 + 5 = 57 \text{ ft}^2$$

(continued on the next page)

Algebra Lab

Area Under a Curve *Continued*

Method 2

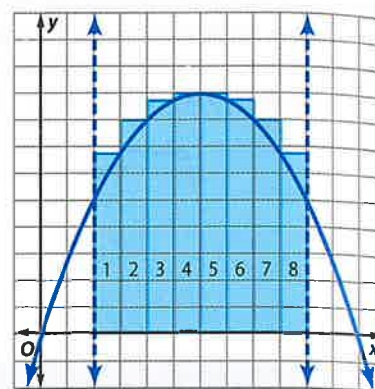
Step 1 Draw a second graph of the equation and divide into regions. Divide the archway into rectangles as shown in blue.

Step 2 Find the area of the regions.

Rectangle	1	2	3	4	5	6	7	8
Width (ft)	1	1	1	1	1	1	1	1
Height (ft)	6.75	8	8.75	9	9	8.75	8	6.75
Area (ft ²)	6.75	8	8.75	9	9	8.75	8	6.75

The approximate area of the archway is the sum of the areas of the rectangles.

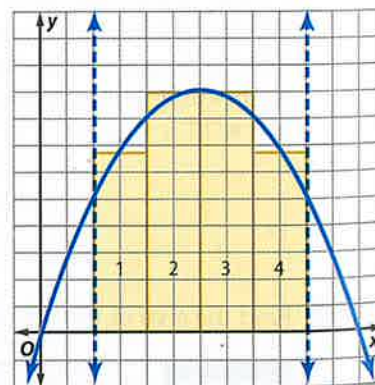
$$6.75 + 8 + 8.75 + 9 + 9 + 8.75 + 8 + 6.75 = 65 \text{ ft}^2$$



Both Method 1 and Method 2 illustrate how to approximate the area under a curve over a specified interval.

Analyze the Results

1. Is the area of the regions calculated using Method 1 greater than or less than the actual area of the archway? Explain your reasoning.
2. Is the area of the regions calculated using Method 2 greater than or less than the actual area of the archway? Explain your reasoning.
3. Compare the area estimates for both methods. How could you find the best estimate for the area inside the archway? Explain your reasoning.
4. The diagram shows a third method for finding an estimate of the area of the archway. Is this estimate for the area greater than or less than the actual area? How does this estimate compare to the other two estimates of the area?



Exercises

Estimate the area described by any method. Make a table of values, draw graphs with rectangles, and make a table for the areas of the rectangles. Compare each estimate to the actual area.

5. the area under the curve for $y = -x^2 + 4$, from $x = -2$ to $x = 2$, and above the x -axis
6. the area under the curve for $y = x^3$, from $x = 0$ to $x = 4$, and above the x -axis
7. the area under the curve for $y = x^2$, from $x = -3$ to $x = 3$, and above the x -axis

Infinite Geometric Series



Then

- You found sums of finite geometric series.

Now

- Find sums of infinite geometric series.
- Write repeating decimals as fractions.

Why?

- With their opponent on the 10-yard line, the defense is penalized half the distance to the goal, placing the ball on the 5-yard line. If they continue to be penalized in this way, where will the ball eventually be placed? Will they ever reach the goal line? How many total penalty yards will the defense have incurred? These questions can be answered by looking at infinite geometric series.

New Vocabulary
 infinite geometric series
 convergent series
 divergent series
 infinity

Common Core State Standards

- Mathematical Practices**
- Attend to precision.
 - Look for and express regularity in repeated reasoning.

1 Infinite Geometric Series An **infinite geometric series** has an infinite number of terms. A series that has a sum is a **convergent series**, because its sum converges to a specific value. A series that does not have a sum is a **divergent series**.

When you evaluated the sum S_n of an infinite geometric series for the first n terms, you were finding the partial sum of the series. It is also possible to find the sum of an entire series. In the application above, it seems that the ball will eventually reach the goal line, and the defense will be penalized a total of 10 yards. This value is the actual sum of the infinite series $5 + 2.5 + 1.25 + \dots$. The graph of S_n for $1 \leq n \leq 10$ is shown on the left below. As n increases, S_n approaches 10.

KeyConcept Convergent and Divergent Series

Convergent Series		Divergent Series	
Words	The sum approaches a finite value.	Words	The sum does not approach a finite value.
Ratio	$ r < 1$	Ratio	$ r \geq 1$
Example	$5 + 2.5 + 1.25 + \dots$	Example	$\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \dots$
<p>The graph shows the sum of terms for a convergent series. The x-axis is labeled 'Term' and ranges from 0 to 9. The y-axis is labeled 'Sum of Terms' and ranges from 0 to 9. The points start at (1, 5) and approach a horizontal asymptote at y = 10 as the term number increases.</p>		<p>The graph shows the sum of terms for a divergent series. The x-axis is labeled 'Term' and ranges from 0 to 9. The y-axis is labeled 'Sum of Terms' and ranges from 0 to 1.8. The points start at (1, 0.0625) and increase rapidly, showing that the sum does not approach a finite value.</p>	

Example 1 Convergent and Divergent Series

Determine whether each infinite geometric series is *convergent* or *divergent*.

a. $54 + 36 + 24 + \dots$

Find the value of r .

$r = \frac{36}{54}$ or $\frac{2}{3}$; since $-1 < \frac{2}{3} < 1$, the series is convergent.



b. $8 + 12 + 18 + \dots$

$r = \frac{12}{8}$ or 1.5; since $1.5 > 1$, the series is divergent.

Guided Practice

1A. $2 + 3 + 4.5 + \dots$

1B. $100 + 50 + 25 + \dots$

StudyTip

Absolute Value Recall that $|r| < 1$ means $-1 < r < 1$.

When $|r| < 1$, the value of r^n will approach 0 as n increases. Therefore, the partial sums of the infinite geometric series will approach $\frac{a_1 - a_1(0)}{1 - r}$ or $\frac{a_1}{1 - r}$.

KeyConcept Sum of an Infinite Geometric Series

The sum S of an infinite geometric series with $|r| < 1$ is given by

$$S = \frac{a_1}{1 - r}$$

If $|r| \geq 1$, the series has no sum.

When an infinite geometric series is divergent, $|r| \geq 1$ and the series has no sum because the absolute value of r^n will increase infinitely as n increases.

The table at the right shows the partial sums for the divergent series $4 + 16 + 64 + \dots$. As n increases, S_n increases rapidly without limit.

n	S_n
5	1364
10	1,398,100
15	1,431,655,764

Example 2 Sum of an Infinite Series

Find the sum of each infinite series, if it exists.

Determine whether each infinite geometric series is *convergent* or *divergent*.

a. $\frac{2}{3} + \frac{6}{15} + \frac{18}{75} + \dots$

Step 1 Find the value of r to determine if the sum exists.

$$r = \frac{6}{15} \div \frac{2}{3} \text{ or } \frac{3}{5} \quad \text{Divide consecutive terms.}$$

Since $|\frac{3}{5}| < 1$, the sum exists.

Step 2 Use the formula to find the sum.

$$S = \frac{a_1}{1 - r} \quad \text{Sum formula}$$

$$= \frac{\frac{2}{3}}{1 - \frac{3}{5}} \quad a_1 = \frac{2}{3} \text{ and } r = \frac{3}{5}$$

$$= \frac{2}{3} \div \frac{2}{5} \text{ or } \frac{5}{3} \quad \text{Simplify.}$$

b. $6 + 9 + 13.5 + 20.25 + \dots$

$r = \frac{9}{6}$ or 1.5; since $|1.5| \geq 1$, the series diverges and the sum does not exist.

Guided Practice

2A. $4 - 2 + 1 - 0.5 + \dots$

2B. $16 + 20 + 25 + \dots$

StudyTip

Convergence and Divergence A series converges when the absolute value of a term is smaller than the absolute value of the previous term. An infinite arithmetic series will always be divergent.



Sigma notation can be used to represent infinite series. If a sequence goes to **infinity**, it continues without end. The infinity symbol ∞ is placed above the \sum to indicate that a series is infinite.



Example 3 Infinite Series in Sigma Notation

Find $\sum_{k=1}^{\infty} 18\left(\frac{4}{5}\right)^{k-1}$.

$$\begin{aligned}
 S &= \frac{a_1}{1-r} && \text{Sum formula} \\
 &= \frac{18}{1-\frac{4}{5}} && a_1 = 18 \text{ and } r = \frac{4}{5} \\
 &= \frac{18}{\frac{1}{5}} \text{ or } 90 && \text{Simplify.}
 \end{aligned}$$

Guided Practice

3. Find $\sum_{k=1}^{\infty} 12\left(\frac{3}{4}\right)^{k-1}$.

2 Repeating Decimals A repeating decimal is the sum of an infinite geometric series. For instance, $0.\overline{45} = 0.454545\dots$ or $0.45 + 0.0045 + 0.000045 + \dots$. The formula for the sum of an infinite series can be used to convert the decimal to a fraction.



Example 4 Write a Repeating Decimal as a Fraction

Write $0.\overline{63}$ as a fraction.

Method 1 Use the sum of an infinite series.

$$\begin{aligned}
 0.\overline{63} &= 0.63 + 0.0063 + \dots \\
 &= \frac{63}{100} + \frac{63}{10,000} + \dots \\
 S &= \frac{a_1}{1-r} && \text{Sum formula} \\
 &= \frac{\frac{63}{100}}{1-\frac{1}{100}} && a_1 = \frac{63}{100} \text{ and } r = \frac{1}{100} \\
 &= \frac{63}{99} \text{ or } \frac{7}{11} && \text{Simplify.}
 \end{aligned}$$

Method 2 Use algebraic properties.

$$\begin{aligned}
 x &= 0.\overline{63} && \text{Let } x = 0.\overline{63}. \\
 x &= 0.636363\dots && \text{Write as a repeating decimal.} \\
 100x &= 63.636363\dots && \text{Multiply each side by 100.} \\
 99x &= 63 && \text{Subtract } x \text{ from } 100x \text{ and } 0.\overline{63} \text{ from } 63.\overline{63}. \\
 x &= \frac{63}{99} \text{ or } \frac{7}{11} && \text{Divide each side by 99.}
 \end{aligned}$$

Guided Practice

4. Write $0.\overline{21}$ as a fraction.

Problem-Solving Tip

CCSS Sense-Making In many cases, it is possible to solve a problem in more than one way. Use the method with which you are most comfortable.

Study Tip

Repeating Decimals Every repeating decimal is a rational number and can be written as a fraction.



Example 1 Determine whether each infinite geometric series is *convergent* or *divergent*.

1. $16 - 8 + 4 - \dots$
2. $32 - 48 + 72 - \dots$
3. $0.5 + 0.7 + 0.98 + \dots$
4. $1 + 1 + 1 + \dots$

Example 2 Find the sum of each infinite series, if it exists.

5. $440 + 220 + 110 + \dots$
6. $520 + 130 + 32.5 + \dots$
7. $\frac{1}{4} + \frac{3}{8} + \frac{9}{16} + \dots$
8. $\frac{32}{9} + \frac{16}{3} + 8 + \dots$
9. **CCSS SENSE-MAKING** A certain drug has a half-life of 8 hours after it is administered to a patient. What percent of the drug is still in the patient's system after 24 hours?

Example 3 Find the sum of each infinite series, if it exists.

10. $\sum_{k=1}^{\infty} 5 \cdot 4^{k-1}$
11. $\sum_{k=1}^{\infty} (-2) \cdot (0.5)^{k-1}$
12. $\sum_{k=1}^{\infty} 3 \cdot \left(\frac{4}{5}\right)^{k-1}$
13. $\sum_{k=1}^{\infty} \frac{1}{2} \cdot \left(\frac{3}{4}\right)^{k-1}$

Example 4 Write each repeating decimal as a fraction.

14. $0.\overline{35}$
15. $0.\overline{642}$

Practice and Problem Solving

Extra Practice is on page R10

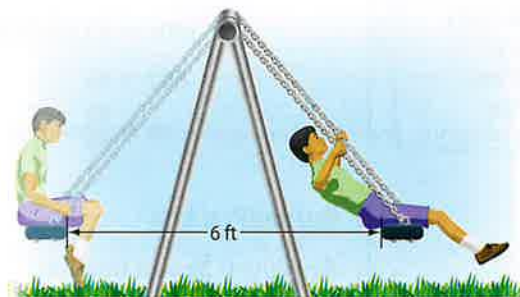
Example 1 Determine whether each infinite geometric series is *convergent* or *divergent*.

16. $21 + 63 + 189 + \dots$
17. $480 + 360 + 270 + \dots$
18. $\frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \dots$
19. $\frac{5}{6} + \frac{10}{9} + \frac{40}{27} + \dots$
20. $0.1 + 0.01 + 0.001 + \dots$
21. $0.008 + 0.08 + 0.8 + \dots$

Example 2 Find the sum of each infinite series, if it exists.

22. $18 + 21.6 + 25.92 + \dots$
23. $-3 - 4.2 - 5.88 - \dots$
24. $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \dots$
25. $\frac{12}{5} + \frac{6}{5} + \frac{3}{5} + \dots$
26. $21 + 14 + \frac{28}{3} + \dots$
27. $32 + 40 + 50 + \dots$

28. **SWINGS** If Kerry does not push any harder after his initial swing, the distance traveled per swing will decrease by 10% with each swing. If his initial swing traveled 6 feet, find the total distance traveled when he comes to rest.



Example 3

Find the sum of each infinite series, if it exists.

29. $\sum_{k=1}^{\infty} \frac{4}{3} \cdot \left(\frac{5}{4}\right)^{k-1}$

30. $\sum_{k=1}^{\infty} \frac{1}{4} \cdot 3^{k-1}$

31. $\sum_{k=1}^{\infty} \frac{5}{3} \cdot \left(\frac{3}{7}\right)^{k-1}$

32. $\sum_{k=1}^{\infty} \frac{2}{3} \cdot \left(\frac{4}{3}\right)^{k-1}$

33. $\sum_{k=1}^{\infty} \frac{8}{3} \cdot \left(\frac{5}{6}\right)^{k-1}$

34. $\sum_{k=1}^{\infty} \frac{1}{8} \cdot \left(\frac{1}{12}\right)^{k-1}$

Example 4

Write each repeating decimal as a fraction.

35. $0.3\overline{21}$

36. $0.1\overline{45}$

37. $2.\overline{18}$

38. $4.\overline{96}$

39. $0.1\overline{214}$

40. $0.4\overline{336}$

41. **FANS** A fan is running at 10 revolutions per second. After it is turned off, its speed decreases at a rate of 75% per second. Determine the number of revolutions completed by the fan after it is turned off.
42. **CCSS PRECISION** Kamiko deposited \$5000 into an account at the beginning of the year. The account earns 8% interest each year.
- How much money will be in the account after 20 years? (*Hint:* Let $5000(1 + 0.08)^1$ represent the end of the first year.)
 - Is this series *convergent* or *divergent*? Explain.
43. **RECHARGEABLE BATTERIES** A certain rechargeable battery is advertised to recharge back to 99.9% of its previous capacity with every charge. If its initial capacity is 8 hours of life, how many total hours should the battery last?

Find the sum of each infinite series, if it exists.

44. $\frac{7}{5} + \frac{21}{20} + \frac{63}{80} + \dots$

45. $\frac{15}{4} + \frac{5}{2} + \frac{5}{3} + \dots$

46. $-\frac{16}{9} + \frac{4}{3} - 1 + \dots$

47. $\frac{15}{8} + \frac{5}{2} + \frac{10}{3} + \dots$

48. $\frac{21}{16} + \frac{7}{4} + \frac{7}{3} + \dots$

49. $-\frac{18}{7} + \frac{12}{7} - \frac{8}{7} + \dots$

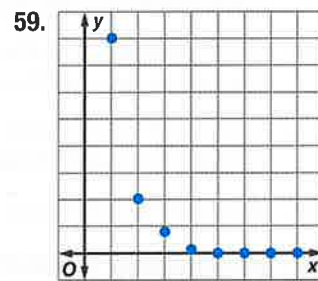
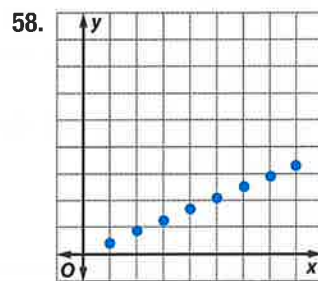
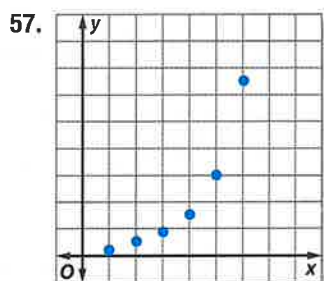
50. **MULTIPLE REPRESENTATIONS** In this problem, you will use a square of paper that is at least 8 inches on a side.
- Concrete** Let the square be one unit. Cut away one half of the square. Call this piece Term 1. Next, cut away one half of the remaining sheet of paper. Call this piece Term 2. Continue cutting the remaining paper in half and labeling the pieces with a term number as long as possible. List the fractions represented by the pieces.
 - Numerical** If you could cut the squares indefinitely, you would have an infinite series. Find the sum of the series.
 - Verbal** How does the sum of the series relate to the original square of paper?
51. **PHYSICS** In a physics experiment, a steel ball on a flat track is accelerated, and then allowed to roll freely. After the first minute, the ball has rolled 120 feet. Each minute the ball travels only 40% as far as it did during the preceding minute. How far does the ball travel?
52. **PENDULUMS** A pendulum travels 12 centimeters on its first swing and 95% of that distance on each swing thereafter. Find the total distance traveled by the pendulum when it comes to rest.
53. **TOYS** If a rubber ball can bounce back to 95% of its original height, what is the total vertical distance that it will travel if it is dropped from an elevation of 30 feet?
54. **CARS** During a maintenance inspection, a tire is removed from a car and spun on a diagnostic machine. When the machine is turned off, the spinning tire completes 20 revolutions the first second and 98% of the revolutions each additional second. How many revolutions does the tire complete before it stops spinning?



55. **ECONOMICS** A state government decides to stimulate its economy by giving \$500 to every adult. The government assumes that everyone who receives the money will spend 80% on consumer goods and that the producers of these goods will in turn spend 80% on consumer goods. How much money is generated for the economy for every \$500 that the government provides?

56. **SCIENCE MUSEUM** An exhibit at a science museum offers visitors the opportunity to experiment with the motion of an object on a spring. One visitor pulls the object down and lets it go. The object travels 1.2 feet upward before heading back the other way. Each time the object changes direction, it decreases its distance by 20% when compared to the previous direction. Find the total distance traveled by the object.

Match each graph with its corresponding description.



- a. converging geometric series
b. diverging geometric series
c. converging arithmetic series
d. diverging arithmetic series

H.O.T. Problems Use Higher-Order Thinking Skills

60. **ERROR ANALYSIS** Emmitt and Austin are asked to find the sum of $1 - 1 + 1 - \dots$. Is either of them correct? Explain your reasoning.

Emmitt

The sum is 0 because the sum of each pair of terms in the sequence is 0.

Austin

There is no sum because $|r| \geq 1$, and the series diverges.

61. **PROOF** Derive the formula for the sum of an infinite geometric series.
62. **CHALLENGE** For what values of b does $3 + 9b + 27b^2 + 81b^3 + \dots$ have a sum?
63. **REASONING** When does an infinite geometric series have a sum, and when does it not have a sum? Explain your reasoning.
64. **CCSS ARGUMENTS** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

If the absolute value of a term of any geometric series is greater than the absolute value of the previous term, then the series is divergent.

65. **OPEN ENDED** Write an infinite series with a sum that converges to 9.
66. **OPEN ENDED** Write $3 - 6 + 12 - \dots$ using sigma notation in two different ways.
67. **WRITING IN MATH** Explain why an arithmetic series is always divergent.





You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as n increases, a_n approaches 0. The value that the terms of a sequence approach, in this case 0, is called the **limit** of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

You can use a TI-83/84 Plus graphing calculator to help find the limits of infinite sequences.

Activity

Find the limit of the geometric sequence $1, \frac{1}{4}, \frac{1}{16}, \dots$

Step 1 Enter the sequence.

The formula for this sequence is $a_n = \left(\frac{1}{4}\right)^{n-1}$.

- Position the cursor on L1 in the **STAT EDIT 1: Edit...** screen and enter the formula **seq(N,N,1,10,1)**. This generates the values 1, 2, ..., 10 of the index N .

KEYSTROKES: **[STAT]** **[ENTER]** **[↑]** **[2nd]** **[STAT]** **[▶]** **5** **[X,T,θ,n]** **,**
[X,T,θ,n] **,** **1** **,** **10** **,** **1** **)** **[ENTER]**

- Position the cursor on L2 and enter the formula **seq((1/4)^(N-1),N,1,10,1)**. This generates the first ten terms of the sequence.

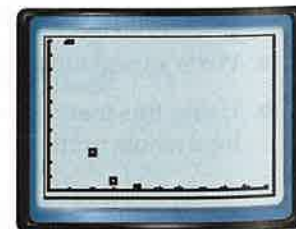
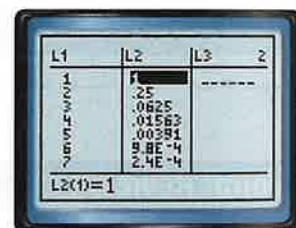
KEYSTROKES: **[▶]** **[↑]** **[2nd]** **[STAT]** **[▶]** **5** **(** **1** **÷** **4** **)** **^** **(** **[X,T,θ,n]**
- **1** **)** **,** **[X,T,θ,n]** **,** **1** **,** **10** **,** **1** **)** **[ENTER]**

Notice that as n increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for $n \geq 6$ the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

Step 2 Graph the sequence.

Use **STAT PLOT** to graph the sequence. Use L1 as the **Xlist** and L2 as the **Ylist**.

The graph also shows that, as n increases, the terms approach 0. In fact, for $n \geq 3$, the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.



[0, 10] scl: 1 by [0, 1] scl: 0.1

Exercises

Find the limit of each sequence.

1. $a_n = \left(\frac{1}{3}\right)^n$

2. $a_n = \left(-\frac{1}{3}\right)^n$

3. $a_n = 5^n$

4. $a_n = \frac{1}{n^2}$

5. $a_n = \frac{3^n}{3^n + 1}$

6. $a_n = \frac{n^2}{n + 2}$

CHAPTER 10 Mid-Chapter Quiz

Lessons 10-1 through 10-4

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Explain your reasoning. (Lesson 10-1)

1. $5, -3, -12, -22, -33, \dots$

2. $\frac{1}{5}, \frac{7}{10}, \frac{6}{5}, \frac{17}{10}, \frac{11}{5}, \dots$

3. **HOUSING** Laura is a real estate agent. She needs to sell 15 houses in 6 months. (Lesson 10-1)

a. By the end of the first 2 months she has sold 4 houses. If she sells 2 houses each month for the rest of the 6 months, will she meet her goal? Explain.

b. If she has sold 5 houses by the end of the first month, how many will she have to sell on average each month in order to meet her goal?

4. **GEOMETRY** The figures below show a pattern of filled squares and white squares. (Lesson 10-1)

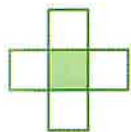


Figure 1

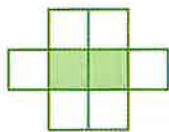


Figure 2

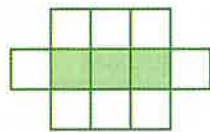


Figure 3

a. Write an equation representing the n th number in this pattern where n is the number of white squares.

b. Is it possible to have exactly 84 white squares in an arrangement? Explain.

Find the indicated term of each arithmetic sequence.

(Lesson 10-2)

5. $a_1 = 10, d = -5, n = 9$

6. $a_1 = -8, d = 4, n = 99$

Find the sum of each arithmetic series. (Lesson 10-2)

7. $-15 + (-11) + (-7) + \dots + 53$

8. $a_1 = -12, d = 8, n = 22$

9. $\sum_{k=11}^{50} (-3k + 1)$

10. **MULTIPLE CHOICE** What is the sum of the first 50 odd numbers? (Lesson 10-2)

A 2550

B 2500

C 2499

D 2401

Find the indicated term for each geometric sequence.

(Lesson 10-3)

11. $a_2 = 8, r = 2, a_8 = ?$

12. $a_3 = 0.5, r = 8, a_{10} = ?$

13. **MULTIPLE CHOICE** What are the geometric means of the sequence below? (Lesson 10-3)

$0.5, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, 2048$

F 512.375, 1024.25, 1536.125

G 683, 1365.5, 2048

H 2, 8, 32

J 4, 32, 256

14. **INCOME** Peter works for a house building company for 4 months per year. He starts out making \$3000 per month. At the end of each month, his salary increases by 5%. How much money will he make in those 4 months? (Lesson 10-3)

Evaluate the sum of each geometric series. (Lesson 10-3)

15. $\sum_{k=1}^8 3 \cdot 2^{k-1}$

16. $\sum_{k=1}^9 4 \cdot (-1)^{k-1}$

17. $\sum_{k=1}^{20} -2\left(\frac{2}{3}\right)^{k-1}$

Find the sum of each infinite series, if it exists. (Lesson 10-4)

18. $\sum_{n=1}^{\infty} 9 \cdot 2^{n-1}$

19. $\sum_{n=1}^{\infty} (4) \cdot (0.5)^{n-1}$

20. $\sum_{n=1}^{\infty} 12 \cdot \left(\frac{2}{3}\right)^{n-1}$

Then

- You explored compositions of functions.

Now

- 1 Recognize and use special sequences.
- 2 Iterate functions.

Why?

- The female honeybee is produced after the queen mates with a male, so the female has two parents, a male and a female. The male honeybee, however, is produced by the queen's unfertilized eggs and thus has only one parent, a female. The family tree for the honeybee follows a special sequence.



Generation	1	2	3	4	5	6
Ancestors	1	1	2	3	5	8



New Vocabulary

Fibonacci sequence
recursive sequence
explicit formula
recursive formula
iteration



Common Core State Standards

Mathematical Practices

- 6 Attend to precision.
8 Look for and express regularity in repeated reasoning.

1 Special Sequences Notice that every term in the list of ancestors is the sum of the previous two terms. This special sequence is called the **Fibonacci sequence**, and it is found in many places in nature. The Fibonacci sequence is an example of a **recursive sequence**. In a recursive sequence, each term is determined by one or more of the previous terms.

The formulas you have used for sequences thus far have been explicit formulas. An **explicit formula** gives a_n as a function of n , such as $a_n = 3n + 1$. The formula that describes the Fibonacci sequence, $a_n = a_{n-2} + a_{n-1}$, is a **recursive formula**, which means that every term will be determined by one or more of the previous terms. An initial term must be given in a recursive formula.

Key Concept Recursive Formulas for Sequences

Arithmetic Sequence $a_n = a_{n-1} + d$, where d is the common difference

Geometric Sequence $a_n = r \cdot a_{n-1}$, where r is the common ratio

Example 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = -3$ and $a_{n+1} = 4a_n - 2$, if $n \geq 1$.

$$a_{n+1} = 4a_n - 2 \quad \text{Recursive formula}$$

$$a_{1+1} = 4a_1 - 2 \quad n = 1$$

$$a_2 = 4(-3) - 2 \text{ or } -14 \quad a_1 = -3$$

$$a_3 = 4(-14) - 2 \text{ or } -58 \quad a_2 = -14$$

$$a_4 = 4(-58) - 2 \text{ or } -234 \quad a_3 = -58$$

$$a_5 = 4(-234) - 2 \text{ or } -938 \quad a_4 = -234$$

The first five terms of the sequence are $-3, -14, -58, -234$, and -938 .

Guided Practice

1. Find the first five terms of the sequence in which $a_1 = 8$ and $a_{n+1} = -3a_n + 6$, if $n \geq 1$.



In order to find a recursive formula, first determine the initial term. Then evaluate the pattern to generate the later terms. The recursive formula that generates a sequence does not include the value of the initial term.



StudyTip

Sequences and Recursive Formulas Like arithmetic and geometric sequences, recursive formulas define functions in which the domain is the set of positive integers, represented by the term number n .

Example 2 Write Recursive Formulas

Write a recursive formula for each sequence.

a. 2, 10, 18, 26, 34, ...

Step 1 Determine whether the sequence is arithmetic or geometric.
The sequence is arithmetic because each term after the first can be found by adding a common difference.

Step 2 Find the common difference.

$$d = 10 - 2 \text{ or } 8$$

Step 3 Write the recursive formula.

$$a_n = a_{n-1} + d \quad \text{Recursive formula for arithmetic sequence}$$

$$a_n = a_{n-1} + 8 \quad d = 8$$

A recursive formula for the sequence is $a_n = a_{n-1} + 8$, $a_1 = 2$.

b. 16, 56, 196, 686, 2401, ...

Step 1 Determine whether the sequence is arithmetic or geometric.
The sequence is geometric because each term after the first can be found after multiplying by a common ratio.

Step 2 Find the common ratio.

$$r = \frac{56}{16} \text{ or } 3.5$$

Step 3 Write the recursive formula.

$$a_n = r \cdot a_{n-1} \quad \text{Recursive formula for geometric sequence}$$

$$a_n = 3.5a_{n-1} \quad r = 3.5$$

A recursive formula for the sequence is $a_n = 3.5a_{n-1}$, $a_1 = 16$.

c. $a_4 = 108$ and $r = 3$

Step 1 Determine whether the sequence is arithmetic or geometric.
Because r is given, the sequence is geometric.

Step 2 Write the recursive formula.

$$a_n = r \cdot a_{n-1} \quad \text{Recursive formula for geometric sequence}$$

$$a_n = 3a_{n-1} \quad r = 3$$

A recursive formula for the sequence is $a_n = 3a_{n-1}$, $a_1 = 4$.

Guided Practice

Write a recursive formula for each sequence.

2A. 8, 20, 50, 125, 312.5, ... 2B. 8, 17, 26, 35, 44, ... 2C. $a_3 = 16$ and $r = 4$





Real-WorldLink

In 2008, the average credit card debt for college students was about \$3173.

Source: USA Today

Real-World Example 3 Use a Recursive Formula

FINANCIAL LITERACY Nate had \$15,000 in credit card debt when he graduated from college. The balance increased by 2% each month due to interest, and Nate could only make payments of \$400 per month. Write a recursive formula for the balance of his account each month. Then determine the balance after five months.

Step 1 Write the recursive formula.

Let a_n represent the balance of the account in the n th month. The initial balance is \$15,000. After one month, **interest is added** and **a payment is made**.

$$\begin{array}{r}
 \text{initial} \\
 \text{balance}
 \end{array}
 +
 \begin{array}{r}
 \text{balance} \\
 \text{times } 2\%
 \end{array}
 -
 \begin{array}{r}
 \text{monthly} \\
 \text{payment}
 \end{array}$$

$$\begin{array}{r}
 a_2 = \\
 a_2 =
 \end{array}
 \begin{array}{r}
 a_1 \\
 1.02a_1 - 400
 \end{array}
 +
 \begin{array}{r}
 (a_1 \times 0.02) \\
 - 400
 \end{array}$$

The formula is $a_n = 1.02a_{n-1} - 400$.

Step 2 Find the next five terms.

$a_n = 1.02a_{n-1} - 400$	Recursive formula
$a_2 = (15,000 \times 1.02) - 400$ or 14,900	$a_1 = 15,000$
$a_3 = (14,900 \times 1.02) - 400$ or 14,798	$a_2 = 14,900$
$a_4 = (14,798 \times 1.02) - 400$ or 14,693.96	$a_3 = 14,798$
$a_5 = (14,693.96 \times 1.02) - 400$ or 14,587.84	$a_4 = 14,693.96$
$a_6 = (14,587.84 \times 1.02) - 400$ or 14,479.60	$a_5 = 14,587.84$

After the fifth month, the balance will be \$14,479.60.

GuidedPractice

3. Write a recursive formula for a \$10,000 debt, at 2.5% interest per month, with a \$600 monthly payment. Then find the first five balances.

Review Vocabulary

Composition of functions

A function is performed, and then a second function is performed on the result of the first function.

2 Iteration **Iteration** is the process of repeatedly composing a function with itself. Consider x_0 . The first iterate is $f(x_0)$, the second iterate is $f(f(x_0))$, the third iterate is $f(f(f(x_0)))$, and so on.

Iteration can be used to recursively generate a sequence. Start with the initial value x_0 . Let $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, and so on.

Example 4 Iterate a Function

Find the first three iterates x_1 , x_2 , and x_3 of $f(x) = 5x + 4$ for an initial value of $x_0 = 2$.

$x_1 = f(x_0)$	Iterate the function.
$= 5(2) + 4$ or 14	$x_0 = 2$
$x_2 = f(x_1)$	Iterate the function.
$= 5(14) + 4$ or 74	$x_1 = 14$
$x_3 = f(x_2)$	Iterate the function.
$= 5(74) + 4$ or 374	$x_2 = 74$

The first three iterates are 14, 74, and 374.

GuidedPractice

4. Find the first three iterates x_1 , x_2 , and x_3 of $f(x) = -3x + 8$ for an initial value of $x_0 = 6$.



**Example 1** Find the first five terms of each sequence described.

1. $a_1 = 16, a_{n+1} = a_n + 4$

2. $a_1 = -3, a_{n+1} = a_n + 8$

3. $a_1 = 5, a_{n+1} = 3a_n + 2$

4. $a_1 = -4, a_{n+1} = 2a_n - 6$

Example 2 Write a recursive formula for each sequence.

5. 3, 8, 18, 38, 78, ...

6. 5, 14, 41, 122, 365, ...

Example 37. **FINANCING** Ben financed a \$1500 rowing machine to help him train for the college rowing team. He could only make a \$100 payment each month, and his bill increased by 1% due to interest at the end of each month.

a. Write a recursive formula for the balance owed at the end of each month.

b. Find the balance owed after the first four months.

c. How much interest has accumulated after the first six months?

Example 4 Find the first three iterates of each function for the given initial value.

8. $f(x) = 5x + 2, x_0 = 8$

9. $f(x) = -4x + 2, x_0 = 5$

10. $f(x) = 6x + 3, x_0 = -4$

11. $f(x) = 8x - 4, x_0 = -6$

Practice and Problem Solving

Extra Practice is on page R10.

Example 1**CCSS PERSEVERANCE** Find the first five terms of each sequence described.

12. $a_1 = 10, a_{n+1} = 4a_n + 1$

13. $a_1 = -9, a_{n+1} = 2a_n + 8$

14. $a_1 = 12, a_{n+1} = a_n + n$

15. $a_1 = -4, a_{n+1} = 2a_n + n$

16. $a_1 = 6, a_{n+1} = 3a_n - n$

17. $a_1 = -2, a_{n+1} = 5a_n + 2n$

18. $a_1 = 7, a_2 = 10, a_{n+2} = 2a_n + a_{n+1}$

19. $a_1 = 4, a_2 = 5, a_{n+2} = 4a_n - 2a_{n+1}$

20. $a_1 = 4, a_2 = 3x, a_n = a_{n-1} + 4a_{n-2}$

21. $a_1 = 3, a_2 = 2x, a_n = 4a_{n-1} - 3a_{n-2}$

22. $a_1 = 2, a_2 = x + 3, a_n = a_{n-1} + 6a_{n-2}$

23. $a_1 = 1, a_2 = x, a_n = 3a_{n-1} + 6a_{n-2}$

Example 2 Write a recursive formula for each sequence.

24. 16, 10, 7, 5.5, 4.75, ...

25. 32, 12, 7, 5.75, ...

26. 4, 15, 224, 50, 175, ...

27. 1, 2, 9, 730, ...

28. 9, 33, 129, 513, ...

29. 480, 128, 40, 18, ...

30. 393, 132, 45, 16, ...

31. 68, 104, 176, 320, ...

Example 332. **FINANCIAL LITERACY** Mr. Edwards and his company deposit \$20,000 into his retirement account at the end of each year. The account earns 8% interest before each deposit.

a. Write a recursive formula for the balance in the account at the end of each year.

b. Determine how much is in the account at the end of each of the first 8 years.

Example 4

Find the first three iterates of each function for the given initial value.

33. $f(x) = 12x + 8, x_0 = 4$

34. $f(x) = -9x + 1, x_0 = -6$

35. $f(x) = -6x + 3, x_0 = 8$

36. $f(x) = 8x + 3, x_0 = -4$

37. $f(x) = -3x^2 + 9, x_0 = 2$

38. $f(x) = 4x^2 + 5, x_0 = -2$

39. $f(x) = 2x^2 - 5x + 1, x_0 = 6$

40. $f(x) = -0.25x^2 + x + 6, x_0 = 8$


41. $f(x) = x^2 + 2x + 3, x_0 = \frac{1}{2}$

42. $f(x) = 2x^2 + x + 1, x_0 = -\frac{1}{2}$



43. **FRACTALS** Consider the figures at the right. The number of blue triangles increases according to a specific pattern.



- Write a recursive formula for the number of blue triangles in the sequence of figures.
 - How many blue triangles will be in the sixth figure?
44. **FINANCIAL LITERACY** Miguel's monthly car payment is \$234.85. The recursive formula $b_n = 1.005b_{n-1} - 234.85$ describes the balance left on the loan after n payments. Find the balance of the \$10,000 loan after each of the first eight payments.
45. **CONSERVATION** Suppose a lake is populated with 10,000 fish. A year later, 80% of the fish have died or been caught, and the lake is replenished with 10,000 new fish. If the pattern continues, will the lake eventually run out of fish? If not, will the population of the lake converge to any particular value? Explain.
46. **GEOMETRY** Consider the pattern at the right.
- 
- Write a sequence of the total number of triangles in the first six figures.
 - Write a recursive formula for the number of triangles.
 - How many triangles will be in the tenth figure?
47. **SPREADSHEETS** Consider the sequence with $x_0 = 20,000$ and $f(x) = 0.3x + 5000$.
- Enter x_0 in cell A1 of your spreadsheet. Enter " $= (0.3)*(A1) + 5000$ " in cell A2. What answer does it provide?
 - Copy cell A2, highlight cells A3 through A70, and paste. What do you notice about the sequence?
 - How do spreadsheets help analyze recursive sequences?
48. **VIDEO GAMES** The final monster in Helena's video game has 100 health points. During the final battle, the monster regains 10% of its health points after every 10 seconds. If Helena can inflict damage to the monster that takes away 10 health points every 10 seconds without getting hurt herself, will she ever kill the monster? If so, when?

H.O.T. Problems Use Higher-Order Thinking Skills

49. **CCSS CRITIQUE** Marcus and Armando are finding the first three iterates of $f(x) = 5x - 3$ for an initial value of $x_0 = 4$. Is either of them correct? Explain.

Marcus

$$f(4) = 5(4) - 3 \text{ or } 17$$

$$f(17) = 5(17) - 3 \text{ or } 82$$

The first three iterates are 4, 17, and 82.

Armando

$$f(4) = 5(4) - 3 \text{ or } 17$$

$$f(17) = 5(17) - 3 \text{ or } 82$$

$$f(82) = 5(82) - 3 \text{ or } 407$$

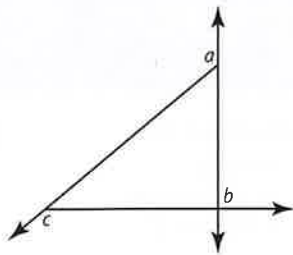
The first three iterates are 17, 82, and 407.

50. **CHALLENGE** Find a recursive formula for 5, 23, 98, 401, ...
51. **REASONING** Is the statement "If the first three terms of a sequence are identical, then the sequence is not recursive" sometimes, always, or never true? Explain your reasoning.
52. **OPEN ENDED** Write a function for which the first three iterates are 9, 19, and 39.
53. **WRITING IN MATH** Why is it useful to represent a sequence with an explicit or recursive formula?



Standardized Test Practice

54. **GEOMETRY** In the figure shown, $a + b + c = ?$



- A 180°
 B 270°
 C 360°
 D 450°
55. **EXTENDED RESPONSE** Bill launches a model rocket from ground level. The rocket's height h in meters is given by the equation $h = -4.9t^2 + 56t$, where t is the time in seconds after the launch.
- What is the maximum height the rocket will reach?
 - How long after it is launched will the rocket reach its maximum height? Round to the nearest tenth of a second.
 - How long after it is launched will the rocket land? Round to the nearest tenth of a second.

56. Which of the following is true about the graphs of $y = 3(x - 4)^2 + 5$ and $y = 3(x + 4)^2 + 5$?

- F Their vertices are maximums.
 G The graphs have the same shape with different vertices.
 H The graphs have different shapes with different vertices.
 J One graph has a vertex that is a maximum, while the other graph has a vertex that is a minimum.

57. Which factors could represent the length times the width?

$$A = 16x^4 - 25y^2$$

- A $(4x - 5y)(4x - 5y)$
 B $(4x + 5y)(4x - 5y)$
 C $(4x^2 - 5y)(4x^2 + 5y)$
 D $(4x^2 + 5y)(4x^2 + 5y)$

Spiral Review

Write each repeating decimal as a fraction. (Lesson 10-4)

58. $0.\overline{7}$

59. $5.\overline{126}$

60. $6.\overline{259}$

61. **SPORTS** Adrahan is training for a marathon, about 26 miles. He begins by running 2 miles. Then, when he runs every other day, he runs one and a half times the distance he ran the time before. (Lesson 10-3)
- Write the first five terms of a sequence describing his training schedule.
 - When will he exceed 26 miles in one run?
 - When will he have run 100 total miles?

State whether the events are *independent* or *dependent*. (Lesson 0-6)

62. tossing a penny and rolling a number cube
 63. choosing first and second place in an academic competition

Skills Review

Find each product.

64. $(y + 4)(y + 3)$

65. $(x - 2)(x + 6)$

66. $(a - 8)(a + 5)$

67. $(4h + 5)(h + 7)$

68. $(9p - 1)(3p - 2)$

69. $(2g + 7)(5g - 8)$





When a payment is made on a loan, part of the payment is used to cover the interest that has accumulated since the last payment. The rest is used to reduce the *principal*, or original amount of the loan. This process is called *amortization*. You can use a spreadsheet to analyze the payments, interest, and balance on a loan. A table that shows this kind of information is called an *amortization schedule*.

Example

LOANS Gloria just bought a new computer for \$695. The store is letting her make monthly payments of \$60.78 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest on the remaining balance will be $\frac{9\%}{12}$ or 0.75%. You can find the balance after a payment by multiplying the balance after the previous payment by $1 + 0.0075$ or 1.0075 and then subtracting 60.78.

In a spreadsheet, the column of numbers represents the number of payments, and Column B shows the balance. Enter the interest rate and monthly payment in cells in Column A so that they can be easily updated if the information changes.

The spreadsheet at the right shows the formulas for the balances after each of the first six payments. After six months, Gloria still owes \$355.28.

	A	B	C
1	Interest Rate	=695*(1+A2)-A5	
2	0.0075	=B1*(1+A2)-A5	
3		=B2*(1+A2)-A5	
4	Monthly payment	=B3*(1+A2)-A5	
5	60.78	=B4*(1+A2)-A5	
6		=B5*(1+A2)-A5	
7			

Model and Analyze

- Let b_n be the balance left on Gloria's loan after n months. Write an equation relating b_n and b_{n+1} .
- Payments at the beginning of a loan go more toward interest than payments at the end. What percent of Gloria's loan remains to be paid after half a year?
- Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?
- Suppose Gloria decides to pay \$70 every month. How long would it take her to pay off the loan?
- Suppose that, based on how much she can afford, Gloria will pay a variable amount each month in addition to the \$60.78. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.
- Ethan has a three-year, \$12,000 motorcycle loan. The annual interest rate is 6%, and his monthly payment is \$365.06. After fifteen months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?

StudyTip

Combinations Recall that both ${}_nC_0$ and ${}_nC_n$ equal 1.

KeyConcept Binomial Theorem

If n is a natural number, then $(a + b)^n =$

$${}_nC_0 a^n b^0 + {}_nC_1 a^{n-1} b^1 + {}_nC_2 a^{n-2} b^2 + \dots + {}_nC_n a^0 b^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} a^{n-k} b^k.$$

To use the theorem, replace n with the value of the exponent. Notice how the terms will follow the pattern of Pascal's triangle, and the coefficients will be symmetric.

Example 2 Use the Binomial Theorem

Expand $(a + b)^7$.

Method 1 Use combinations.

Replace n with 7 in the Binomial Theorem.

$$\begin{aligned}(a + b)^7 &= {}_7C_0 a^7 + {}_7C_1 a^6 b + {}_7C_2 a^5 b^2 + {}_7C_3 a^4 b^3 + {}_7C_4 a^3 b^4 + {}_7C_5 a^2 b^5 + {}_7C_6 a b^6 + {}_7C_7 b^7 \\ &= a^7 + \frac{7!}{6!} a^6 b + \frac{7!}{2!5!} a^5 b^2 + \frac{7!}{3!4!} a^4 b^3 + \frac{7!}{4!3!} a^3 b^4 + \frac{7!}{5!2!} a^2 b^5 + \frac{7!}{6!} a b^6 + b^7 \\ &= a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7a b^6 + b^7\end{aligned}$$

Method 2 Use Pascal's triangle.

Use the Binomial Theorem to determine exponents, but instead of finding the coefficients by using combinations, look at the seventh row of Pascal's triangle.

6		1		6		15		20		15		6		1			
7		1		7		21		35		35		21		7		1	
8	1		8		28		56		70		56		28		8		1

$$(a + b)^7 = a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7a b^6 + b^7$$

GuidedPractice

2. Expand $(x + y)^{10}$.

When the binomial to be expanded has coefficients other than 1, the coefficients will no longer be symmetric. In these cases, you may want to use the Binomial Theorem.

Example 3 Coefficients Other Than 1

Expand $(5a - 4b)^4$.

$$\begin{aligned}(5a - 4b)^4 &= {}_4C_0 (5a)^4 + {}_4C_1 (5a)^3 (-4b) + {}_4C_2 (5a)^2 (-4b)^2 + {}_4C_3 (5a) (-4b)^3 + {}_4C_4 (-4b)^4 \\ &= 625a^4 + \frac{4!}{3!} (125a^3) (-4b) + \frac{4!}{2!2!} (25a^2) (16b^2) + \frac{4!}{3!} (5a) (-64b^3) + 256b^4 \\ &= 625a^4 - 2000a^3 b + 2400a^2 b^2 - 1280ab^3 + 256b^4\end{aligned}$$

GuidedPractice

3. Expand $(3x + 2y)^5$.

Sometimes you may need to find only one term in a binomial expansion. To do this, you can use the summation formula for the Binomial Theorem, $\sum_{k=0}^n \frac{n!}{k!(n-k)!} a^{n-k} b^k$.

StudyTip

Graphing Calculator

You can calculate ${}_nC_r$ by using a graphing calculator.

Press **MATH** and choose PRB 3.



Example 4 Determine a Single Term

Find the fifth term of $(y + z)^{11}$.

Step 1 Use the Binomial Theorem to write the expansion in sigma notation.

$$(y + z)^{11} = \sum_{k=0}^{11} \frac{11!}{k!(11-k)!} y^{11-k} z^k$$

Step 2 $\frac{11!}{k!(11-k)!} y^{11-k} z^k = \frac{11!}{4!(11-4)!} y^{11-4} z^4$
 $= 330y^7z^4$

For the fifth term, $k = 4$.

$C(11, 4) = 330$

Guided Practice

4. Find the sixth term of $(c + d)^{10}$.

Concept Summary Binomial Expansion

In a binomial expansion of $(a + b)^n$,

- there are $n + 1$ terms.
- n is the exponent of a in the first term and b in the last term.
- in successive terms, the exponent of a decreases by 1, and the exponent of b increases by 1.
- the sum of the exponents in each term is n .
- the coefficients are symmetric.

Check Your Understanding

 = Step-by-Step Solutions begin on page R14.



Examples 1–3 Expand each binomial.

1. $(c + d)^5$

2. $(g + h)^7$

3. $(x - 4)^6$

4. $(2y - z)^5$

5. $(x + 3)^5$

6. $(y - 4z)^4$

7. **GENETICS** If a woman is equally as likely to have a baby boy or a baby girl, use binomial expansion to determine the probability that 5 of her 6 children are girls. Do not consider identical twins.

Example 4 Find the indicated term of each expression.

8. fourth term of $(b + c)^9$

 9. fifth term of $(x + 3y)^8$

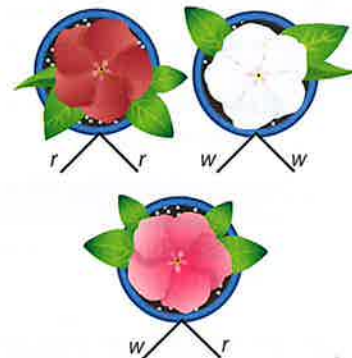
10. third term of $(a - 4b)^6$

11. sixth term of $(2c - 3d)^8$

12. last term of $(5x + y)^5$

13. first term of $(3a + 8b)^5$

14. **CCSS MODELING** The color of a particular flower is determined by the combination of two genes, also called *alleles*. If the flower has two red alleles r , the flower is red. If the flower has two white alleles w , the flower is white. If the flower has one allele of each color, the flower will be pink. In a lab, two pink flowers are mated and eventually produce 1000 offspring. How many of the 1000 offspring will be pink?



Examples 1–3 Expand each binomial.

15. $(a - b)^6$

16. $(c - d)^7$

17. $(x + 6)^6$

18. $(y - 5)^7$

19. $(2a + 4b)^4$

20. $(3a - 4b)^5$

21. COMMITTEES If an equal number of men and women applied to be on a community planning committee and the committee needs a total of 10 people, find the probability that 7 of the members will be women. Assume that committee members will be chosen randomly.

22. BASEBALL If a pitcher is just as likely to throw a ball as a strike, find the probability that 11 of his first 12 pitches are balls.

Example 4 Find the indicated term of each expression.

23. third term of $(x + 2z)^7$

24. fourth term of $(y - 3x)^6$

25. seventh term of $(2a - 2b)^8$

26. sixth term of $(4x + 5y)^6$

27. fifth term of $(x - 4)^9$

28. fourth term of $(c + 6)^8$

Expand each binomial.

29. $\left(x + \frac{1}{2}\right)^5$

30. $\left(x - \frac{1}{3}\right)^4$

31. $\left(2b + \frac{1}{4}\right)^5$

32. $\left(3c + \frac{1}{3}\right)^5$

33. CCSS SENSE-MAKING In $\frac{n!}{k!(n-k)!} p^k q^{n-k}$, let p represent the likelihood of a success and q represent the likelihood of a failure.

- If a place-kicker makes 70% of his kicks within 40 yards, find the likelihood that he makes 9 of his next 10 attempts from within 40 yards.
- If a quarterback completes 60% of his passes, find the likelihood that he completes 8 of his next 10 attempts.
- If a team converts 30% of their two-point conversions, find the likelihood that they convert 2 of their next 5 conversions.

H.O.T. Problems Use Higher-Order Thinking Skills

- CHALLENGE** Find the sixth term of the expansion of $(\sqrt{a} + \sqrt{b})^{12}$. Explain your reasoning.
- REASONING** Explain how the terms of $(x + y)^n$ and $(x - y)^n$ are the same and how they are different.
- REASONING** Determine whether the following statement is *true* or *false*. Explain your reasoning.
The eighth and twelfth terms of $(x + y)^{20}$ have the same coefficients.
- OPEN ENDED** Write a power of a binomial for which the second term of the expansion is $6x^4y$.
- WRITING IN MATH** Explain how to write out the terms of Pascal's triangle.



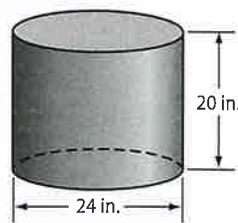
Standardized Test Practice

39. **PROBABILITY** A desk drawer contains 7 sharpened red pencils, 5 sharpened yellow pencils, 3 unsharpened red pencils, and 5 unsharpened yellow pencils. If a pencil is taken from the drawer at random, what is the probability that it is yellow, given that it is one of the sharpened pencils?

A $\frac{5}{12}$
B $\frac{7}{20}$
C $\frac{5}{8}$
D $\frac{1}{5}$

40. **GRIDDED RESPONSE** Two people are 17.5 miles apart. They begin to walk toward each other along a straight line at the same time. One walks at the rate of 4 miles per hour, and the other walks at the rate of 3 miles per hour. In how many hours will they meet?

41. **GEOMETRY** Christie has a cylindrical block that she needs to paint for an art project.



What is the surface area of the cylinder in square inches rounded to the nearest square inch?

- F 1960 H 5127
G 2413 J 6635
42. Which of the following is a linear function?
- A $y = \frac{x+3}{x+2}$
B $y = (3x+2)^2$
C $y = \frac{x+3}{2}$
D $y = |3x| + 2$

Spiral Review

Find the first five terms of each sequence. (Lesson 10-5)

43. $a_1 = -2, a_{n+1} = a_n + 5$

44. $a_1 = 3, a_{n+1} = 4a_n - 10$

45. $a_1 = 4, a_{n+1} = 3a_n - 6$

Find the sum of each infinite geometric series, if it exists. (Lesson 10-4)

46. $-6 + 3 - \frac{3}{2} + \dots$

47. $\frac{3}{4} + \frac{1}{4} + \frac{1}{12} + \dots$

48. $\sqrt{3} + 3 + \sqrt{27} + \dots$

49. **TRAVEL** A trip between two towns takes 4 hours under ideal conditions. The first 150 miles of the trip is on an interstate, and the last 130 miles is on a highway with a speed limit that is 10 miles per hour less than on the interstate. (Lesson 8-6)

- a. If x represents the speed limit on the interstate, write expressions for the time spent at that speed and for the time spent on the other highway.
- b. Write and solve an equation to find the speed limits on the two highways.

Skills Review

State whether each statement is *true* or *false* when $n = 1$. Explain.

50. $\frac{(n+1)(n+1)}{2} = 2$

51. $3n + 5$ is even.

52. $n^2 - 1$ is odd.





Recall that an arrangement or selection of objects in which order is not important is called a *combination*. For example, selecting 2 snacks from a choice of 6 is a combination of 6 objects taken 2 at a time and can be written ${}_6C_2$ or $C(6, 2)$.



**Common Core State Standards
Content Standards**

A.APR.5 Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

Activity

A contestant on a game show has the opportunity to win up to five prizes, one for each of five rounds of the game. If the contestant wins a round, he or she may choose one prize. Determine the number of ways that prizes can be chosen.

Step 1 If a contestant does not win any rounds, he or she receives 0 prizes. This represents 5 items taken 0 at a time.

$${}_nC_r = \frac{n!}{(n-r)!r!} \quad \text{Definition of combination}$$

$$\begin{aligned} {}_5C_0 &= \frac{5!}{(5-0)!0!} && n = 5 \text{ and } r = 0 \\ &= \frac{120}{120(1)} \text{ or } 1 && 5! = 120 \text{ and } 0! = 1 \end{aligned}$$

There is 1 way to receive 0 prizes.

If a contestant wins one round, any one of the prizes can be selected. If a contestant wins two rounds, two prizes can be chosen. If three rounds are won, three prizes can be chosen, and so on. In how many ways can 1 prize be chosen? 2 prizes? 3, 4, and 5 prizes? We can determine these answers by examining Pascal's triangle.

Step 2 Examine Pascal's triangle.

List Rows 0 through 5 of Pascal's triangle.

Row 0				1									
Row 1			1		1								
Row 2			1		2		1						
Row 3			1		3		3		1				
Row 4			1		4		6		4		1		
Row 5			1		5		10		10		5		1

The number of ways one prize can be chosen from 5 can be determined by looking at Row 5. The first number in Row 5 represents the number of ways to choose 0 prizes, the second number represents the number of ways to choose 1 prize, and so on.

Analyze the Result

1. Make a conjecture about how the numbers in one of the rows can be used to find the number of ways that 0, 1, 2, 3, 4, ..., n objects can be selected from n objects.
2. Suppose the rules of the game are changed so that there are 6 rounds and 6 prizes from which to choose. Find the number of ways that 0, 1, 2, 3, 4, 5, or 6 prizes can be chosen. Which row of Pascal's triangle can be used to find the answers?
3. Use Pascal's triangle to find ${}_8C_0$, ${}_8C_1$, ${}_8C_2$, ${}_8C_3$, ${}_8C_4$, ${}_8C_5$, ${}_8C_6$, ${}_8C_7$, and ${}_8C_8$. State the row number that you used to find the answers.

Proof by Mathematical Induction



Then

You have proved the sum of an arithmetic series.

Now

- 1 Prove statements by using mathematical induction.
- 2 Disprove statements by finding a counterexample.

Why?

When dominoes are set up closely and the first domino is knocked down, the rest of the dominoes come tumbling down. All that is needed with this setup is for the first domino to fall, and the rest will follow. The same is true with mathematical induction.

ABC **New Vocabulary**
mathematical induction
induction hypothesis

CCSS **Common Core State Standards**

Content Standards
A.SSE.1.b Interpret complicated expressions by viewing one or more of their parts as a single entity.
A.APR.5 Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

Mathematical Practices
1 Make sense of problems and persevere in solving them.
3 Construct viable arguments and critique the reasoning of others.

1 Mathematical Induction **Mathematical induction** is a method of proving statements involving natural numbers.

Key Concept Mathematical Induction

To prove that a statement is true for all natural numbers n ,

- Step 1** Show that the statement is true for $n = 1$.
- Step 2** Assume that the statement is true for some natural number k . This assumption is called the **induction hypothesis**.
- Step 3** Show that the statement is true for the next natural number $k + 1$.



Example 1 Prove Summation

Prove that $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$.

Step 1 When $n = 1$, the left side of the equation is 1^3 or 1.
The right side is $\frac{1^2(1+1)^2}{4}$ or 1. Thus, the statement is true for $n = 1$.

Step 2 Assume that $1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4}$ for a natural number k .

Step 3 Show that the given statement is true for $n = k + 1$.

$$\begin{aligned}
 1^3 + 2^3 + 3^3 + \dots + k^3 &= \frac{k^2(k+1)^2}{4} && \text{Inductive hypothesis} \\
 1^3 + 2^3 + \dots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 && \text{Add } (k+1)^3 \text{ to each side.} \\
 &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} && \text{The LCD is 4.} \\
 &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} && \text{Factor.} \\
 &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} && \text{Simplify.} \\
 &= \frac{(k+1)^2(k+2)^2}{4} && \text{Factor.}
 \end{aligned}$$

The last expression is the statement to be proved, where n has been replaced by $k + 1$. This proves the conjecture.

Guided Practice

1. Prove that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Along with summation, mathematical induction can be used to prove divisibility.

StudyTip

CCSS Regularity r is a whole number used to show divisibility in proof. When a value equals $4r$, then it must be divisible by 4.

Example 2 Prove Divisibility

Prove that $8^n - 1$ is divisible by 7 for all natural numbers n .

Step 1 When $n = 1$, $8^n - 1 = 8^1 - 1$ or 7. Since 7 is divisible by 7, the statement is true for $n = 1$.

Step 2 Assume that $8^k - 1$ is divisible by 7 for some natural number k . This means that there is a natural number r such that $8^k - 1 = 7r$.

Step 3 Show that the statement is true for $n = k + 1$.

$$8^k - 1 = 7r \quad \text{Inductive hypothesis}$$

$$8^k = 7r + 1 \quad \text{Add 1 to each side.}$$

$$8(8^k) = 8(7r + 1) \quad \text{Multiply each side by 8.}$$

$$8^{k+1} = 56r + 8 \quad \text{Simplify.}$$

$$8^{k+1} - 1 = 56r + 7 \quad \text{Subtract 1 from each side.}$$

$$8^{k+1} - 1 = 7(8r + 1) \quad \text{Factor.}$$

Since r is a natural number, $8r + 1$ is a natural number and $7(8r + 1)$ is divisible by 7. Therefore, $8^{k+1} - 1$ is divisible by 7.

This proves that $8^n - 1$ is divisible by 7 for all natural numbers n .

GuidedPractice

2. Prove that $7^n - 1$ is divisible by 6 for all natural numbers n .

2 Counterexamples Statements can be proved false by using mathematical induction. An easier method is by finding a counterexample, which is a specific case in which the statement is false.

ReviewVocabulary

counterexample One of the synonyms of *counter* is to *contradict*, so a counterexample is an example that contradicts a hypothesis.

Example 3 Use a Counterexample to Disprove

Find a counterexample to disprove the statement that $2^n + 2n^2$ is divisible by 4 for any natural number n .

Test different values of n .

n	$2^n + 2n^2$	Divisible by 4?
1	$2^1 + 2(1)^2 = 2 + 2$ or 4	yes
2	$2^2 + 2(2)^2 = 4 + 8$ or 12	yes
3	$2^3 + 2(3)^2 = 8 + 18$ or 26	no

The value $n = 3$ is a counterexample for the statement.

GuidedPractice

3. Find a counterexample to disprove $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(3n-1)}{2}$.





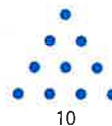
Example 1

Prove that each statement is true for all natural numbers.

1. $1 + 3 + 5 + \dots + (2n - 1) = n^2$

2. $1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$

3. **NUMBER THEORY** A number is *triangular* if it can be represented visually by a triangular array.



a. The first triangular number is 1. Find the next 5 triangular numbers.

b. Write a formula for the n th triangular number.

c. Prove that the sum of the first n triangular numbers equals $\frac{n(n + 1)(n + 2)}{6}$.

Example 2

Prove that each statement is true for all natural numbers.

4. $10^n - 1$ is divisible by 9.

5. $4^n - 1$ is divisible by 3.

Example 3

Find a counterexample to disprove each statement.

6. $3^n + 1$ is divisible by 4.

7. $2^n + 3^n$ is divisible by 4.

Practice and Problem Solving

Extra Practice is on page R10.

Example 1

CCSS ARGUMENTS Prove that each statement is true for all natural numbers.

8. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

9. $2 + 5 + 8 + \dots + (3n - 1) = \frac{n(3n + 1)}{2}$

10. $1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$

11. $1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$

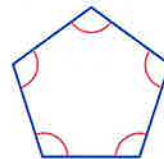
12. $1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n - 1)}{2}$

13. $3 + 7 + 11 + \dots + (4n - 1) = 2n^2 + n$

14. $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$

15. $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$

16. **GEOMETRY** According to the Interior Angle Sum Formula, if a convex polygon has n sides, then the sum of the measures of the interior angles of a polygon equals $180(n - 2)$. Prove this formula for $n \geq 3$ using mathematical induction and geometry.



Example 2

Prove that each statement is true for all natural numbers.

17. $5^n + 3$ is divisible by 4.

18. $9^n - 1$ is divisible by 8.

19. $12^n + 10$ is divisible by 11.

20. $13^n + 11$ is divisible by 12.

Example 3

Find a counterexample to disprove each statement.

21. $1 + 2 + 3 + \dots + n = n^2$

22. $1 + 8 + 27 + \dots + n^3 = (2n + 2)^2$

23. $n^2 - n + 15$ is prime.

24. $n^2 + n + 23$ is prime.



- 25 NATURE** The terms of the Fibonacci sequence are found in many places in nature. The number of spirals of seeds in sunflowers is a Fibonacci number, as is the number of spirals of scales on a pinecone. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, Each element after the first two is found by adding the previous two terms. If f_n stands for the n th Fibonacci number, prove that $f_1 + f_2 + \dots + f_n = f_{n+2} - 1$.

Prove that each statement is true for all natural numbers or find a counterexample.

26. $7^n + 5$ is divisible by 6. 27. $18^n - 1$ is divisible by 17.
 28. $n^2 + 21n + 7$ is a prime number. 29. $n^2 + 3n + 3$ is a prime number.
 30. $500 + 100 + 20 + \dots + 4 \cdot 5^{4-n} = 625 \left(1 - \frac{1}{5^n}\right)$
 31. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$
 32. **CCSS PERSEVERANCE** Refer to the figures below.



Figure 1



Figure 2



Figure 3

- a. There is a total of 5 squares in the second figure. How many squares are there in the third figure?
 b. Write a sequence for the first five figures.
 c. How many squares are there in a standard 8×8 checkerboard?
 d. Write a formula to represent the number of squares in an $n \times n$ grid.

H.O.T. Problems Use Higher-Order Thinking Skills

33. **CHALLENGE** Suggest a formula to represent $2 + 4 + 6 + \dots + 2n$, and prove your hypothesis using mathematical induction.

REASONING Determine whether the following statements are *true* or *false*. Explain.

34. If you cannot find a counterexample to a statement, then it is true.
 35. If a statement is true for $n = k$ and $n = k + 1$, then it is also true for $n = 1$.

36. **CHALLENGE** Prove $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

37. **REASONING** Find a counterexample to $x^3 + 30 > x^2 + 20x$.

38. **OPEN ENDED** Write a sequence, the formula that produces it, and determine the formula for the sum of the terms of the sequence. Then prove the formula with mathematical induction.

39. **WRITING IN MATH** Explain how the concept of dominoes can help you understand the power of mathematical induction.

40. **WRITING IN MATH** Provide a real-world example other than dominoes that describes mathematical induction.



Standardized Test Practice

41. Which of the following is a counterexample to the statement below?

$$n^2 + n - 11 \text{ is prime.}$$

- A $n = -6$ C $n = 5$
B $n = 4$ D $n = 6$

42. **PROBABILITY** Latisha wants to create a 7-character password. She wants to use an arrangement of the first 3 letters of her first name (lat), followed by an arrangement of the 4 digits in 1986, the year she was born. How many possible passwords can she create in this way?

- F 72 H 288
G 144 J 576

43. **GRIDDED RESPONSE** A gear that is 8 inches in diameter turns a smaller gear that is 3 inches in diameter. If the larger gear makes 36 revolutions, how many revolutions does the smaller gear make in that time?

44. **SHORT RESPONSE** Write an equation for the n th line. Show how it fits the pattern for each given line in the list.

Line 1: $1 \times 0 = 1 - 1$

Line 2: $2 \times 1 = 4 - 2$

Line 3: $3 \times 2 = 9 - 3$

Line 4: $4 \times 3 = 16 - 4$

Line 5: $5 \times 4 = 25 - 5$

Spiral Review

Find the indicated term of each expansion. (Lesson 10-6)

45. fourth term of $(x + 2y)^6$

46. fifth term of $(a + b)^6$

47. fourth term of $(x - y)^9$

48. **BIOLOGY** In a particular forest, scientists are interested in how the population of wolves will change over the next two years. One model for animal population is the Verhulst population model, $p_{n+1} = p_n + rp_n(1 - p_n)$, where n represents the number of time periods that have passed, p_n represents the percent of the maximum sustainable population that exists at time n , and r is the growth factor. (Lesson 10-5)

- a. To find the population of the wolves after one year, evaluate $p_1 = 0.45 + 1.5(0.45)(1 - 0.45)$.
- b. Explain what each number in the expression in part a represents.
- c. The current population of wolves is 165. Find the new population by multiplying 165 by the value in part a.

Find the exact solution(s) of each system of equations. (Lesson 9-7)

49. $x^2 + y^2 - 18x + 24y + 200 = 0$
 $4x + 3y = 0$

50. $4x^2 + y^2 = 16$
 $x^2 + 2y^2 = 4$

Skills Review

Evaluate each expression.

51. $P(8, 2)$

52. $P(9, 1)$

53. $P(12, 6)$

54. $C(5, 2)$

55. $C(8, 4)$

56. $C(20, 17)$

57. $P(12, 2)$

58. $P(7, 2)$

59. $C(8, 6)$

60. $C(9, 4) \cdot C(5, 3)$

61. $C(6, 1) \cdot C(4, 1)$

62. $C(10, 5) \cdot C(8, 4)$



Study Guide

Key Concepts

Arithmetic Sequences and Series (Lessons 10-1 and 10-2)

- The n th term a_n of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is any positive integer.
- The sum S_n of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$.

Geometric Sequences and Series (Lessons 10-3 and 10-4)

- The n th term a_n of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1 \cdot r^{n-1}$, where n is any positive integer.
- The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, where $r \neq 1$.
- The sum S of an infinite geometric series with $-1 < r < 1$ is given by $S_n = \frac{a_1}{1 - r}$.

Recursion and Iteration (Lesson 10-5)

- In a recursive formula, each term is formulated from one or more previous terms.

The Binomial Theorem (Lesson 10-6)

- The Binomial Theorem:

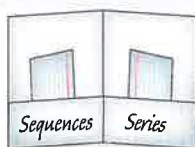
$$(a + b)^n = \sum_{k=0}^n \frac{n!}{(n-k)!k!} a^{n-k} b^k$$

Mathematical Induction (Lesson 10-7)

- Mathematical induction is a method of proof used to prove statements about the positive integers.

FOLDABLES Study Organizer

Be sure the Key Concepts are noted in your Foldable.



Key Vocabulary

- arithmetic means (p. 667)
- arithmetic sequence (p. 659)
- arithmetic series (p. 668)
- common difference (p. 659)
- common ratio (p. 661)
- convergent series (p. 683)
- divergent series (p. 683)
- explicit formula (p. 692)
- Fibonacci sequence (p. 692)
- finite sequence (p. 659)
- geometric means (p. 675)
- geometric sequence (p. 661)
- geometric series (p. 676)
- induction hypothesis (p. 705)
- infinite geometric series (p. 659)
- infinite sequence (p. 659)
- infinity (p. 685)
- iteration (p. 694)
- mathematical induction (p. 694)
- partial sum (p. 668)
- Pascal's triangle (p. 699)
- recursive formula (p. 692)
- recursive sequence (p. 692)
- sequence (p. 659)
- series (p. 668)
- sigma notation (p. 669)
- term (p. 659)

Vocabulary Check

State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.

- An infinite geometric series that has a sum is called a convergent series.
- Mathematical induction is the process of repeatedly composing a function with itself.
- The arithmetic means of a sequence are the terms between any two non-successive terms of an arithmetic sequence.
- A term is a list of numbers in a particular order.
- The sum of the first n terms of a series is called the partial sum.
- The formula $a_n = a_{n-2} + a_{n-1}$ is a recursive formula.
- A geometric sequence is a sequence in which every term is determined by adding a constant value to the previous term.
- An infinite geometric series that does not have a sum is called a partial sum.
- Eleven and 17 are two geometric means between 5 and 23 in the sequence 5, 11, 17, 23.
- Using the Binomial Theorem, $(x - 2)^4$ can be expanded to $x^4 - 8x^3 + 24x^2 - 32x + 16$.



Lesson-by-Lesson Review

10-1 Sequences as Functions

Find the indicated term of each arithmetic sequence.

11. $a_1 = 9, d = 3, n = 14$
12. $a_1 = -3, d = 6, n = 22$
13. $a_1 = 10, d = -4, n = 9$
14. $a_1 = -1, d = -5, n = 18$

Example 1

Find the 11th term of an arithmetic sequence if $a_1 = -15$ and $d = 6$.

$$a_n = a_1 + (n - 1)d \quad \text{Formula for the } n\text{th term}$$

$$a_{11} = -15 + (11 - 1)6 \quad n = 11, a_1 = -15, d = 6$$

$$a_{11} = 45 \quad \text{Simplify.}$$

10-2 Arithmetic Sequences and Series

Find the arithmetic means in each sequence.

15. $-12, _, _, _, 8$
16. $15, _, _, 29$
17. $12, _, _, _, -8$
18. $72, _, _, _, 24$
19. **BANKING** Carson saves \$40 every 2 months. If he saves at this rate for two years, how much will he have at the end of two years?

Find S_n for each arithmetic series.

20. $a_1 = 16, a_n = 48, n = 6$
21. $a_1 = 8, a_n = 96, n = 20$
22. $9 + 14 + 19 + \dots + 74$
23. $16 + 7 + -2 + \dots + -65$
24. **DRAMA** Laura has a drama performance in 12 days. She plans to practice her lines each night. On the first night she rehearses her lines 2 times. The next night she rehearses her lines 4 times. The third night she rehearses her lines 6 times. On the eleventh night, how many times has she rehearsed her lines?

Find the sum of each arithmetic series.

25. $\sum_{k=5}^{21} (3k - 2)$
26. $\sum_{k=0}^{10} (6k - 1)$
27. $\sum_{k=4}^{12} (-2k + 5)$

Example 2

Find the two arithmetic means between 3 and 39.

$$a_n = a_1 + (n - 1)d \quad \text{Formula for the } n\text{th term}$$

$$a_4 = 3 + (4 - 1)d \quad n = 4, a_1 = 3$$

$$39 = 3 + 3d \quad a_4 = 39$$

$$12 = d \quad \text{Simplify.}$$

The arithmetic means are 3 + 12 or 15 and 15 + 12 or 27.

Example 3

Find S_n for the arithmetic series with $a_1 = 18, a_n = 56$, and $n = 8$.

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{Sum formula}$$

$$S_8 = \frac{8}{2}(18 + 56) \quad n = 8, a_1 = 18, a_n = 56$$

$$= 296 \quad \text{Simplify.}$$

Example 4

Evaluate $\sum_{k=3}^{15} 5k + 1$.

Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. There are 13 terms, $a_1 = 5(3) + 1$ or 16, and $a_{13} = 5(15) + 1$ or 76.

$$S_{13} = \frac{13}{2}(16 + 76) \\ = 598$$

10-3 Geometric Sequences and Series

Find the indicated term for each geometric sequence.

28. $a_1 = 5, r = 2, n = 7$

29. $a_1 = 11, r = 3, n = 3$

30. $a_1 = 128, r = -\frac{1}{2}, n = 5$

31. a_8 for $\frac{1}{8}, \frac{3}{8}, \frac{9}{8}, \dots$

Find the geometric means in each sequence.

32. 6, __, __, 162

33. 8, __, __, __, 648

34. -4, __, __, 108

35. **SAVINGS** Nolan has a savings account with a current balance of \$1500. What would be Nolan's account balance after 4 years if he receives 5% interest annually?

Find S_n for each geometric series.

36. $a_1 = 15, r = 2, n = 4$

37. $a_1 = 9, r = 4, n = 6$

38. $5 - 10 + 20 - \dots$ to 7 terms

39. $243 + 81 + 27 + \dots$ to 5 terms

Evaluate the sum of each geometric series.

40. $\sum_{k=1}^7 3 \cdot (-2)^{k-1}$

41. $\sum_{k=1}^8 -1 \left(\frac{2}{3}\right)^{k-1}$

42. **ADVERTISING** Natalie is handing out fliers to advertise the next student council meeting. She hands out fliers to 4 people. Then, each of those 4 people hand out 4 fliers to 4 other people. Those 4 then hand out 4 fliers to 4 new people. If Natalie is considered the first round, how many people will have been given fliers after 4 rounds?

Example 5

Find the sixth term of a geometric sequence for which $a_1 = 9$ and $r = 4$.

$$a_n = a_1 \cdot r^{n-1} \quad \text{Formula for the } n\text{th term}$$

$$a_6 = 9 \cdot 4^{6-1} \quad n = 6, a_1 = 9, r = 4$$

$$a_6 = 9216$$

The sixth term is 9216.

Example 6

Find two geometric means between 1 and 27.

$$a_n = a_1 \cdot r^{n-1} \quad \text{Formula for the } n\text{th term}$$

$$a_4 = 1 \cdot r^{4-1} \quad n = 4 \text{ and } a_1 = 1$$

$$27 = r^3 \quad a_4 = 27$$

$$3 = r \quad \text{Simplify.}$$

The geometric means are 1(3) or 3 and 3(3) or 9.

Example 7

Find the sum of a geometric series for which $a_1 = 3, r = 5$, and $n = 11$.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \quad \text{Sum formula}$$

$$S_{11} = \frac{3 - 3 \cdot 5^{11}}{1 - 5} \quad n = 11, a_1 = 3, r = 5$$

$$S_{11} = 36,621,093 \quad \text{Use a calculator.}$$

Example 8

Evaluate $\sum_{k=1}^6 2 \cdot (4)^{k-1}$.

$$S_6 = \frac{2 - 2 \cdot 4^6}{1 - 4} \quad n = 6, a_1 = 2, r = 4$$

$$= \frac{-8190}{-3} \quad \text{Simplify.}$$

$$= 2730 \quad \text{Simplify.}$$

10-4 Infinite Geometric Series

Find the sum of each infinite series, if it exists.

43. $a_1 = 8, r = \frac{3}{4}$

44. $\frac{5}{6} - \frac{20}{18} + \frac{80}{54} - \frac{320}{162} + \dots$

45. $\sum_{k=1}^{\infty} 3\left(\frac{1}{2}\right)^{k-1}$

46. **PHYSICAL SCIENCE** Maddy drops a ball off of a building that is 60 feet high. Each time the ball bounces, it bounces back to $\frac{2}{3}$ its previous height. If the ball continues to follow this pattern, what will be the total distance that the ball travels?

Example 9

Find the sum of the infinite geometric series for which $a_1 = 15$ and $r = \frac{1}{3}$.

$$S = \frac{a_1}{1-r} \quad \text{Sum formula}$$

$$= \frac{15}{1-\frac{1}{3}} \quad a_1 = 15, r = \frac{1}{3}$$

$$= \frac{15}{\frac{2}{3}} \text{ or } 22.5 \quad \text{Simplify.}$$

10-5 Recursion and Iteration

Find the first five terms of each sequence.

47. $a_1 = -3, a_{n+1} = a_n + 4$

48. $a_1 = 5, a_{n+1} = 2a_n - 5$

49. $a_1 = 1, a_{n+1} = a_n + 5$

50. **SAVINGS** Sari has a savings account with a \$12,000 balance. She has a 5% interest rate that is compounded monthly. Every month Sari adds \$500 to the account. The recursive formula $b_n = 1.05b_{n-1} + 500$ describes the balance in Sari's savings account after n months. Find the balance of Sari's account after 3 months. Round your answer to the nearest penny.

Find the first three iterates of each function for the given initial value.

51. $f(x) = 2x + 1, x_0 = 3$

52. $f(x) = 5x - 4, x_0 = 1$

53. $f(x) = 6x - 1, x_0 = 2$

54. $f(x) = 3x + 1, x_0 = 4$

Example 10

Find the first five terms of the sequence in which $a_1 = 1, a_{n+1} = 3a_n + 2$.

$$a_{n+1} = 3a_n + 2 \quad \text{Recursive formula}$$

$$a_{1+1} = 3a_1 + 2 \quad n = 1$$

$$a_2 = 3(1) + 2 \text{ or } 5 \quad a_1 = 1$$

$$a_{2+1} = 3a_2 + 2 \quad n = 2$$

$$a_3 = 3(5) + 2 \text{ or } 17 \quad a_2 = 5$$

$$a_{3+1} = 3a_3 + 2 \quad n = 3$$

$$a_4 = 3(17) + 2 \text{ or } 53 \quad a_3 = 17$$

$$a_{4+1} = 3a_4 + 2 \quad n = 4$$

$$a_5 = 3(53) + 2 \text{ or } 161 \quad a_4 = 53$$

The first five terms of the sequence are 1, 5, 17, 53, and 161.

Example 11

Find the first three iterates of the function $f(x) = 3x - 2$ for the initial value of $x_0 = 2$.

$$x_1 = f(x_0) \quad x_2 = f(x_1) \quad x_3 = f(x_2)$$

$$= f(2) \quad = f(4) \quad = f(10)$$

$$= 3(2) - 2 \quad = 3(4) - 2 \quad = 3(10) - 2$$

$$= 4 \quad = 10 \quad = 28$$

The first three iterates are 4, 10, and 28.

10-6 The Binomial Theorem

Expand each binomial.

55. $(a + b)^3$

56. $(y - 3)^7$

57. $(3 - 2z)^5$

58. $(4a - 3b)^4$

59. $\left(x - \frac{1}{4}\right)^5$

Find the indicated term of each expression.

60. third term of $(a + 2b)^8$

61. sixth term of $(3x + 4y)^7$

62. second term of $(4x - 5)^{10}$

Example 12

Expand $(x - 3y)^4$.

$$\begin{aligned} (x - 3y)^4 &= x^4 + {}_4C_1x^3(-3y) + {}_4C_2x^2(-3y)^2 + {}_4C_3(-3y)^3 + {}_4C_4(-3y)^4 \\ &= x^4 + \frac{4!}{3!1!}x^3(-3y) + \frac{4!}{2!2!}x^2(9y^2) + \frac{4!}{3!1!}x(-27y^3) + 81y^4 \\ &= x^4 + -12x^3y + 54x^2y^2 + -108xy^3 + 81y^4 \end{aligned}$$

Example 13

Find the fourth term of $(x + y)^8$.

Use the Binomial Theorem to write the expansion in sigma notation.

$$(x + y)^8 = \sum_{k=0}^8 \frac{8!}{k!(8-k)!} x^{8-k} y^k$$

For the fourth term, $k = 3$.

$$\begin{aligned} \frac{8!}{k!(8-k)!} x^{8-k} y^k &= \frac{8!}{3!(8-3)!} x^{8-3} y^3 \\ &= 56x^5y^3 \end{aligned}$$

10-7 Proof by Mathematical Induction

Prove that each statement is true for all positive integers.

63. $2 + 6 + 12 + \dots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$

64. $7^n - 1$ is divisible by 6.

65. $5^n - 1$ is divisible by 4.

Find a counterexample for each statement.

66. $8^n + 3$ is divisible by 11.

67. $6^{n+1} - 2$ is divisible by 17.

68. $n^2 + 2n + 4$ is prime.

69. $n + 19$ is prime.

Example 14

Prove that $9^n + 3$ is divisible by 4.

Step 1 When $n = 1$, $9^n + 3 = 9^1 + 3$ or 12. Since 12 divided by 4 is 3, the statement is true for $n = 1$.

Step 2 Assume that $9^k + 3$ is divisible by 4 for some positive integer k . This means that $9^k + 3 = 4r$ for some whole number r .

Step 3

$$\begin{aligned} 9^k + 3 &= 4r \\ 9^k &= 4r - 3 \\ 9^{k+1} &= 36r - 27 \\ 9^{k+1} + 3 &= 36r - 27 + 3 \\ 9^{k+1} + 3 &= 36r - 24 \\ 9^{k+1} + 3 &= 4(9r - 6) \end{aligned}$$

Since r is a whole number, $9r - 6$ is a whole number. Thus, $9^{k+1} + 3$ is divisible by 4, so the statement is true for $n = k + 1$.

Therefore, $9^n + 3$ is divisible by 4 for all positive integers n .

CHAPTER 10 Practice Test

- Find the next 4 terms of the arithmetic sequence 81, 72, 63, ...
- Find the 25th term of an arithmetic sequence for which $a_1 = 9$ and $d = 5$.
- MULTIPLE CHOICE** What is the eighth term in the arithmetic sequence that begins 18, 20.2, 22.4, 24.6, ...?
 - 26.8
 - 29
 - 31.2
 - 33.4
- Find the four arithmetic means between -9 and 11 .
- Find the sum of the arithmetic series for which $a_1 = 11$, $n = 14$, and $a_n = 22$.
- MULTIPLE CHOICE** What is the next term in the geometric sequence below?

$$10, \frac{5}{2}, \frac{5}{8}, \frac{5}{32}, \dots$$
 - $\frac{5}{8}$
 - $\frac{5}{32}$
 - $\frac{5}{128}$
 - $\frac{5}{256}$
- Find the three geometric means between 6 and 1536.
- Find the sum of the geometric series for which $a_1 = 15$, $r = \frac{2}{3}$, and $n = 5$.

Find the sum of each series, if it exists.

- $\sum_{k=2}^{12} (3k - 1)$
- $\sum_{k=1}^{\infty} \frac{1}{2^k} (3^k)$
- $45 + 37 + 29 + \dots + -11$
- $\frac{1}{8} + \frac{2}{24} + \frac{4}{72} + \dots$

- Write $0.\overline{65}$ as a fraction.

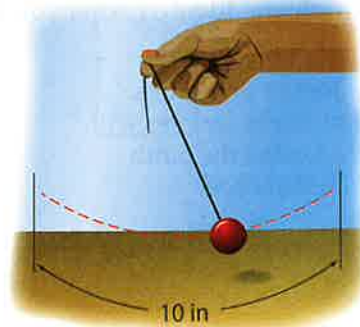
Find the first five terms of each sequence.

- $a_1 = -1, a_{n+1} = 3a_n + 5$
- $a_1 = 4, a_{n+1} = a_n + n$
- MULTIPLE CHOICE** What are the first 3 iterates of $f(x) = -5x + 4$ for an initial value of $x_0 = 3$?
 - 3, -11 , 59
 - -11 , 59 , -291
 - -1 , -6 , -11
 - 59 , -291 , 1459
- Expand $(2a - 3b)^4$.
- What is the coefficient of the fifth term of $(m + 3n)^6$?
- Find the fourth term of the expansion of $(c + d)^9$.

Prove that each statement is true for all positive integers.

- $1 + 6 + 36 + \dots + 6^{n-1} = \frac{1}{5}(6^n - 1)$.
- $11^n - 1$ is divisible by 10.
- Find a counterexample for the following statement.

$$2^n + 4^n \text{ is divisible by } 4.$$
- SCHOOL** There are an equal number of girls and boys in Mr. Marshall's science class. He needs to choose 8 students to represent his class at the science fair. What is the probability that 5 are boys?
- PENDULUM** Laurie swings a pendulum. The distance traveled per swing decreases by 15% with each swing. If the pendulum initially traveled 10 inches, find the total distance traveled when the pendulum comes to a rest.



Look For a Pattern

One of the most common problem-solving strategies is to look for a pattern. The ability to recognize patterns, model them algebraically, and extend them is a valuable problem-solving tool.

Strategies for Looking For a Pattern

Step 1

Identify the pattern.

- Compare the numbers, shapes, or graphs in the pattern.
- Ask yourself: How are the terms of the pattern related?
- Ask yourself: Are there any common operations that lead from one term to the next?

Step 2

Generalize the pattern.

- Write a rule using words to describe how the terms of the pattern are generated.
- Assign variables and write an algebraic expression to model the pattern if appropriate.

Step 3

Find missing terms, extend the pattern, and solve the problem.

- Use your pattern or your rule to find missing terms and/or extend the pattern to solve the problem.
- Check your answer to make sure it makes sense.



Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Use the sequence of squares shown. How many squares will be needed to make the ninth figure of the sequence?

A 55

C 74

B 65

D 82



Figure 1



Figure 2



Figure 3

Read the problem statement carefully. You are given three figures of a sequence and asked to find how many squares will be needed to make the ninth figure.

Look for a pattern in the figures of squares. Count the number of squares in each figure.



Write an expression to model this pattern.

Words The number of squares is equal to the square of the figure number plus one.

Variable Let n represent the figure number.

Equation $a_n = n^2 + 1$

Use your expression to extend the pattern and find the number of squares in the ninth figure.

$$a_9 = 9^2 + 1 = 82$$

So, the ninth figure will have 82 squares. The correct answer is D.

Exercises

Read each problem. Use a pattern to solve the problem.

- The numbers below form a famous mathematical sequence of numbers known as the Fibonacci sequence. What is the next Fibonacci number in the sequence?

1, 1, 2, 3, 5, 8, 13, 21, ...

- A 36
- B 34
- C 31
- D 29

- What is the missing number in the table?

n	a_n
1	0
2	2
3	6
4	12
5	??
6	30

- F 17
- G 18
- H 20
- J 21

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

10. What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$$

11. **GRIDDED RESPONSE** Consider the pattern below. Into how many pieces will the sixth figure of the pattern be divided?



Figure 1
1 piece

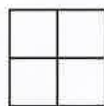


Figure 2
4 pieces

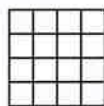
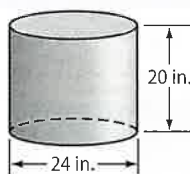


Figure 3
16 pieces

12. Use the Binomial Theorem to expand the expression $(c + d)^6$.
13. **GRIDDED RESPONSE** Kara has a cylindrical container that she needs to fill with dirt so she can plant some flowers.



What is the volume of the cylinder in cubic inches rounded to the nearest cubic inch?

14. Bacteria in a culture are growing exponentially with time, as shown in the table.

Hours	Bacteria
0	1000
1	2000
2	4000

Write an equation to express the number of bacteria, y , with respect to time, t .

15. **GRIDDED RESPONSE** What is the value of $f[g(6)]$ if $f(x) = 2x + 4$ and $g(x) = x^2 + 5$?

Extended Response

Record your answers on a sheet of paper. Show your work.

16. Prove that the sum of any two odd integers is even.
17. The endpoints of a diameter of a circle are at $(-1, 0)$ and $(5, -8)$.
- What are the coordinates of the center of the circle? Explain your method.
 - Find the radius of the circle. Explain your method.
 - Write an equation of the circle.
18. A cyclist travels from Centerville to Springfield in 2.5 hours. If she increases her speed, she can make the trip in 2 hours.
- Does this situation represent a direct or inverse variation? Explain your reasoning.
 - If the trip from Centerville to Springfield takes 2.5 hours when traveling at 12 miles per hour, what must the speed be to make the trip in 2 hours?

Need ExtraHelp?

If you missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Go to Lesson...	10-2	1-1	10-2	4-7	10-3	2-4	10-3	3-1	10-4	3-6	10-5	10-6	5-7	7-8	6-1	10-7	9-3	8-5

