CHAPTER 13

Trigonometric Identities and Equations

Then

You graphed trigonometric functions and determined the period, amplitude, phase shifts, and vertical shifts.

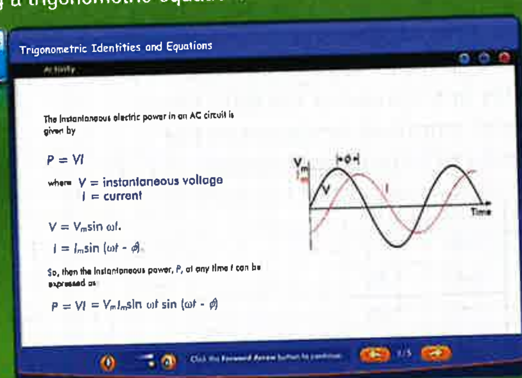
Now

You will:

- Use and verify trigonometric identities.
- Use the sum and difference of angles identities.
- Use the double- and half-angle identities.
- Solve trigonometric equations.

Why? ▲

ELECTRONICS Many aspects of electronics can be modeled by trigonometric functions. Radio, television, cellular telephones, and wireless Internet all communicate through radio waves that are modeled by trigonometric functions. The amount of power in an electronic gadget can be found by using a trigonometric equation.



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Get Ready for the Chapter

Diagnose Readiness | You have two options for checking prerequisite skills.

1 Textbook Option

Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Factor completely. If the polynomial is not factorable, write *prime*.

1. $-16a^2 + 4a$

2. $5x^2 - 20$

3. $x^3 + 9$

4. $2y^2 - y - 15$

5. GEOMETRY The area of a rectangular piece of cardboard is $x^2 + 6x + 8$ square inches. If the cardboard has a length of $(x + 4)$ inches, what is the width?

Solve each equation by factoring.

6. $x^2 + 6x = 0$

7. $x^2 + 2x - 35 = 0$

8. $x^2 - 9 = 0$

9. $x^2 - 7x + 12 = 0$

10. GARDENING Peyton is building a flower bed in her back yard. The area of the flower bed will be 42 square feet. Find the possible values for x.

Find the exact value of each trigonometric function.

11. sin 45°

12. cos 225°

13. tan 150°

14. sin 120°

15. RIDES The distance from the highest point of a Ferris wheel to the ground can be found by multiplying 90 feet by sin 90°. What is the height of the Ferris wheel when it is halfway between the tallest point and the ground?

QuickReview

Example 1

Factor $x^3 + 2x^2 - 24x$ completely.

$x^3 + 2x^2 - 24x = x(x^2 + 2x - 24)$

The product of the coefficients of the x terms must be -24, and their sum must be 2. The product of 6 and -4 is -24 and their sum is 2.

$x(x^2 + 2x - 24) = x(x + 6)(x - 4)$

Example 2

Solve $x^2 + 6x + 5 = 0$ by factoring.

$x^2 + 6x + 5 = 0$ Original equation

$(x + 5)(x + 1) = 0$ Factor.

$x + 5 = 0$ or $x + 1 = 0$

$x = -5$ $x = -1$

The solution set is $\{-5, -1\}$.

Example 3

Find the exact value of cos 135°.

The reference angle is 180° − 135° or 45°.

cos 45° is $\frac{\sqrt{2}}{2}$. Since 135° is in the second quadrant, cos 135° $= -\frac{\sqrt{2}}{2}$.

2 Online Option

Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

Get Started on the Chapter

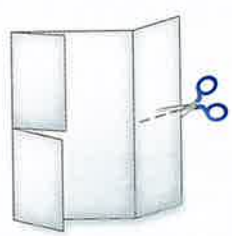
You will learn several new concepts, skills, and vocabulary terms as you study Chapter 13. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

FOLDABLES StudyOrganizer

Trigonometric Identities and Equations Make this Foldable to help you organize your Chapter 13 notes about trigonometric identities and equations. Begin with one sheet of 11″ × 17″ paper and four sheets of grid paper.

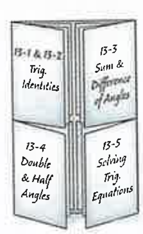
1 **Fold** the short sides of the 11″ × 17″ paper to meet in the middle.

2 **Cut** each tab in half as shown.



3 **Cut** four sheets of grid paper in half and fold the half-sheets in half.

4 **Insert** two folded half-sheets under each of the four tabs and staple along the fold. Label each tab as shown.



NewVocabulary

English		Español
trigonometric identity	p. 873	identidad trigonométrica
quotient identity	p. 873	identidad de cociente
reciprocal identity	p. 873	identidad recíproca
Pythagorean identity	p. 873	identidad Pitagórica
cofunction identity	p. 873	identidad de función conjunta
negative angle identity	p. 873	identidad negativa de ángulo
trigonometric equation	p. 901	ecuación trigonométrica

ReviewVocabulary

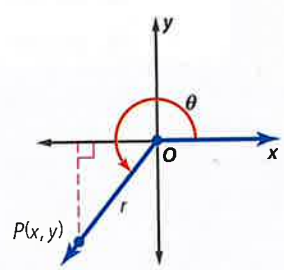
formula fórmula a mathematical sentence that expresses the relationship between certain quantities

identity identidad an equality that remains true regardless of the values of any variables that are in it

trigonometric functions funciones rigonométricas For any angle, with measure θ, a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, the trigonometric functions of θ are as follows.

$\sin \theta = \frac{y}{r}$ $\cos \theta = \frac{x}{r}$ $\tan \theta = \frac{y}{x}$

$\csc \theta = \frac{r}{y}$ $\sec \theta = \frac{r}{x}$ $\cot \theta = \frac{x}{y}$



LESSON

13-1 Trigonometric Identities

Then

- You evaluated trigonometric functions.

Now

1. Use trigonometric identities to find trigonometric values.
2. Use trigonometric identities to simplify expressions.

Why?

- The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface is related to the distance R in feet from the light source. The formula $\sec \theta = \frac{I}{ER^2}$, where I is the intensity of the light source measured in candles and θ is the angle between the light beam and a line perpendicular to the surface, can be used in situations in which lighting is important, as in photography.

NewVocabulary

trigonometric identity

Common Core State Standards

Content Standards

F.TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Mathematical Practices

2 Reason abstractly and quantitatively.

7 Look for and make use of structure.

1 Find Trigonometric Values

The equation above can also be written as $E = \frac{I \cos \theta}{R^2}$. This is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

If you can show that a specific value of the variable in an equation makes the equation false, then you have produced a *counterexample*. It only takes one counterexample to prove that an equation is not an identity.

KeyConcept Basic Trigonometric Identities

Quotient Identities

$\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cos \theta \neq 0$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$, $\sin \theta \neq 0$

Reciprocal Identities

$\sin \theta = \frac{1}{\csc \theta}$, $\csc \theta \neq 0$	$\csc \theta = \frac{1}{\sin \theta}$, $\sin \theta \neq 0$
$\cos \theta = \frac{1}{\sec \theta}$, $\sec \theta \neq 0$	$\sec \theta = \frac{1}{\cos \theta}$, $\cos \theta \neq 0$
$\tan \theta = \frac{1}{\cot \theta}$, $\cot \theta \neq 0$	$\cot \theta = \frac{1}{\tan \theta}$, $\tan \theta \neq 0$

Pythagorean Identities

$\cos^2 \theta + \sin^2 \theta = 1$	$\tan^2 \theta + 1 = \sec^2 \theta$	$\cot^2 \theta + 1 = \csc^2 \theta$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$

Negative Angle Identities

$\sin(-\theta) = -\sin \theta$	$\cos(-\theta) = \cos \theta$	$\tan(-\theta) = -\tan \theta$

The negative angle identities are sometimes called *odd-even* identities.

The identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$ is true except for angle measures such as 90°, 270°, … , 90° + k180°, where k is an integer. The cosine of each of these angle measures is 0, so $\tan \theta$ is not defined when $\cos \theta = 0$. An identity similar to this is $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

You can use trigonometric identities to find exact values of trigonometric functions. You can find approximate values by using a graphing calculator.

Example 1 Use Trigonometric Identities

a. Find the exact value of $\cos\theta$ if $\sin\theta = \frac{1}{4}$ and $90° < \theta < 180°$.

$\cos^2\theta + \sin^2\theta = 1$ — Pythagorean identity

$\cos^2\theta = 1 - \sin^2\theta$ — Subtract $\sin^2\theta$ from each side.

$\cos^2\theta = 1 - \left(\frac{1}{4}\right)^2$ — Substitute $\frac{1}{4}$ for $\sin\theta$.

$\cos^2\theta = 1 - \frac{1}{16}$ — Square $\frac{1}{4}$.

$\cos^2\theta = \frac{15}{16}$ — Subtract: $\frac{16}{16} - \frac{1}{16} = \frac{15}{16}$.

$\cos\theta = \pm\frac{\sqrt{15}}{4}$ — Take the square root of each side.

Since θ is in the second quadrant, $\cos\theta$ is negative. Thus, $\cos\theta = -\frac{\sqrt{15}}{4}$.

CHECK Use a calculator to find an approximate answer.

Step 1 Find Arcsin $\frac{1}{4}$.

$\sin^{-1}\frac{1}{4} \approx 14.48°$ — Use a calculator.

Because $90° < \theta < 180°$, $\theta \approx 180° - 14.48°$ or about $165.52°$.

Step 2 Find $\cos\theta$.

Replace θ with $165.52°$.

$\cos 165.52° \approx -0.97$

Step 3 Compare with the exact value.

$-\frac{\sqrt{15}}{4} \stackrel{?}{\approx} 0.97$

$-0.968 \approx 0.97$ ✓

b. Find the exact value of $\csc\theta$ if $\cot\theta = -\frac{3}{5}$ and $270° < \theta < 360°$.

$\cot^2\theta + 1 = \csc^2\theta$ — Pythagorean identity

$\left(-\frac{3}{5}\right)^2 + 1 = \csc^2\theta$ — Substitute $-\frac{3}{5}$ for $\cot\theta$.

$\frac{9}{25} + 1 = \csc^2\theta$ — Square $-\frac{3}{5}$.

$\frac{34}{25} = \csc^2\theta$ — Add: $\frac{9}{25} + \frac{25}{25} = \frac{34}{25}$.

$\pm\frac{\sqrt{34}}{5} = \csc\theta$ — Take the square root of each side.

Since θ is in the fourth quadrant, $\csc\theta$ is negative. Thus, $\csc\theta = -\frac{\sqrt{34}}{5}$.

Guided Practice

1A. Find $\sin\theta$ if $\cos\theta = \frac{1}{3}$ and $270° < \theta < 360°$.

1B. Find $\sec\theta$ if $\sin\theta = -\frac{2}{7}$ and $180° < \theta < 270°$.

Study Tip

Quadrants Here is a table to help you remember which ratios are positive and which are negative in each quadrant.

Function	+	−
$\sin\theta$	1, 2	3, 4
$\cos\theta$	1, 4	2, 3
$\tan\theta$	1, 3	2, 4
$\csc\theta$	1, 2	3, 4
$\sec\theta$	1, 4	2, 3
$\cot\theta$	1, 3	2, 4

2 Simplify Expressions

Simplifying an expression that contains trigonometric functions means that the expression is written as a numerical value or in terms of a single trigonometric function, if possible.

StudyTip

Simplifying It is often easiest to write all expressions in terms of sine and/or cosine.

Example 2 Simplify an Expression

Simplify $\frac{\sin \theta \csc \theta}{\cot \theta}$.

$$\frac{\sin \theta \csc \theta}{\cot \theta} = \frac{\cancel{\sin \theta}\left(\frac{1}{\cancel{\sin \theta}}\right)}{\frac{1}{\tan \theta}}$$ $\csc \theta = \frac{1}{\sin \theta}$ and $\cot \theta = \frac{1}{\tan \theta}$

$$= \frac{1}{\frac{1}{\tan \theta}}$$ $\frac{\sin \theta}{\sin \theta} = 1$

$$= \frac{1}{1} \cdot \frac{\tan \theta}{1} \text{ or } \tan \theta$$ $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

GuidedPractice

Simplify each expression.

2A. $\frac{\tan^2 \theta \csc^2 \theta - 1}{\sec^2 \theta}$

2B. $\frac{\sec \theta}{\sin \theta}(1 - \cos^2 \theta)$

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Math HistoryLink

Aryabhatta (476–550 A.D.) Among Indian mathematicians, Aryabhatta is probably the most famous. His name is closely associated with trigonometry. He was the first to introduce inverse trigonometric functions and spherical trigonometry. Aryabhatta also calculated approximations for pi and trigonometric functions.

Simplifying trigonometric expressions can be helpful when solving real-world problems.

Real-World Example 3 Simplify and Use an Expression

LIGHTING **Refer to the beginning of the lesson.**

a. Solve the formula in terms of E.

$\sec \theta = \frac{I}{ER^2}$ Original equation

$ER^2 \sec \theta = I$ Multiply each side by ER^2.

$ER^2 \frac{1}{\cos \theta} = I$ $\frac{1}{\cos \theta} = \sec \theta$

$\frac{E}{\cos \theta} = \frac{I}{R^2}$ Divide each side by R^2.

$E = \frac{I \cos \theta}{R^2}$ Multiply each side by $\cos \theta$.

b. Is the equation in part a equivalent to $R^2 = \frac{I \tan \theta \cos \theta}{E}$? Explain.

$R^2 = \frac{I \tan \theta \cos \theta}{E}$ Original equation

$ER^2 = I \tan \theta \cos \theta$ Multiply each side by E.

$E = \frac{I \tan \theta \cos \theta}{R^2}$ Divide each side by R^2.

$E = \frac{I\left(\frac{\sin \theta}{\cos \theta}\right) \cos \theta}{R^2}$ $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$E = \frac{I \sin \theta}{R^2}$ Simplify.

No; the equations are not equivalent. $R^2 = \frac{I \tan \theta \cos \theta}{E}$ simplifies to $E = \frac{I \sin \theta}{R^2}$.

GuidedPractice

3. Rewrite $\cot^2 \theta - \tan^2 \theta$ in terms of $\sin \theta$.

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Example 1 Find the exact value of each expression if $0° < \theta < 90°$.

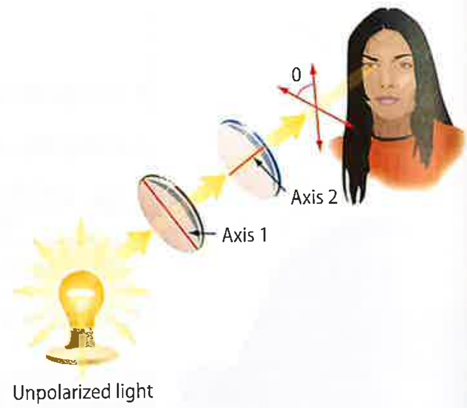
1. If $\cot \theta = 2$, find $\tan \theta$.
2. If $\sin \theta = \frac{4}{5}$, find $\cos \theta$.
3. If $\cos \theta = \frac{2}{3}$, find $\sin \theta$.
4. If $\cos \theta = \frac{2}{3}$, find $\csc \theta$.

Example 2 Simplify each expression.

5. $\tan \theta \cos^2 \theta$
6. $\csc^2 \theta - \cot^2 \theta$
7. $\frac{\cos \theta \csc \theta}{\tan \theta}$

Example 3

8. **CCSS PERSEVERANCE** When unpolarized light passes through polarized sunglass lenses, the intensity of the light is cut in half. If the light then passes through another polarized lens with its axis at an angle of θ to the first, the intensity of the light is again diminished. The intensity of the emerging light can be found by using the formula $I = I_0 - \frac{I_0}{\csc^2 \theta}$, where I_0 is the intensity of the light incoming to the second polarized lens, I is the intensity of the emerging light, and θ is the angle between the axes of polarization.



   a. Simplify the formula in terms of $\cos \theta$.

   b. Use the simplified formula to determine the intensity of light that passes through a second polarizing lens with axis at 30° to the original.

Practice and Problem Solving

Extra Practice is on page R13.

Example 1 Find the exact value of each expression if $0° < \theta < 90°$.

9. If $\cos \theta = \frac{3}{5}$, find $\csc \theta$.
10. If $\sin \theta = \frac{1}{2}$, find $\tan \theta$.
11. If $\sin \theta = \frac{3}{5}$, find $\cos \theta$.
12. If $\tan \theta = 2$, find $\sec \theta$.

Find the exact value of each expression if $180° < \theta < 270°$.

13. If $\cos \theta = -\frac{3}{5}$, find $\csc \theta$.
14. If $\sec \theta = -3$, find $\tan \theta$.
15. If $\cot \theta = \frac{1}{4}$, find $\csc \theta$.
16. If $\sin \theta = -\frac{1}{2}$, find $\cos \theta$.

Find the exact value of each expression if $270° < \theta < 360°$.

17. If $\cos \theta = \frac{5}{13}$, find $\sin \theta$.
18. If $\tan \theta = -1$, find $\sec \theta$.
19. If $\sec \theta = \frac{5}{3}$, find $\cos \theta$.
20. If $\csc \theta = -\frac{5}{3}$, find $\cos \theta$.

Example 2 Simplify each expression.

21. $\sec \theta \tan^2 \theta + \sec \theta$
22. $\cos\left(\frac{\pi}{2} - \theta\right)\cot \theta$
23. $\cot \theta \sec \theta$
24. $\sin \theta (1 + \cot^2 \theta)$
25. $\sin\left(\frac{\pi}{2} - \theta\right)\sec \theta$
26. $\frac{\cos(-\theta)}{\sin(-\theta)}$

Example 3

27 **ELECTRONICS** When there is a current in a wire in a magnetic field, such as in a hairdryer, a force acts on the wire. The strength of the magnetic field can be determined using the formula $B = \frac{F \csc \theta}{I\ell}$, where F is the force on the wire, I is the current in the wire, ℓ is the length of the wire, and θ is the angle the wire makes with the magnetic field. Rewrite the equation in terms of $\sin \theta$. (*Hint:* Solve for F.)

Simplify each expression.

28. $\frac{1 - \sin^2 \theta}{\sin^2 \theta}$

29. $\tan \theta \csc \theta$

30. $\frac{1}{\sin^2 \theta} - \frac{\cos^2 \theta}{\sin^2 \theta}$

31. $2(\csc^2 \theta - \cot^2 \theta)$

32. $(1 + \sin \theta)(1 - \sin \theta)$

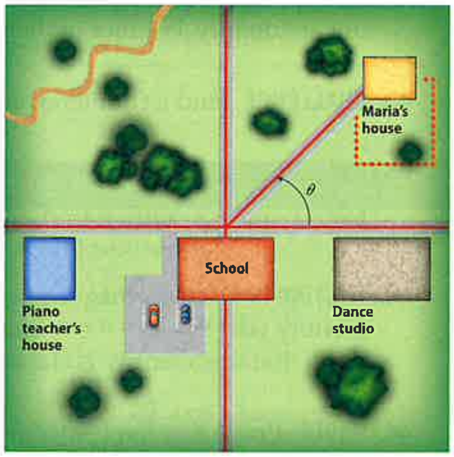
33. $2 - 2 \sin^2 \theta$

34. **SUN** The ability of an object to absorb energy is related to a factor called the emissivity e of the object. The emissivity can be calculated by using the formula $e = \frac{W \sec \theta}{AS}$, where W is the rate at which a person's skin absorbs energy from the Sun, S is the energy from the Sun in watts per square meter, A is the surface area exposed to the Sun, and θ is the angle between the Sun's rays and a line perpendicular to the body.

a. Solve the equation for W. Write your answer using only $\sin \theta$ or $\cos \theta$.

b. Find W if $e = 0.80$, $\theta = 40°$, $A = 0.75\ \text{m}^2$, and $S = 1000\ W/\text{m}^2$. Round to the nearest hundredth.

35. **CCSS MODELING** The map shows some of the buildings in Maria's neighborhood that she visits on a regular basis. The sine of the angle θ formed by the roads connecting the dance studio, the school, and Maria's house is $\frac{4}{9}$.



a. What is the cosine of the angle?

b. What is the tangent of the angle?

c. What are the sine, cosine, and tangent of the angle formed by the roads connecting the piano teacher's house, the school, and Maria's house?

36. **MULTIPLE REPRESENTATIONS** In this problem, you will use a graphing calculator to determine whether an equation may be a trigonometric identity. Consider the trigonometric identity $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$.

a. Tabular Copy and complete the table below.

θ	0°	30°	45°	60°
$\tan^2 \theta - \sin^2 \theta$				
$\tan^2 \theta \sin^2 \theta$				

b. Graphical Use a graphing calculator to graph $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ as two separate functions. Sketch the graph.

c. Analytical If the graphs of the two functions do not match, then the equation is not an identity. Do the graphs coincide?

d. Analytical Use a graphing calculator to determine whether the equation $\sec^2 x - 1 = \sin^2 x \sec^2 x$ may be an identity. (Be sure your calculator is in degree mode.)

37. **SKIING** A skier of mass m descends a θ-degree hill at a constant speed. When Newton's laws are applied to the situation, the following system of equations is produced: $F_n - mg \cos \theta = 0$ and $mg \sin \theta - \mu_k F_n = 0$, where g is the acceleration due to gravity, F_n is the normal force exerted on the skier, and μ_k is the coefficient of friction. Use the system to define μ_k as a function of θ.

Simplify each expression.

38. $\dfrac{\tan\left(\frac{\pi}{2} - \theta\right)\sec \theta}{1 - \csc^2 \theta}$

39. $\dfrac{\cos\left(\frac{\pi}{2} - \theta\right) - 1}{1 + \sin(-\theta)}$

40. $\dfrac{\sec \theta \sin \theta + \cos\left(\frac{\pi}{2} - \theta\right)}{1 + \sec \theta}$

41. $\dfrac{\cot \theta \cos \theta}{\tan(-\theta) \sin\left(\frac{\pi}{2} - \theta\right)}$

H.O.T. Problems Use Higher-Order Thinking Skills

42. CCSS **CRITIQUE** Clyde and Rosalina are debating whether an equation from their homework assignment is an identity. Clyde says that since he has tried ten specific values for the variable and all of them worked, it must be an identity. Rosalina argues that specific values could only be used as counterexamples to prove that an equation is not an identity. Is either of them correct? Explain your reasoning.

43. **CHALLENGE** Find a counterexample to show that $1 - \sin x = \cos x$ is *not* an identity.

44. **REASONING** Demonstrate how the formula about illuminance from the beginning of the lesson can be rewritten to show that $\cos \theta = \dfrac{ER^2}{I}$.

45. **WRITING IN MATH** Pythagoras is most famous for the Pythagorean Theorem. The identity $\cos^2 \theta + \sin^2 \theta = 1$ is an example of a Pythagorean identity. Why do you think that this identity is classified in this way?

46. **PROOF** Prove that $\tan(-a) = -\tan a$ by using the quotient and negative angle identities.

47. **OPEN ENDED** Write two expressions that are equivalent to $\tan \theta \sin \theta$.

48. **REASONING** Explain how you can use division to rewrite $\sin^2 \theta + \cos^2 \theta = 1$ as $1 + \cot^2 \theta = \csc^2 \theta$.

49. **CHALLENGE** Find $\cot \theta$ if $\sin \theta = \frac{3}{5}$ and $90° \leq \theta < 180°$.

50. **ERROR ANALYSIS** Jordan and Ebony are simplifying $\dfrac{\sin^2 \theta}{\cos^2 \theta + \sin^2 \theta}$. Is either of them correct? Explain your reasoning.

Jordan

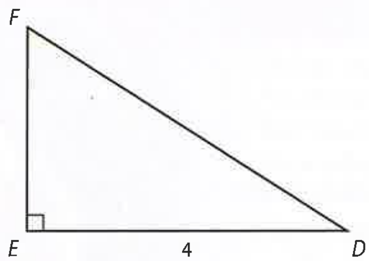
$$\frac{\sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} = \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta}$$
$$= \tan^2 \theta + 1$$
$$= \sec^2 \theta$$

Ebony

$$\frac{\sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} = \frac{\sin^2 \theta}{1}$$
$$= \sin^2 \theta$$

Standardized Test Practice

51. Refer to the figure below. If $\cos D = 0.8$, what is the length of $\overline{DF}$?



A 5
B 4
C 3.2
D $\frac{4}{5}$

52. PROBABILITY There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

F 4
G 6
H 8
J 12

53. SAT/ACT Ella is 6 years younger than Amanda. Zoe is twice as old as Amanda. The total of their ages is 54. Which equation can be used to find Amanda's age?

A $x + (x - 6) + 2(x - 6) = 54$
B $x - 6x + (x + 2) = 54$
C $x - 6 + 2x = 54$
D $x + (x - 6) + 2x = 54$
E $2(x + 6) + (x + 6) + x = 54$

54. Which of the following functions represents exponential growth?

F $y = (0.3)^x$
G $y = (1.3)^x$
H $y = x^3$
J $y = x^{\frac{1}{3}}$

Spiral Review

Find each value. Write angle measures in radians. Round to the nearest hundredth. (Lesson 12-9)

55. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$

56. $\text{Sin}^{-1}\frac{\pi}{2}$

57. $\text{Arctan}\,\frac{\sqrt{3}}{3}$

58. $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$

59. $\sin\left(\text{Arctan}\,\frac{\sqrt{3}}{3}\right)$

60. $\cos\left(\text{Arcsin}\,\frac{3}{5}\right)$

61. PHYSICS A weight is attached to a spring and suspended from the ceiling. At equilibrium, the weight is located 4 feet above the floor. The weight is pulled down 1 foot and released. Write the equation for the distance d of the weight above the floor as a function of time t seconds assuming the weight returns to its lowest position every 4 seconds. (Lesson 12-8)

Evaluate the sum of each geometric series. (Lesson 10-3)

62. $\sum_{k=1}^{5} \frac{1}{4} \cdot 2^{k-1}$

63. $\sum_{k=1}^{7} 81\left(\frac{1}{3}\right)^{k-1}$

64. $\sum_{k=1}^{8} \frac{1}{3} \cdot 5^{k-1}$

Skills Review

Solve each equation.

65. $a + 1 = \frac{6}{a}$

66. $\frac{9}{t-3} = \frac{t-4}{t-3} + \frac{1}{4}$

67. $\frac{5}{x+1} - \frac{1}{3} = \frac{x+2}{x+1}$

LESSON 13-2 Verifying Trigonometric Identities

Then

- You used identities to find trigonometric values and simplify expressions.

Now

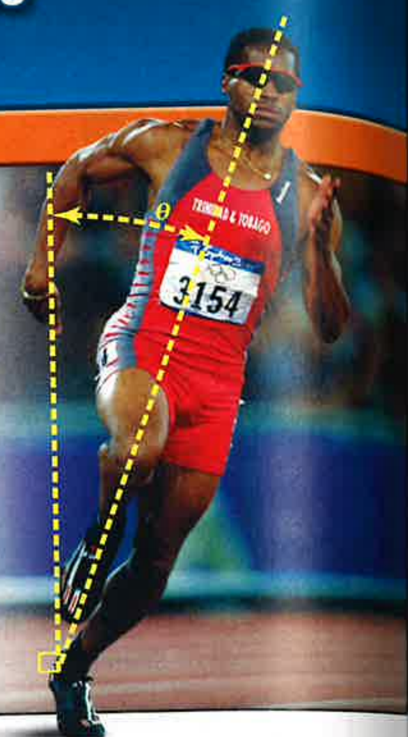
1. Verify trigonometric identities by transforming one side of an equation into the form of the other side.
2. Verify trigonometric identities by transforming each side of the equation into the same form.

Why?

- While running on a circular track, Lamont notices that his body is not perpendicular to the ground. Instead, it leans away from a vertical position. The nonnegative acute angle θ that Lamont's body makes with the vertical is called the *angle of incline* and is described by the equation $\tan \theta = \frac{v^2}{gR}$.

  This is not the only equation that describes the angle of incline in terms of trigonometric functions. Another such equation is $\sin \theta = \cos \frac{v^2}{gR}\theta$, where $0 \leq \theta \leq 90°$.

  Are these two equations completely independent of one another or are they merely different versions of the same relationship?



Common Core State Standards

Content Standards

F.TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Mathematical Practices

1 Make sense of problems and persevere in solving them.

8 Look for and express regularity in repeated reasoning.

1 Transform One Side of an Equation

You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. If you wish to show that an identity, you need to show that it is true for all values of θ.

KeyConcept Verifying Identities by Transforming One Side

Step 1 Simplify one side of an equation until the two sides of the equation are the same. It is often easier to work with the more complicated side of the equation.

Step 2 Transform that expression into the form of the simpler side.

PT

Example 1 Transform One Side of an Equation

Verify that $\frac{\sin^2 \theta}{1 - \cos \theta} = 1 + \cos \theta$ is an identity.

$$\frac{\sin^2 \theta}{1 - \cos \theta} \stackrel{?}{=} 1 + \cos \theta \qquad \text{Original equation}$$

$$\frac{1 + \cos \theta}{1 + \cos \theta} \cdot \frac{\sin^2 \theta}{1 - \cos \theta} \stackrel{?}{=} 1 + \cos \theta \qquad \text{Multiply the numerator and denominator by } 1 + \cos \theta.$$

$$\frac{\sin^2 \theta(1 + \cos \theta)}{1 - \cos^2 \theta} \stackrel{?}{=} 1 + \cos \theta \qquad (1 + \cos \theta)(1 - \cos \theta) = 1 - \cos^2 \theta$$

$$\frac{\sin^2 \theta(1 + \cos \theta)}{\sin^2 \theta} \stackrel{?}{=} 1 + \cos \theta \qquad \sin^2 \theta = 1 - \cos^2 \theta$$

$$1 + \cos \theta = 1 + \cos \theta \ ✓ \qquad \text{Divide the numerator and denominator by } \sin^2 \theta.$$

GuidedPractice

1. Verify that $\cot^2 \theta - \cos^2 \theta = \cot^2 \theta \cos^2 \theta$ is an identity.

When you verify a trigonometric identity, you are really working backward. In Example 1, consider the last step $1 + \cos\theta = 1 + \cos\theta$. Since that step is clearly true, you can conclude that the next-to-last step is also true, and so on, all the way back to the original equation.

Standardized Test Example 2 Simplify an Expression

PT

$\frac{\cos\theta \csc\theta}{\tan\theta} =$

A $\cot\theta$ **B** $\csc\theta$ **C** $\cot^2\theta$ **D** $\csc^2\theta$

Read the Test Item

Find an expression that is always equal to the given expression. Notice that all of the answer choices involve either $\cot\theta$ or $\csc\theta$. So work toward eliminating the other trigonometric functions.

Solve the Test Item

Transform the given expression to match one of the choices.

$$\frac{\cos\theta\csc\theta}{\tan\theta} = \frac{\cos\theta\left(\frac{1}{\sin\theta}\right)}{\frac{\sin\theta}{\cos\theta}}$$ $\csc\theta = \frac{1}{\sin\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}}$$ Multiply.

$$= \frac{\cos\theta}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta}$$ Invert the denominator and multiply.

$$= \cot\theta \cdot \cot\theta$$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$= \cot^2\theta$$ Multiply.

The answer is C.

WatchOut!

CCSS Perseverance Verifying an identity is like checking the solution of an equation. You must simplify one or both sides separately until they are the same.

Test-TakingTip

Checking Answers Verify your answer by choosing values for θ. Then evaluate the original expression and compare to your answer choice.

GuidedPractice

2. $\tan^2\theta(\cot^2\theta - \cos^2\theta) =$

F $\cot^2\theta$ **G** $\tan^2\theta$ **H** $\cos^2\theta$ **J** $\sin^2\theta$

2 Transform Each Side of an Equation

Sometimes it is easier to transform each side of an equation separately into a common form. The following suggestions may be helpful as you verify trigonometric identities.

KeyConcept Suggestions for Verifying Identities

- Substitute one or more basic trigonometric identities to simplify the expression.
- Factor or multiply as necessary. You may have to multiply both the numerator and denominator by the same trigonometric expression.
- Write each side of the identity in terms of sine and cosine only. Then simplify each side as much as possible.
- The properties of equality do not apply to identities as with equations. Do not perform operations to the quantities on each side of an unverified identity.

Example 3 Verify by Transforming Each Side

Verify that $1 - \tan^4 \theta = 2\sec^2 \theta - \sec^4 \theta$ is an identity.

$1 - \tan^4 \theta \stackrel{?}{=} 2\sec^2 \theta - \sec^4 \theta$	Original equation
$(1 - \tan^2 \theta)(1 + \tan^2 \theta) \stackrel{?}{=} \sec^2 \theta\,(2 - \sec^2 \theta)$	Factor each side.
$[1 - (\sec^2 \theta - 1)]\sec^2 \theta \stackrel{?}{=} (2 - \sec^2 \theta)\sec^2 \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$(2 - \sec^2 \theta)\sec^2 \theta = (2 - \sec^2 \theta)\sec^2 \theta$ ✓	Simplify.

GuidedPractice

3. Verify that $\csc^2 \theta - \cot^2 \theta = \cot \theta \tan \theta$ is an identity.

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Examples 1–3 CCSS **PRECISION** **Verify that each equation is an identity.**

1. $\cot \theta + \tan \theta = \dfrac{\sec^2 \theta}{\tan \theta}$

2. $\cos^2 \theta = (1 + \sin \theta)(1 - \sin \theta)$

3. $\sin \theta = \dfrac{\sec \theta}{\tan \theta + \cot \theta}$

4. $\tan^2 \theta = \dfrac{1 - \cos^2 \theta}{\cos^2 \theta}$

5. $\tan^2 \theta \csc^2 \theta = 1 + \tan^2 \theta$

6. $\tan^2 \theta = (\sec \theta + 1)(\sec \theta - 1)$

Example 2

7. **MULTIPLE CHOICE** Which expression can be used to form an identity with $\dfrac{\tan^2 \theta + 1}{\tan^2 \theta}$?

A $\sin^2 \theta$ **B** $\cos^2 \theta$ **C** $\tan^2 \theta$ **D** $\csc^2 \theta$

Practice and Problem Solving

Extra Practice is on page R13.

Example 1 **Verify that each equation is an identity.**

8. $\cos^2 \theta + \tan^2 \theta \cos^2 \theta = 1$

9. $\cot \theta\,(\cot \theta + \tan \theta) = \csc^2 \theta$

10. $1 + \sec^2 \theta \sin^2 \theta = \sec^2 \theta$

11. $\sin \theta \sec \theta \cot \theta = 1$

12. $\dfrac{1 - \cos \theta}{1 + \cos \theta} = (\csc \theta - \cot \theta)^2$

13. $\dfrac{1 - 2\cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta - \cot \theta$

14. $\tan \theta = \dfrac{\sec \theta}{\csc \theta}$

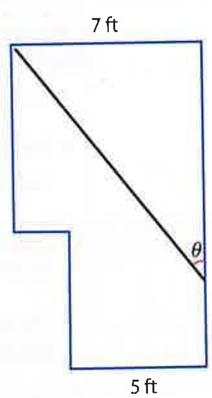
15. $\cos \theta = \sin \theta \cot \theta$

16. $(\sin \theta - 1)(\tan \theta + \sec \theta) = -\cos \theta$

17. $\cos \theta \cos(-\theta) - \sin \theta \sin(-\theta) = 1$

Example 2

18. **LADDER** Some students derived an expression for the length of a ladder that, when carried flat, could fit around a corner from a 5-foot-wide hallway into a 7-foot-wide hallway, as shown. They determined that the maximum length ℓ of a ladder that would fit was given by $\ell(\theta) = \dfrac{7\sin \theta + 5\cos \theta}{\sin \theta \cos \theta}$. When their teacher worked the problem, she concluded that $\ell(\theta) = 7\sec \theta + 5\csc \theta$. Are the two expressions equivalent?



Example 3 **Verify that each equation is an identity.**

19. $\sec\theta - \tan\theta = \frac{1-\sin\theta}{\cos\theta}$

20. $\frac{1+\tan\theta}{\sin\theta+\cos\theta} = \sec\theta$

21. $\sec\theta\csc\theta = \tan\theta + \cot\theta$

22. $\sin\theta + \cos\theta = \frac{2\sin^2\theta - 1}{\sin\theta - \cos\theta}$

23. $(\sin\theta+\cos\theta)^2 = \frac{2+\sec\theta\csc\theta}{\sec\theta\csc\theta}$

24. $\frac{\cos\theta}{1-\sin\theta} = \frac{1+\sin\theta}{\cos\theta}$

25. $\csc\theta - 1 = \frac{\cot^2\theta}{\csc\theta+1}$

26. $\cos\theta\cot\theta = \csc\theta - \sin\theta$

27. $\sin\theta\cos\theta\tan\theta + \cos^2\theta = 1$

28. $(\csc\theta - \cot\theta)^2 = \frac{1-\cos\theta}{1+\cos\theta}$

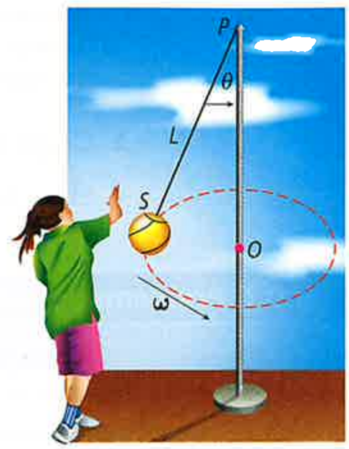
29. $\csc^2\theta = \cot^2\theta + \sin\theta\csc\theta$

30. $\frac{\sec\theta - \csc\theta}{\csc\theta\sec\theta} = \sin\theta - \cos\theta$

31. $\sin^2\theta + \cos^2\theta = \sec^2\theta - \tan^2\theta$

32. $\sec\theta - \cos\theta = \tan\theta\sin\theta$

33. CCSS **SENSE-MAKING** The diagram at the right represents a game of tetherball. As the ball rotates around the pole, a conical surface is swept out by the line segment $\overline{SP}$. A formula for the relationship between the length L of the string and the angle θ that the string makes with the pole is given by the equation $L = \frac{g\sec\theta}{\omega^2}$. Is $L = \frac{g\tan\theta}{\omega^2\sin\theta}$ also an equation for the relationship between L and θ?



34. **RUNNING** A portion of a racetrack has the shape of a circular arc with a radius of 16.7 meters. As a runner races along the arc, the sine of her angle of incline θ is found to be $\frac{1}{4}$. Find the speed of the runner. Use the Angle of Incline Formula given at the beginning of the lesson, $\tan\theta = \frac{v^2}{gR}$, where $g = 9.8$ and R is the radius. (*Hint*: Find $\cos\theta$ first.)

When simplified, would the expression be equal to 1 or −1?

35 $\cot(-\theta)\tan(-\theta)$

36. $\sin\theta\csc(-\theta)$

37. $\sin^2(-\theta) + \cos^2(-\theta)$

38. $\sec(-\theta)\cos(-\theta)$

39. $\sec^2(-\theta) - \tan^2(-\theta)$

40. $\cot(-\theta)\cot\left(\frac{\pi}{2} - \theta\right)$

Simplify the expression to either a constant or a basic trigonometric function.

41. $\frac{\tan\left(\frac{\pi}{2}-\theta\right)\csc\theta}{\csc^2\theta}$

42. $\frac{1+\tan\theta}{1+\cot\theta}$

43. $(\sec^2\theta + \csc^2\theta) - (\tan^2\theta + \cot^2\theta)$

44. $\frac{\sec^2\theta - \tan^2\theta}{\cos^2 x + \sin^2 x}$

45. $\tan\theta\cos\theta$

46. $\cot\theta\tan\theta$

47. $\sec\theta\sin\left(\frac{\pi}{2} - \theta\right)$

48. $\frac{1+\tan^2\theta}{\csc^2\theta}$

49. **PHYSICS** When a firework is fired from the ground, its height y and horizontal displacement x are related by the equation $y = \frac{-gx^2}{2v_0^2\cos^2\theta} + \frac{x\sin\theta}{\cos\theta}$, where v_0 is the initial velocity of the projectile, θ is the angle at which it was fired, and g is the acceleration due to gravity. Rewrite this equation so that $\tan\theta$ is the only trigonometric function that appears in the equation.

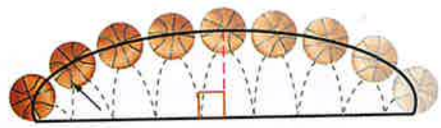
50. ELECTRONICS When an alternating current of frequency f and peak current I_0 passes through a resistance R, the power delivered to the resistance at time t seconds is $P = I_0^2 R \sin^2 2\pi ft$.

a. Write an expression for the power in terms of $\cos^2 2\pi ft$.

b. Write an expression for the power in terms of $\csc^2 2\pi ft$.

51. THROWING A BALL In this problem, you will investigate the path of a ball represented by the equation $h = \frac{v_0^2 \sin^2 \theta}{2g}$, where θ is the measure of the angle between the ground and the path of the ball, v_0 is its initial velocity in meters per second, and g is the acceleration due to gravity. The value of g is 9.8 m/s^2.

a. If the initial velocity of the ball is 47 meters per second, find the height of the ball at 30°, 45°, 60°, and 90°. Round to the nearest tenth.



b. Graph the equation on a graphing calculator.

c. Show that the formula $h = \frac{v_0^2 \tan^2 \theta}{2g \sec^2 \theta}$ is equivalent to the one given above.

H.O.T. Problems Use Higher-Order Thinking Skills

52. CCSS ARGUMENTS Identify the equation that does not belong with the other three. Explain your reasoning.

$\sin^2 \theta + \cos^2 \theta = 1$	$1 + \cot^2 \theta = \csc^2 \theta$
$\sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta$	$\tan^2 \theta + 1 = \sec^2 \theta$

53. CHALLENGE Transform the right side of $\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$ to show that $\tan^2 \theta = \sec^2 \theta - 1$.

54. WRITING IN MATH Explain why you cannot square each side of an equation when verifying a trigonometric identity.

55. REASONING Explain why $\sin^2 \theta + \cos^2 \theta = 1$ is an identity, but $\sin \theta = \sqrt{1 - \cos \theta}$ is not.

56. WRITE A QUESTION A classmate is having trouble trying to verify a trigonometric identity involving multiple trigonometric functions to multiple degrees. Write a question to help her work through the problem.

57. WRITING IN MATH Why do you think expressions in trigonometric identities are often rewritten in terms of sine and cosine?

58. CHALLENGE Let $x = \frac{1}{2} \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Write $f(x) = \frac{x}{\sqrt{1 + 4x^2}}$ in terms of a single trigonometric function of θ.

59. REASONING Justify the three basic Pythagorean identities.

Standardized Test Practice

60. SAT/ACT A small business owner must hire seasonal workers as the need arises. The following list shows the number of employees hired monthly for a 5-month period.

5, 14, 6, 8, 12

If the mean of these data is 9, what is the population standard deviation for these data? (Round your answer to the nearest tenth.)

A 3.5
B 3.9
C 5.7
D 8.6
E 12.3

61. Find the center and radius of the circle with equation $(x - 4)^2 + y^2 - 16 = 0$.

F $C(-4, 0)$; $r = 4$ units
G $C(-4, 0)$; $r = 16$ units
H $C(4, 0)$; $r = 4$ units
J $C(4, 0)$; $r = 16$ units

62. GEOMETRY The perimeter of a right triangle is 36 inches. Twice the length of the longer leg minus twice the length of the shorter leg is 6 inches. What are the lengths of all three sides?

A 3 in., 4 in., 5 in.
B 6 in., 8 in., 10 in.
C 9 in., 12 in., 15 in.
D 12 in., 16 in., 20 in.

63. Simplify $128^{\frac{1}{4}}$.

F $2\sqrt[4]{2}$
G $2\sqrt[4]{8}$
H 4
J $4\sqrt[4]{2}$

Spiral Review

Find the exact value of each expression. (Lesson 13-1)

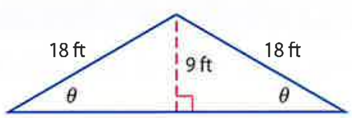
64. $\tan \theta$, if $\cot \theta = 2$; $0° \leq \theta < 90°$

65. $\sin \theta$, if $\cos \theta = \frac{2}{3}$; $0° \leq \theta < 90°$

66. $\csc \theta$, if $\cos \theta = -\frac{3}{5}$; $90° < \theta < 180°$

67. $\cos \theta$, if $\sec \theta = \frac{5}{3}$; $270° < \theta < 360°$

68. ARCHITECTURE The support for a roof is shaped like two right triangles, as shown at the right. Find θ. (Lesson 12-9)



69. FAST FOOD The table shows the probability distribution for value meals ordered at a fast food restaurant on Saturday mornings. Use this information to determine the expected value of the meals ordered. (Lesson 11-3)

Value Meals Ordered				
Meals	\$3	\$4	\$5	\$6
Probability	0.5	0.2	0.1	0.2

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbolas with the given equations. Then graph the hyperbola. (Lesson 9-5)

70. $\frac{y^2}{18} - \frac{x^2}{20} = 1$

71. $\frac{(y + 6)^2}{20} - \frac{(x - 1)^2}{25} = 1$

72. $x^2 - 36y^2 = 36$

Skills Review

Simplify.

73. $\frac{2 + \sqrt{2}}{5 - \sqrt{2}}$

74. $\frac{x + 1}{\sqrt{x^2 - 1}}$

75. $\frac{x - 1}{\sqrt{x} - 1}$

76. $\frac{-2 - \sqrt{3}}{1 + \sqrt{3}}$

LESSON 13-3 Sum and Difference of Angles Identities

Then

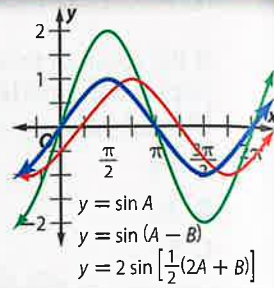
- You found values of trigonometric functions for general angles.

Now

1. Find values of sine and cosine by using sum and difference identities.
2. Verify trigonometric identities by using sum and difference identities.

Why?

- Have you ever been using a wireless Internet provider and temporarily lost the signal? Waves that pass through the same place at the same time cause interference. Interference occurs when two waves combine to have a greater, or smaller, amplitude than either of the component waves.



CCSS Common Core State Standards

Content Standards

F.TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Mathematical Practices

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

1 Sum and Difference Identities

Notice that the third equation shown above involves the sum of A and B. It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find the exact value of sin 15° by evaluating sin (60° − 45°). Formulas exist that can be used to evaluate expressions like $\sin(A - B)$ or $\cos(A + B)$.

KeyConcept Sum and Difference Identities

Sum Identities	Difference Identities
• $\sin(A + B) = \sin A\cos B + \cos A\sin B$	• $\sin(A - B) = \sin A\cos B - \cos A\sin B$
• $\cos(A + B) = \cos A\cos B - \sin A\sin B$	• $\cos(A - B) = \cos A\cos B + \sin A\sin B$
• $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$	• $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Example 1 Find Trigonometric Values

PT

Find the exact value of each expression.

a. **sin 105°**

Use the identity $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

$\sin 105° = \sin(60° + 45°)$ — $A = 60°$ and $B = 45°$

$= \sin 60°\cos 45° + \cos 60°\sin 45°$ — Sum identity

$= \left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right)$ — Evaluate each expression.

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$ or $\frac{\sqrt{6}+\sqrt{2}}{4}$ — Multiply.

b. **cos (−120°)**

Use the identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$.

$\cos(-120) = \cos(60° - 180°)$ — $A = 60°$ and $B = 180°$

$= \cos 60°\cos 180° + \sin 60°\sin 180°$ — Difference identity

$= \frac{1}{2}\cdot(-1) + \frac{\sqrt{3}}{2}\cdot 0$ — Evaluate each expression.

$= -\frac{1}{2}$ — Multiply.

GuidedPractice

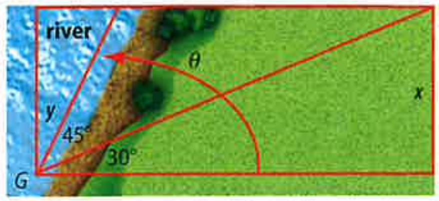
1A. sin 15°

1B. cos (−15°)

You can use the sum and difference of angles identities to solve real-world applications.

Real-World Example 2 Sum and Difference of Angles Identities

A geologist measures the angle between one side of a rectangular lot and the line from her position to the opposite corner of the lot as 30°. She then measures the angle between that line and the line to the point on the property where a river crosses as 45°. She stands 100 yards from the opposite corner of the property. How far is she from the point at which the river crosses the property line?

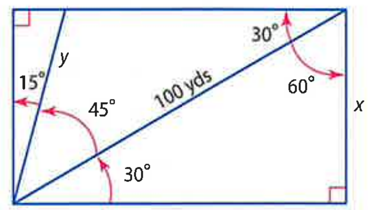


Problem-Solving Tip

Make a Model Make a model to visualize a problem situation. A model can be a drawing or a figure made of different objects, such as algebra tiles or folded paper.

Understand The question asks for the distance between the geologist and the point where the river crosses the property line, or y.

Plan Draw a picture that labels all the things that you know from the information given.



Solve Solve for x.

$\sin 30° = \frac{x}{100}$ Definition of sine

$x = 100 \sin 30°$

$x = 50$ Since the lot is rectangular, opposite sides are equal.

Now look at the triangle on the far left and solve for y.

$\cos 15° = \frac{50}{y}$ Definition of cosine

$\cos (45° - 30°) = \frac{50}{y}$ $15 = 45 - 30$

$\cos 45° \cos 30° + \sin 45° \sin 30° = \frac{50}{y}$ Difference identity

$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{50}{y}$ Evaluate.

$\frac{\sqrt{6} + \sqrt{2}}{4} = \frac{50}{y}$ Simplify.

$(\sqrt{6} + \sqrt{2})y = 200$ Cross products

$$y = \frac{200}{(\sqrt{6} + \sqrt{2})} \cdot \frac{(\sqrt{6} - \sqrt{2})}{(\sqrt{6} - \sqrt{2})}$$

$$y = 50(\sqrt{6} - \sqrt{2})$$

$y = 50\sqrt{6} - 50\sqrt{2}$ or about 51.8

The geologist is about 51.8 yards from the point where the river crosses the property line.

Check Use a calculator to find the Arccos of $\frac{50}{51.8} \approx 15°$. ✓

Guided Practice

2. The harmonic motion of an object can be described by $x = 4 \cos \left(2\pi t - \frac{\pi}{4}\right)$, where x is distance from the equilibrium point in inches and t is time in minutes. Find the exact distance from the equilibrium point at 45 seconds.

StudyTip

CCSS Sense-Making Make a list of the trigonometric values for the angles between 0° and 360° for which the sum and difference identities can be easily used. Use your list as a reference.

2 Verify Trigonometric Identities

You can also use the sum and difference identities to verify identities.

Example 3 Verify Trigonometric Identities

Verify that each equation is an identity.

a. $\cos(90° - \theta) = \sin\theta$

$\cos(90° - \theta) \stackrel{?}{=} \sin\theta$	Original equation
$\cos 90° \cos\theta + \sin 90° \sin\theta \stackrel{?}{=} \sin\theta$	Sum identity
$0 \cdot \cos\theta + 1 \cdot \sin\theta \stackrel{?}{=} \sin\theta$	Evaluate each expression.
$\sin\theta = \sin\theta$ ✓	Simplify.

b. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

$\sin\left(\theta + \frac{\pi}{2}\right) \stackrel{?}{=} \cos\theta$	Original equation
$\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2} \stackrel{?}{=} \cos\theta$	Sum identity
$\sin\theta \cdot 0 + \cos\theta \cdot 1 \stackrel{?}{=} \cos\theta$	Evaluate each expression.
$\cos\theta = \cos\theta$ ✓	Simplify.

GuidedPractice

3A. $\sin(90° - \theta) = \cos\theta$

3B. $\cos(90° + \theta) = -\sin\theta$

Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

Example 1 **Find the exact value of each expression.**

1. $\cos 165°$

2. $\cos 105°$

3. $\cos 75°$

4. $\sin(-30°)$

5. $\sin 135°$

6. $\sin(-210°)$

Example 2

7. **CCSS MODELING** Refer to the beginning of the lesson. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude. The first signal can be modeled by the equation $y = 20\sin(3\theta + 45°)$. The second signal can be modeled by the equation $y = 20\sin(3\theta + 225°)$.

a. Find the sum of the two functions.

b. What type of interference results when signals modeled by the two equations are combined?

Example 3 **Verify that each equation is an identity.**

8. $\sin(90° + \theta) = \cos\theta$

9. $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

10. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta$

11. $\sin(\theta + \pi) = -\sin\theta$

Practice and Problem Solving

Extra Practice is on page R13.

Example 1 **Find the exact value of each expression.**

12. $\sin 165°$ **13.** $\cos 135°$ **14.** $\cos \frac{7\pi}{12}$

15. $\sin \frac{\pi}{12}$ **16.** $\tan 195°$ **17.** $\cos\left(-\frac{\pi}{12}\right)$

Example 2 **18. ELECTRONICS** In a certain circuit carrying alternating current, the formula $c = 2 \sin(120t)$ can be used to find the current c in amperes after t seconds.

a. Rewrite the formula using the sum of two angles.

b. Use the sum of angles formula to find the exact current at $t = 1$ second.

Example 3 **Verify that each equation is an identity.**

19. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$ **20.** $\cos(60° + \theta) = \sin(30° - \theta)$

21. $\cos(180° + \theta) = -\cos \theta$ **22.** $\tan(\theta + 45°) = \frac{1 + \tan \theta}{1 - \tan \theta}$

23. CCSS **REASONING** The monthly high temperatures for Minneapolis, Minnesota, can be modeled by the equation $y = 31.65 \sin\left(\frac{\pi}{6}x - 2.09\right) + 52.35$, where the months x are represented by January = 1, February = 2, and so on. The monthly low temperatures for Minneapolis can be modeled by the equation $y = 30.15 \sin\left(\frac{\pi}{6}x - 2.09\right) + 32.95$.

a. Write a new function by adding the expressions on the right side of each equation and dividing the result by 2.

b. What is the meaning of the function you wrote in part **a**?

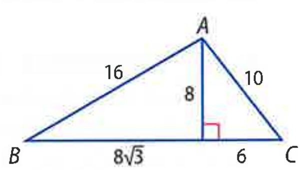
Find the exact value of each expression.

24. $\tan 165°$ **25.** $\sec 1275°$ **26.** $\sin 735°$

27. $\tan \frac{23\pi}{12}$ **28.** $\csc \frac{5\pi}{12}$ **29.** $\cot \frac{113\pi}{12}$

30. FORCE In the figure at the right, the effort F necessary to hold a safe in position on a ramp is given by $F = \frac{W(\sin A + \mu \cos A)}{\cos A - \mu \sin A}$, where W is the weight of the safe and $\mu = \tan \theta$. Show that $F = W \tan(A + \theta)$.

31. QUILTING As part of a quilt that is being made, the quilter places two right triangular swatches together to make a new triangular piece. One swatch has sides 6 inches, 8 inches, and 10 inches long. The other swatch has sides 8 inches, $8\sqrt{3}$ inches, and 16 inches long. The pieces are placed with the sides of eight inches against each other, as shown in the figure, to form triangle ABC.



a. What is the exact value of the sine of angle BAC?

b. What is the exact value of the cosine of angle BAC?

c. What is the measure of angle BAC?

d. Is the new triangle formed from the two triangles also a right triangle?

32. **OPTICS** When light passes symmetrically through a prism, the index of refraction n of the glass with respect to air is $n = \dfrac{\sin\left[\frac{1}{2}(a+b)\right]}{\sin \frac{b}{2}}$, where a is the measure of the deviation angle and b is the measure of the prism apex angle.

a. Show that for the prism shown, $n = \sqrt{3}\sin\frac{a}{2} + \cos\frac{a}{2}$.

b. Find n for the prism shown.

33. **MULTIPLE REPRESENTATIONS** In this problem, you will disprove the hypothesis that $\sin(A+B) = \sin A + \sin B$.

a. Tabular Copy and complete the table.

A	B	$\sin A$	$\sin B$	$\sin(A+B)$	$\sin A + \sin B$
30°	90°				
45°	60°				
60°	45°				
90°	30°				

b. Graphical Assume that B is always 15° less than A. Use a graphing calculator to graph $y = \sin(x + x - 15)$ and $y = \sin x + \sin(x - 15)$ on the same screen.

c. Analytical Determine whether $\cos(A+B) = \cos A + \cos B$ is an identity. Explain your reasoning.

Verify that each equation is an identity.

34. $\sin(A+B) = \dfrac{\tan A + \tan B}{\sec A \sec B}$

35. $\cos(A+B) = \dfrac{1 - \tan A \tan B}{\sec A \sec B}$

36. $\sec(A-B) = \dfrac{\sec A \sec B}{1 + \tan A \tan B}$

37. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

H.O.T. Problems Use Higher-Order Thinking Skills

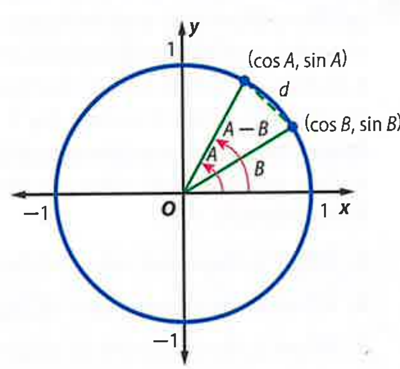
38. **REASONING** Simplify the following expression without expanding any of the sums or differences.

$$\sin\left(\frac{\pi}{3} - \theta\right)\cos\left(\frac{\pi}{3} + \theta\right) - \cos\left(\frac{\pi}{3} - \theta\right)\sin\left(\frac{\pi}{3} + \theta\right)$$

39. **WRITING IN MATH** Use the information at the beginning of the lesson and in Exercise 7 to explain how the sum and difference identities are used to describe wireless Internet interference. Include an explanation of the difference between constructive and destructive interference.

40. **CHALLENGE** Derive an identity for $\cot(A+B)$ in terms of $\cot A$ and $\cot B$.

41. **CCSS ARGUMENTS** The figure shows two angles A and B in standard position on the unit circle. Use the Distance Formula to find d, where $(x_1, y_1) = (\cos B, \sin B)$ and $(x_2, y_2) = (\cos A, \sin A)$.



42. **OPEN ENDED** Consider the following theorem. *If A, B, and C are the angles of an oblique triangle, then tan A + tan B + tan C = tan A tan B tan C.* Choose values for A, B, and C. Verify that the conclusion is true for your specific values.

Standardized Test Practice

43. GRIDDED RESPONSE The mean of seven numbers is 0. The sum of three of the numbers is -9. What is the sum of the remaining four numbers?

44. The variables a, b, c, d, and f are integers in a sequence, where $a = 2$ and $b = 12$. To find the next term, double the last term and add that result to one less than the next-to-last term. For example, $c = 25$, because $2(12) = 24$, $2 - 1 = 1$, and $24 + 1 = 25$. What is the value of f?

A 74

B 144

C 146

D 256

45. SAT/ACT Solve $x^2 - 5x < 14$.

F $\{x \mid -7 < x < 2\}$

G $\{x \mid x < -7 \text{ or } x > 2\}$

H $\{x \mid -2 < x < 7\}$

J $\{x \mid x < -2 \text{ or } x > 7\}$

K $\{x \mid x > -2 \text{ and } x < 7\}$

46. PROBABILITY A math teacher is randomly distributing 15 yellow pencils and 10 green pencils. What is the probability that the first pencil she hands out will be yellow and the second pencil will be green?

A $\frac{1}{24}$

B $\frac{1}{4}$

C $\frac{2}{5}$

D $\frac{23}{25}$

Spiral Review

Verify that each equation is an identity. (Lesson 13-2)

47. $\frac{\sin \theta}{\tan \theta} + \frac{\cos \theta}{\cot \theta} = \cos \theta + \sin \theta$

48. $\sec \theta (\sec \theta - \cos \theta) = \tan^2 \theta$

Simplify each expression. (Lesson 13-1)

49. $\sin \theta \csc \theta - \cos^2 \theta$

50. $\cos^2 \theta \sec \theta \csc \theta$

51. $\cos \theta + \sin \theta \tan \theta$

52. GUITAR When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz (Hz). (Lesson 12-6)

a. Find the period of this function.

b. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit, and let the minimum distance below this position have a value of 1 unit.

Prove that each statement is true for all positive integers. (Lesson 10-7)

53. $4^n - 1$ is divisible by 3.

54. $5^n + 3$ is divisible by 4.

Skills Review

Solve each equation.

55. $7 + \sqrt{4x + 8} = 9$

56. $\sqrt{y + 21} - 1 = \sqrt{y + 12}$

57. $\sqrt{4z + 1} = 3 + \sqrt{4z - 2}$

Mid-Chapter Quiz

Lessons 13-1 through 13-3

Simplify each expression. (Lesson 13-1)

1. $\cot\theta \sec\theta$

2. $\dfrac{1-\cos^2\theta}{\sin^2\theta}$

3. $\dfrac{1}{\cos\theta} - \dfrac{\sin^2\theta}{\cos\theta}$

4. $\cos\left(\dfrac{\pi}{2} - \theta\right)\csc\theta$

5. HISTORY In 1861, the United States 34-star flag was adopted. For this flag, $\tan\theta = \dfrac{31.5}{51}$. Find $\sin\theta$.

Find the value of each expression. (Lesson 13-1)

6. $\sin\theta$, if $\cos\theta = \frac{3}{5}$; $0° < \theta < 90°$

7. $\csc\theta$, if $\cot\theta = \frac{1}{2}$; $270° < \theta < 360°$

8. $\tan\theta$, if $\sec\theta = \frac{4}{3}$; $0° < \theta < 90°$

9. MULTIPLE CHOICE Which of the following is equivalent to $\dfrac{\cos\theta}{1-\sin^2\theta}$? (Lesson 13-1)

A $\cos\theta$

B $\csc\theta$

C $\tan\theta$

D $\sec\theta$

10. AMUSEMENT PARKS Suppose a child on a merry-go-round is seated on an outside horse. The diameter of the merry-go-round is 16 meters. The angle of inclination is represented by the equation $\tan\theta = \dfrac{v^2}{gR}$, where R is the radius of the circular path, v is the speed in meters per second, and g is 9.8 meters per second squared. (Lesson 13-1)

a. If the sine of the angle of inclination of the child is $\frac{1}{5}$, what is the angle of inclination made by the child?

b. What is the velocity of the merry-go-round?

c. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider?

Verify that each of the following is an identity. (Lesson 13-2)

11. $\cot^2\theta + 1 = \dfrac{\cot\theta}{\cos\theta \cdot \sin\theta}$

12. $\dfrac{\cos\theta\csc\theta}{\cot\theta} = 1$

13. $\dfrac{\sin\theta\tan\theta}{1-\cos\theta} = (1+\cos\theta)\sec\theta$

14. $\tan\theta(1-\sin\theta) = \dfrac{\cos\theta\sin\theta}{1+\sin\theta}$

15. COMPUTER The front of a computer monitor is usually measured along the diagonal of the screen as shown below. (Lesson 13-2)

a. Find h.

b. Using the diagram shown, show that $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Verify that each of the following is an identity. (Lesson 13-2)

16. $\tan^2\theta + 1 = \dfrac{\tan\theta}{\cos\theta \cdot \sin\theta}$

17. $\dfrac{\sin\theta \cdot \sec\theta}{\sec\theta - 1} = (\sec\theta + 1)\cot\theta$

18. $\sin^2\theta \cdot \tan^2\theta = \tan^2\theta - \sin^2\theta$

19. $\cot\theta(1-\cos\theta) = \dfrac{\cos\theta \cdot \sin\theta}{1+\cos\theta}$

Find the exact value of each expression. (Lesson 13-3)

20. $\cos 105°$

21. $\sin(-135°)$

22. $\tan 15°$

23. $\cot 75°$

24. MULTIPLE CHOICE What is the exact value of $\cos\frac{5\pi}{12}$? (Lesson 13-3)

F $\sqrt{2}$

G $\dfrac{\sqrt{6}+\sqrt{2}}{2}$

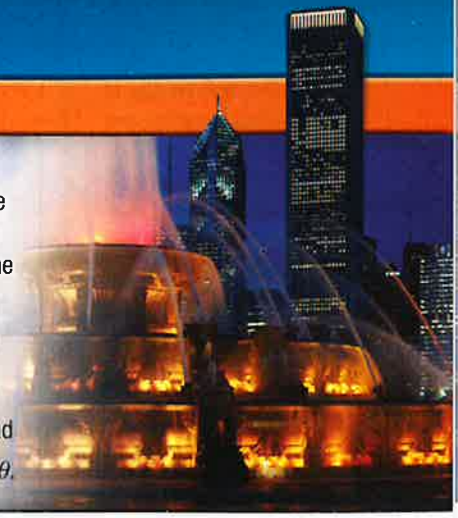
H $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

J $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

25. Verify that $\cos 30°\cos\theta + \sin 30°\sin\theta = \sin 60°\cos\theta + \cos 60°\sin\theta$ is an identity. (Lesson 13-3)

LESSON

13-4 Double-Angle and Half-Angle Identities



Then

- You found values of sine and cosine by using sum and difference identities.

Now

- **1** Find values of sine and cosine by using double-angle identities.
- **2** Find values of sine and cosine by using half-angle identities.

Why?

- Chicago's Buckingham Fountain contains jets placed at specific angles that shoot water into the air to create arcs. When a stream of water shoots into the air with velocity v at an angle of θ with the horizontal, the model predicts that the water will travel a horizontal distance of $D = \frac{v^2}{g}\sin 2\theta$ and reach a maximum height of $H = \frac{v^2}{2g}\sin^2\theta$. The ratio of H to D helps determine the total height and width of the fountain. Express $\frac{H}{D}$ as a function of θ.

CCSS Common Core State Standards

Content Standards

F.TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Mathematical Practices

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

1 Double-Angle Identities

It is sometimes useful to have identities to find the value of a function of twice an angle or half an angle.

KeyConcept Double-Angle Identities

The following identities hold true for all values of θ.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

PT

Example 1 Double-Angle Identities

Find the exact value of $\sin 2\theta$ if $\sin\theta = \frac{2}{3}$ and θ is between 0° and 90°.

Step 1 Use the identity $\sin 2\theta = 2\sin\theta\cos\theta$ to find the value of $\cos\theta$.

$\cos^2\theta = 1 - \sin^2\theta$ — $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \left(\frac{2}{3}\right)^2$ — $\sin\theta = \frac{2}{3}$

$\cos^2\theta = \frac{5}{9}$ — Subtract.

$\cos\theta = \pm\frac{\sqrt{5}}{3}$ — Take the square root of each side.

Since θ is in the first quadrant, cosine is positive. Thus, $\cos\theta = \frac{\sqrt{5}}{3}$.

Step 2 Find $\sin 2\theta$.

$\sin 2\theta = 2\sin\theta\cos\theta$ — Double-angle identity

$= 2\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right)$ — $\sin\theta = \frac{2}{3}$ and $\cos\theta = \frac{\sqrt{5}}{3}$

$= \frac{4\sqrt{5}}{9}$ — Multiply.

GuidedPractice

1. Find the exact value of $\sin 2\theta$ if $\cos\theta = -\frac{1}{3}$ and $90° < \theta < 180°$.

Example 2 Double-Angle Identities

Find the exact value of each expression if $\sin \theta = \frac{2}{3}$ and θ is between 0° and 90°.

a. $\cos 2\theta$

Since we know the values of $\cos \theta$ and $\sin \theta$, we can use any of the double-angle identities for cosine. We will use the identity $\cos 2\theta = 1 - 2\sin^2 \theta$.

$\cos 2\theta = 1 - 2\sin^2 \theta$ — Double-angle identity

$= 1 - 2\left(\frac{2}{3}\right)^2$ or $\frac{1}{9}$ — $\sin \theta = \frac{2}{3}$

b. $\tan 2\theta$

Step 1 Find $\tan \theta$ to use the double-angle identity for $\tan 2\theta$.

$\tan \theta = \frac{\sin \theta}{\cos \theta}$ — Definition of tangent

$= \frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}}$ — $\sin \theta = \frac{2}{3}$ and $\cos \theta = \frac{\sqrt{5}}{3}$

$= \frac{2}{\sqrt{5}}$ or $\frac{2\sqrt{5}}{5}$ — Rationalize the denominator.

Step 2 Find $\tan 2\theta$.

$\tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta}$ — Double-angle identity

$= \frac{2\left(\frac{2\sqrt{5}}{5}\right)}{1 - \left(\frac{2\sqrt{5}}{5}\right)^2}$ — $\tan \theta = \frac{2\sqrt{5}}{5}$

$= \frac{2\left(\frac{2\sqrt{5}}{5}\right)}{\frac{25}{25} - \frac{20}{25}}$ — Square the denominator.

$= \frac{\frac{4\sqrt{5}}{5}}{\frac{1}{5}}$ — Simplify.

$= \frac{4\sqrt{5}}{5} \cdot \frac{5}{1}$ or $4\sqrt{5}$ — $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

GuidedPractice

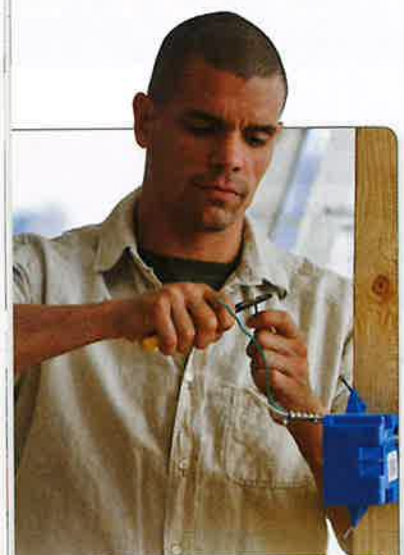
Find the exact value of each expression if $\cos \theta = -\frac{1}{3}$ and $90° < \theta < 180°$.

2A. $\cos 2\theta$

2B. $\tan 2\theta$

StudyTip

Deriving Formulas You can use the identity for $\sin (A + B)$ to find the sine of twice an angle θ, $\sin 2\theta$, and the identity for $\cos (A + B)$ to find the cosine of twice an angle θ, $\cos 2\theta$.



Real-WorldCareer

Electrician An electrician specializes in the wiring of electrical components. Electricians serve an apprenticeship lasting 3–5 years. Schooling in electrical theory and building codes is required. Certification requires work experience and a passing score on a written test.

JGI/Blend Images/Getty Images

2 Half-Angle Identities

It is sometimes useful to have identities to find the value of a function of half an angle.

KeyConcept Half-Angle Identities

The following identities hold true for all values of θ.

$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$ $\quad \cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$ $\quad \tan \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}, \cos \theta \neq -1$

StudyTip

Choosing the Sign In the first step of the solution, you may want to determine the quadrant in which the terminal side of $\frac{\theta}{2}$ will lie. Then you can use the correct sign from that point on.

ReadingMath

Plus or Minus The first sign of the half-angle identity is read *plus or minus*. Unlike with the double-angle identities, you must determine the sign.

Example 3 Half-Angle Identities

a. Find the exact value of $\cos \frac{\theta}{2}$ if $\sin \theta = -\frac{4}{5}$ and θ is in the third quadrant.

$\cos^2 \theta = 1 - \sin^2 \theta$	Use a Pythagorean identity to find $\cos \theta$.
$\cos^2 \theta = 1 - \left(-\frac{4}{5}\right)^2$	$\sin \theta = -\frac{4}{5}$
$\cos^2 \theta = 1 - \frac{16}{25}$	Evaluate exponent.
$\cos^2 \theta = \frac{9}{25}$	Subtract.
$\cos \theta = \pm\frac{3}{5}$	Take the square root of each side.

Since θ is in the third quadrant, $\cos \theta = -\frac{3}{5}$.

$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$	Half-angle identity
$= \pm\sqrt{\frac{1 - \frac{3}{5}}{2}}$	$\cos \theta = -\frac{3}{5}$
$= \pm\sqrt{\frac{1}{5}}$	Simplify.
$= \pm\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$ or $\pm\frac{\sqrt{5}}{5}$	Rationalize the denominator.

If θ is between 180° and 270°, $\frac{\theta}{2}$ is between 90° and 135°. So, $\cos \frac{\theta}{2}$ is $-\frac{\sqrt{5}}{5}$.

b. Find the exact value of cos 67.5°.

$\cos 67.5° = \cos \frac{135°}{2}$	$67.5° = \frac{135°}{2}$
$= \sqrt{\frac{1 + \cos 135°}{2}}$	$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$
$= \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$	67.5° is in Quadrant I; the value is positive.
$= \sqrt{\frac{\frac{2}{2} - \frac{\sqrt{2}}{2}}{2}}$	$1 = \frac{2}{2}$
$= \sqrt{\frac{\frac{2 - \sqrt{2}}{2}}{2}}$	Subtract fractions.
$= \sqrt{\frac{2 - \sqrt{2}}{2} \cdot \frac{1}{2}}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
$= \sqrt{\frac{2 - \sqrt{2}}{4}}$	Multiply.
$= \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{4}}$	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
$= \frac{\sqrt{2 - \sqrt{2}}}{2}$	Simplify.

GuidedPractice

3. Find the exact value of $\sin \frac{\theta}{2}$ if $\sin \theta = \frac{2}{3}$ and θ is in the second quadrant.

Real-WorldLink

The City Hall Park Fountain in New York City is located in the heart of Manhattan in front of City Hall.

Source: Fodor's

Real-World Example 4 Simplify Using Double-Angle Identities

FOUNTAIN **Refer to the beginning of the lesson. Find $\frac{H}{D}$.**

$\frac{H}{D} = \dfrac{\frac{v^2}{2g}\sin^2\theta}{\frac{v^2}{g}\sin 2\theta}$	Original equation
$= \dfrac{\frac{v^2\sin^2\theta}{2g}}{\frac{v^2\sin 2\theta}{g}}$	Simplify the numerator and denominator.
$= \frac{v^2\sin^2\theta}{2g} \cdot \frac{g}{v^2\sin 2\theta}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
$= \frac{\sin^2\theta}{2\sin 2\theta}$	Simplify.
$= \frac{\sin^2\theta}{4\sin\theta\cos\theta}$	$\sin 2\theta = 2\sin\theta\cos\theta$
$= \frac{1}{4} \cdot \frac{\sin\theta}{\cos\theta}$	Simplify.
$= \frac{1}{4}\tan\theta$	$\frac{\sin\theta}{\cos\theta} = \tan\theta$

GuidedPractice

Find each value.

4A. $\sin 135°$

4B. $\cos \frac{7\pi}{8}$

Recall that you can use the sum and difference identities to verify identities. Double- and half-angle identities can also be used to verify identities.

Example 5 Verify Identities

Verify that $\frac{\cos 2\theta}{1+\sin 2\theta} = \frac{\cot\theta - 1}{\cot\theta + 1}$ is an identity.

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cot\theta - 1}{\cot\theta + 1}$	Original equation
$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \dfrac{\frac{\cos\theta}{\sin\theta} - 1}{\frac{\cos\theta}{\sin\theta} + 1}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$	Multiply numerator and denominator by $\sin\theta$.
$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} \cdot \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}$	Multiply the right side by 1.
$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta}$	Multiply.
$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos^2\theta - \sin^2\theta}{1 + 2\cos\theta\sin\theta}$	Simplify.
$\frac{\cos 2\theta}{1+\sin 2\theta} = \frac{\cos 2\theta}{1+\sin 2\theta}$ ✓	$\cos^2\theta - \sin^2\theta = \cos 2\theta$; $2\cos\theta\sin\theta = \sin 2\theta$

GuidedPractice

5. Verify that $4\cos^2 x - \sin^2 2x = 4\cos^4 x$.

Paul Coco/Painet Inc.

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Examples 1–3 CCSS **PRECISION** **Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$.**

1. $\sin \theta = \frac{1}{4}$; $0° < \theta < 90°$
2. $\sin \theta = \frac{4}{5}$; $90° < \theta < 180°$
3. $\cos \theta = -\frac{5}{13}$; $\frac{\pi}{2} < \theta < \pi$
4. $\cos \theta = \frac{3}{5}$; $270° < \theta < 360°$
5. $\tan \theta = -\frac{8}{15}$; $90° < \theta < 180°$
6. $\tan \theta = \frac{5}{12}$; $\pi < \theta < \frac{3\pi}{2}$

Find the exact value of each expression.

7. $\sin \frac{\pi}{8}$
8. $\cos 15°$

Example 4

9. **SOCCER** A soccer player kicks a ball at an angle of 37° with the ground with an initial velocity of 52 feet per second. The distance d that the ball will go in the air if it is not blocked is given by $d = \frac{2v^2 \sin \theta \cos \theta}{g}$. In this formula, g is the acceleration due to gravity and is equal to 32 feet per second squared, and v is the initial velocity.

   a. Simplify this formula by using a double-angle identity.
   b. Using the simplified formula, how far will this ball go?

Example 5 **Verify that each equation is an identity.**

10. $\tan \theta = \frac{1 - \cos 2\theta}{\sin 2\theta}$
11. $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$

Practice and Problem Solving

Extra Practice is on page R13.

Examples 1–3 **Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$.**

12. $\sin \theta = \frac{2}{3}$; $90° < \theta < 180°$
13. $\sin \theta = -\frac{15}{17}$; $\pi < \theta < \frac{3\pi}{2}$
14. $\cos \theta = \frac{3}{5}$; $\frac{3\pi}{2} < \theta < 2\pi$
15. $\cos \theta = \frac{1}{5}$; $270° < \theta < 360°$
16. $\tan \theta = \frac{4}{3}$; $180° < \theta < 270°$
17. $\tan \theta = -2$; $\frac{\pi}{2} < \theta < \pi$

Find the exact value of each expression.

18. $\sin 75°$
19. $\sin \frac{3\pi}{8}$
20. $\cos \frac{7\pi}{12}$
21. $\tan 165°$
22. $\tan \frac{5\pi}{12}$
23. $\tan 22.5°$

24. **GEOGRAPHY** The Mercator projection of the globe is a projection on which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection involves the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.

   a. Write this expression in terms of a trigonometric function of L.
   b. The latitude of Tallahassee, Florida, is 30° north. Find the value of the expression if $L = 30°$.

Example 4

25 **ELECTRONICS** Consider an AC circuit consisting of a power supply and a resistor. If the current I_0 in the circuit at time t is $I_0 \sin t\theta$, then the power delivered to the resistor is $P = I_0^2 R \sin^2 t\theta$, where R is the resistance. Express the power in terms of $\cos 2t\theta$.

Example 5

Verify that each equation is an identity.

26. $\tan 2\theta = \dfrac{2}{\cot \theta - \tan \theta}$

27. $1 + \dfrac{1}{2}\sin 2\theta = \dfrac{\sec \theta + \sin \theta}{\sec \theta}$

28. $\sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2} = \dfrac{\sin \theta}{2}$

29. $\tan \dfrac{\theta}{2} = \dfrac{\sin \theta}{1 + \cos \theta}$

30. **FOOTBALL** Suppose a place kicker consistently kicks a football with an initial velocity of 95 feet per second. Prove that the horizontal distance the ball travels in the air will be the same for $\theta = 45° + A$ as for $\theta = 45° - A$. Use the formula given in Exercise 9.

Find the exact values of sin 2θ, cos 2θ, and tan 2θ.

31. $\cos \theta = \frac{4}{5}$; $0° < \theta < 90°$

32. $\sin \theta = \frac{1}{3}$; $0 < \theta < \frac{\pi}{2}$

33. $\tan \theta = -3$; $90° < \theta < 180°$

34. $\sec \theta = -\frac{4}{3}$; $90° < \theta < 180°$

35. $\csc \theta = -\frac{5}{2}$; $\frac{3\pi}{2} < \theta < 2\pi$

36. $\cot \theta = \frac{3}{2}$; $180° < \theta < 270°$

H.O.T. Problems Use Higher-Order Thinking Skills

37. **CCSS CRITIQUE** Teresa and Nathan are calculating the exact value of sin 15°. Is either of them correct? Explain your reasoning.

Teresa

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\sin(45 - 30) = \sin 45 \cos 30 - \cos 45 \sin 30$

$= \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2}$

$= \dfrac{\sqrt{4}}{4}$

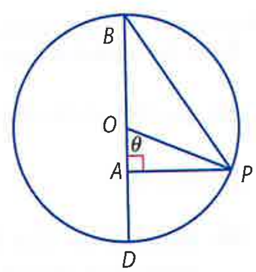
Nathan

$\sin \dfrac{A}{2} = \pm\sqrt{\dfrac{1 - \cos A}{2}}$

$\sin \dfrac{30}{2} = \pm\sqrt{\dfrac{1 - \frac{1}{2}}{2}}$

$= 0.5$

38. **CHALLENGE** Circle O is a unit circle. Use the figure to prove that $\tan \frac{1}{2}\theta = \dfrac{\sin \theta}{1 + \cos \theta}$.



39. **WRITING IN MATH** Write a short paragraph about the conditions under which you would use each of the three identities for $\cos 2\theta$.

40. **PROOF** Use the formula for $\sin(A + B)$ to derive the formula for $\sin 2\theta$, and use the formula for $\cos(A + B)$ to derive the formula for $\cos 2\theta$.

41. **REASONING** Derive the half-angle identities from the double-angle identities.

42. **OPEN ENDED** Suppose a golfer consistently hits the ball so that it leaves the tee with an initial velocity of 115 feet per second and $d = \dfrac{2v^2 \sin \theta \cos \theta}{g}$. Explain why the maximum distance is attained when $\theta = 45°$.

Standardized Test Practice

43. SHORT RESPONSE Angles *C* and *D* are supplementary. The measure of angle *C* is seven times the measure of angle *D*. Find the measure of angle *D* in degrees.

44. SAT/ACT Ms. Romero has a list of the yearly salaries of the staff members in her department. Which measure of data describes the middle income value of the salaries?

A mean
B median
C mode
D range
E standard deviation

45. Identify the domain and range of the function $f(x) = |4x + 1| - 8$.

F $D = \{x \mid -3 \leq x \leq 1\}$, $R = \{y \mid y \geq -8\}$
G $D = \{\text{all real numbers}\}$, $R = \{y \mid y \geq -8\}$
H $D = \{x \mid -3 \leq x \leq 1\}$, $R = \{\text{all real numbers}\}$
J $D = \{\text{all real numbers}\}$, $R = \{\text{all real numbers}\}$

46. GEOMETRY Angel is putting a stone walkway around a circular pond. He has enough stones to make a walkway 144 feet long. If he uses all of the stones to surround the pond, what is the radius of the pond?

A $\frac{12}{\pi}$ ft
B $\frac{72}{\pi}$ ft
C 72π ft
D 144π ft

Spiral Review

Find the exact value of each expression. (Lesson 13-3)

47. $\sin 135°$ **48.** $\cos 105°$ **49.** $\sin 285°$

50. $\cos(-30°)$ **51.** $\sin(-240°)$ **52.** $\cos(-120°)$

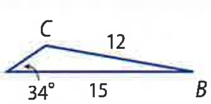
Verify that each equation is an identity. (Lesson 13-2)

53. $\cot\theta + \sec\theta = \frac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta}$

54. $\sin^2\theta + \tan^2\theta = (1 - \cos^2\theta) + \frac{\sec^2\theta}{\csc^2\theta}$

Determine whether each triangle should be solved by beginning with the Law of *Sines* or Law of *Cosines*. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 12-5)

55.



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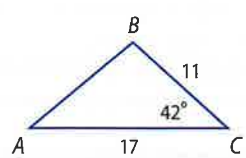

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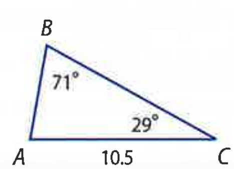


Skills Review

Solve each equation by factoring.

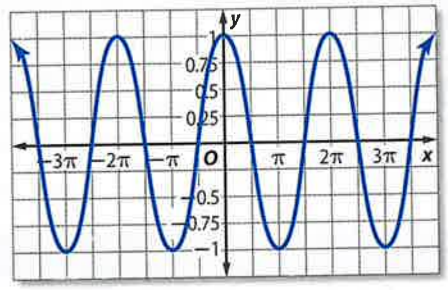
58. $x^2 + 5x - 24 = 0$ **59.** $x^2 - 3x - 28 = 0$ **60.** $x^2 - 4x = 21$

Example 2 Infinitely Many Solutions

Solve $\cos\theta + 1 = 0$ for all values of θ if θ is measured in radians.

$$\cos\theta + 1 = 0$$
$$\cos\theta = -1$$

Look at the graph of $y = \cos\theta$ to find solutions of $\cos\theta = -1$.



The solutions are $\pi, 3\pi, 5\pi$, and so on, and $-\pi, -3\pi, -5\pi$, and so on. The only solution in the interval 0 radians to 2π radians is π. The period of the cosine function is 2π radians. So the solutions can be written as $\pi + 2k\pi$, where k is any integer.

StudyTip

Expressing Solutions as Multiples The expression $\pi + 2k\pi$ includes 3π and its multiples, so it is not necessary to list them separately.

GuidedPractice

2A. Solve $\cos 2\theta + \cos\theta + 1 = 0$ for all values of θ if θ is measured in degrees.

2B. Solve $2\sin\theta = -1$ for all values of θ if θ is measured in radians.

Trigonometric equations are often used to solve real-world problems.

Real-World Example 3 Solve Trigonometric Equations

AMUSEMENT PARKS **Refer to the beginning of the lesson. How long after the Ferris wheel starts will your seat first be 31 meters above the ground?**

$h = 21 - 20\cos 3\pi t$	Original equation
$31 = 21 - 20\cos 3\pi t$	Replace h with 31.
$10 = -20\cos 3\pi t$	Subtract 21 from each side.
$-\frac{1}{2} = \cos 3\pi t$	Divide each side by −20.
$\cos^{-1}\left(-\frac{1}{2}\right) = 3\pi t$	Take the Arccosine.
$\frac{2\pi}{3} = 3\pi t$ or $\frac{4\pi}{3} = 3\pi t$	The Arccosine of $-\frac{1}{2}$ is $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$.
$\frac{2\pi}{3} + 2\pi k = 3\pi t$ or $\frac{4\pi}{3} + 2\pi k = 3\pi t$	k is any integer.
$\frac{2}{9} + \frac{2}{3}k = t$ $\frac{4}{9} + \frac{2}{3}k = t$	Divide each term by 3π.

The least positive value for t is obtained by letting $k = 0$ in the first expression. Therefore, $t = \frac{2}{9}$ of a minute or about 13 seconds.

GuidedPractice

3. How long after the Ferris wheel starts will your seat first be 41 meters above the ground?

2 Extraneous Solutions

Some trigonometric equations have no solution. For example, the equation $\cos \theta = 4$ has no solution because all values of $\cos \theta$ are between -1 and 1, inclusive. Thus, the solution set for $\cos \theta = 4$ is empty.

PT

Example 4 Determine Whether a Solution Exists

Solve each equation.

a. $\mathbf{2 \sin^2 \theta - 3 \sin \theta - 2 = 0}$ **if** $\mathbf{0 \leq \theta \leq 2\pi}$

$2 \sin^2 \theta - 3 \sin \theta - 2 = 0$ — Original equation

$(\sin \theta - 2)(2 \sin \theta + 1) = 0$ — Factor.

$\sin \theta - 2 = 0$ or $2 \sin \theta + 1 = 0$ — Zero Product Property

$\sin \theta = 2$ $\qquad$ $2 \sin \theta = -1$

This is not a solution since all values of $\sin \theta$ are between -1 and 1, inclusive.

$\sin \theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

The solutions are $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

CHECK

$2 \sin \theta - 3 \sin \theta - 2 = 0$

$2 \sin^2 \left(\frac{7\pi}{6}\right) - 3 \sin \left(\frac{7\pi}{6}\right) - 2 \stackrel{?}{=} 0$

$2\left(\frac{1}{4}\right) - 3\left(-\frac{1}{2}\right) - 2 \stackrel{?}{=} 0$

$\frac{1}{2} + \frac{3}{2} - 2 \stackrel{?}{=} 0$

$0 = 0$ ✓

$2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

$2 \sin^2 \left(\frac{11\pi}{6}\right) - 3 \sin \left(\frac{11\pi}{6}\right) - 2 \stackrel{?}{=} 0$

$2\left(\frac{1}{4}\right) - 3\left(-\frac{1}{2}\right) - 2 \stackrel{?}{=} 0$

$\frac{1}{2} + \frac{3}{2} - 2 \stackrel{?}{=} 0$

$0 = 0$ ✓

Problem-Solving Tip

CCSS Regularity Look for patterns in your solutions. Look for pairs of solutions that differ by exactly π or 2π and write your solutions with the simplest possible pattern.

b. $\mathbf{\sin \theta = 1 + \cos \theta}$ **if** $\mathbf{0° \leq \theta < 360°}$

$\sin \theta = 1 + \cos \theta$ — Original equation

$\sin^2 \theta = (1 + \cos \theta)^2$ — Square each side.

$1 - \cos^2 \theta = 1 + 2 \cos \theta + \cos^2 \theta$ — $\sin^2 \theta = 1 - \cos^2 \theta$

$0 = 2 \cos \theta + 2 \cos^2 \theta$ — Set the left side equal to 0.

$0 = 2 \cos \theta (1 + \cos \theta)$ — Factor.

$1 + \cos \theta = 0$ or $2 \cos \theta = 0$ — Zero Product Property

$\cos \theta = -1$ $\qquad$ $\cos \theta = 0$

$\theta = 180$ $\qquad$ $\theta = 90°$ or $270°$

CHECK

$\sin \theta = 1 + \cos \theta$

$\sin 90° \stackrel{?}{=} 1 + \cos 90°$

$1 \stackrel{?}{=} 1 + 0$

$1 = 1$ ✓

$\sin \theta = 1 + \cos \theta$

$\sin 180° \stackrel{?}{=} 1 + \cos 180°$

$0 \stackrel{?}{=} 1 + (-1)$

$0 = 0$ ✓

$\sin \theta = 1 + \cos \theta$

$\sin 270° \stackrel{?}{=} 1 + \cos 270°$

$-1 \stackrel{?}{=} 1 + 0$

$-1 \neq 1$ ✗

The solutions are 90° and 180°.

Guided Practice

4A. $\sin^2 \theta + 2 \cos^2 \theta = 4$

4B. $\cos^2 \theta + 3 = 4 - \sin^2 \theta$

If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.

StudyTip

Solving Trigonometric Equations Remember that *solving a trigonometric equation* means solving for all values of the variable.

Example 5 Solve Trigonometric Equations by Using Identities

Solve $2\sec^2\theta - \tan^4\theta = -1$ for all values of θ if θ is measured in degrees.

$2\sec^2\theta - \tan^4\theta = -1$ — Original equation

$2(1 + \tan^2\theta) - \tan^4\theta = -1$ — $\sec^2\theta = 1 + \tan^2\theta$

$2 + 2\tan^2\theta - \tan^4\theta = -1$ — Distributive Property

$\tan^4\theta - 2\tan^2\theta - 3 = 0$ — Set one side of the equation equal to 0.

$(\tan^2\theta - 3)(\tan^2\theta + 1) = 0$ — Factor.

$\tan^2\theta - 3 = 0$ or $\tan^2\theta + 1 = 0$ — Zero Product Property

$\tan^2\theta = 3$ $\qquad$ $\tan^2\theta = -1$

$\tan\theta = \pm\sqrt{3}$ $\qquad$ This part gives no solutions since $\tan^2\theta$ is never negative.

$\theta = 60° + 180°k$ and $\theta = -60° + 180°k$, where k is any integer. The solutions are $60° + 180°k$ and $-60° + 180°k$.

GuidedPractice

Solve each equation.

5A. $\sin\theta\cot\theta - \cos^2\theta = 0$

5B. $\frac{\cos\theta}{\cot\theta} + 2\sin^2\theta = 0$

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Example 1 CCSS **REGULARITY** **Solve each equation if $0° \le \theta \le 360°$.**

1. $2\sin\theta + 1 = 0$

2. $\cos^2\theta + 2\cos\theta + 1 = 0$

3. $\cos 2\theta + \cos\theta = 0$

4. $2\cos\theta = 1$

5. $\cos\theta = -\frac{\sqrt{3}}{2}$

6. $\sin 2\theta = -\frac{\sqrt{3}}{2}$

7. $\cos 2\theta = 8 - 15\sin\theta$

8. $\sin\theta + \cos\theta = 1$

Example 2 **Solve each equation for all values of θ if θ is measured in radians.**

9. $4\sin^2\theta - 1 = 0$

10. $2\cos^2\theta = 1$

11. $\cos 2\theta\sin\theta = 1$

12. $\sin\frac{\theta}{2} + \cos\frac{\theta}{2} = \sqrt{2}$

13. $\cos 2\theta + 4\cos\theta = -3$

14. $\sin\frac{\theta}{2} + \cos\theta = 1$

Solve each equation for all values of θ if θ is measured in degrees.

15. $\cos 2\theta - \sin^2\theta + 2 = 0$

16. $\sin^2\theta - \sin\theta = 0$

17. $2\sin^2\theta - 1 = 0$

18. $\cos\theta - 2\cos\theta\sin\theta = 0$

19. $\cos 2\theta\sin\theta = 1$

20. $\sin\theta\tan\theta - \tan\theta = 0$

Example 3 **21.** **LIGHT** The number of hours of daylight d in Hartford, Connecticut, may be approximated by the equation $d = 3\sin\frac{2\pi}{365}t + 12$, where t is the number of days after March 21.

a. On what days will Hartford have exactly $10\frac{1}{2}$ hours of daylight?

b. Using the results in part **a**, tell what days of the year have at least $10\frac{1}{2}$ hours of daylight. Explain how you know.

Examples 4–5 **Solve each equation.**

22. $\sin^2 2\theta + \cos^2 \theta = 0$

23. $\tan^2 \theta + 2\tan \theta + 1 = 0$

24. $\cos^2 \theta + 3\cos \theta = -2$

25. $\sin 2\theta - \cos \theta = 0$

26. $\tan \theta = 1$

27. $\cos 8\theta = 1$

28. $\sin \theta + 1 = \cos 2\theta$

29. $2\cos^2 \theta = \cos \theta$

Practice and Problem Solving

Extra Practice is on page R13.

Example 1 **Solve each equation for the given interval.**

30. $\cos^2 \theta = \frac{1}{4}; 0° \leq \theta \leq 360°$

31. $2\sin^2 \theta = 1; 90° < \theta < 270°$

32. $\sin 2\theta - \cos \theta = 0; 0 \leq \theta \leq 2\pi$

33. $3\sin^2 \theta = \cos^2 \theta; 0 \leq \theta \leq \frac{\pi}{2}$

34. $2\sin \theta + \sqrt{3} = 0; 180° < \theta < 360°$

35. $4\sin^2 \theta - 1 = 0; 180° < \theta < 360°$

Example 2 **Solve each equation for all values of θ if θ is measured in radians.**

36. $\cos 2\theta + 3\cos \theta = 1$

37. $2\sin^2 \theta = \cos \theta + 1$

38. $\cos^2 \theta - \frac{3}{2} = \frac{5}{2}\cos \theta$

39. $3\cos \theta - \cos \theta = 2$

Solve each equation for all values of θ if θ is measured in degrees.

40. $\sin \theta - \cos \theta = 0$

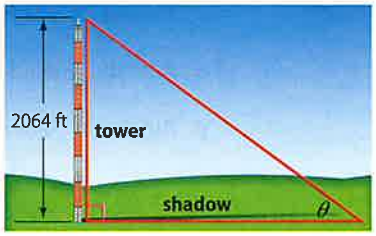
41. $\tan \theta - \sin \theta = 0$

42. $\sin^2 \theta = 2\sin \theta + 3$

43. $4\sin^2 \theta = 4\sin \theta - 1$

Example 3

44. ELECTRONICS One of the tallest structures in the world is a television transmitting tower located near Fargo, North Dakota, with a height of 2064 feet. What is the measure of θ if the length of the shadow is 1 mile?



Examples 4–5 **Solve each equation.**

45. $2\sin^2 \theta = 3\sin \theta + 2$

46. $2\cos^2 \theta + 3\sin \theta = 3$

47. $\sin^2 \theta + \cos 2\theta = \cos \theta$

48. $2\cos^2 \theta = -\cos \theta$

49. CCSS **SENSE-MAKING** Due to ocean tides, the depth y in meters of the River Thames in London varies as a sine function of x, the hour of the day. On a certain day that function was $y = 3\sin\left[\frac{\pi}{6}(x - 4)\right] + 8$, where $x = 0, 1, 2, \ldots, 24$ corresponds to 12:00 midnight, 1:00 A.M., 2:00 A.M., …, 12:00 midnight the next night.

a. What is the maximum depth of the River Thames on that day?

b. At what times does the maximum depth occur?

Solve each equation if θ is measured in radians.

50. $(\cos \theta)(\sin 2\theta) - 2\sin \theta + 2 = 0$

51. $2\sin^2 \theta + (\sqrt{2} - 1)\sin \theta = \frac{\sqrt{2}}{2}$

Solve each equation if θ is measured in degrees.

52. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3}\sin \theta + \cos \theta$

53. $1 - \sin^2 \theta - \cos \theta = \frac{3}{4}$

Solve each equation.

54. $2 \sin \theta = \sin 2\theta$

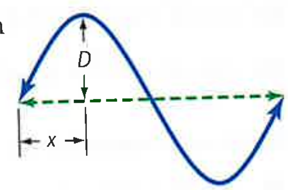
55. $\cos \theta \tan \theta - 2 \cos^2 \theta = -1$

56. DIAMONDS According to Snell's Law, $n_1 \sin i = n_2 \sin r$, where n_1 is the index of refraction of the medium the light is exiting, n_2 is the index of refraction of the medium the light is entering, i is the degree measure of the angle of incidence, and r is the degree measure of the angle of refraction.

a. The index of refraction of a diamond is 2.42, and the index of refraction of air is 1.00. If a beam of light strikes a diamond at an angle of 35°, what is the angle of refraction?

b. Explain how a gemologist might use Snell's Law to determine whether a diamond is genuine.

57 **CCSS PERSEVERANCE** A wave traveling in a guitar string can be modeled by the equation $D = 0.5 \sin (6.5x) \sin (2500t)$, where D is the displacement in millimeters at the position x millimeters from the left end of the string at time t seconds. Find the first positive time when the point 0.5 meter from the left end has a displacement of 0.01 millimeter.



58. MULTIPLE REPRESENTATIONS Consider the trigonometric inequality $\sin \theta \geq \frac{1}{2}$.

a. Tabular Construct a table of values for $0° \leq \theta \leq 360°$. For what values of θ is $\sin \theta \geq \frac{1}{2}$?

b. Graphical Graph $y = \sin \theta$ and $y = \frac{1}{2}$ on the same graph for $0° \leq \theta \leq 360°$. For what values of θ is the graph of $y = \sin \theta$ above the graph of $y = \frac{1}{2}$?

c. Analytic Based on your answers for parts **a** and **b**, solve $\sin \theta \geq \frac{1}{2}$ for all values of θ.

d. Algebraic Solve each inequality if $0 \leq \theta \leq 360°$. Then solve each for all values of θ.

i. $\cos \theta \geq \frac{\sqrt{2}}{2}$

ii. $2 \sin \theta \leq \sqrt{3}$

iii. $-\sin \theta \geq 0$

iv. $\cos \theta - 1 < -\frac{1}{2}$

H.O.T. Problems Use Higher-Order Thinking Skills

59. CHALLENGE Solve $\sin 2x < \sin x$ for $0 \leq x \leq 2\pi$ without a calculator.

60. REASONING Compare and contrast solving trigonometric equations with solving linear and quadratic equations. What techniques are the same? What techniques are different? How many solutions do you expect?

61. WRITING IN MATH Why do trigonometric equations often have infinitely many solutions?

62. OPEN ENDED Write an example of a trigonometric equation that has exactly two solutions if $0° \leq \theta \leq 360°$.

63. CHALLENGE How many solutions in the interval $0° \leq \theta \leq 360°$ should you expect for $a \sin (b\theta + c) = d$, if $a \neq 0$ and b is a positive integer?

Standardized Test Practice

64. EXTENDED RESPONSE Charles received $2500 for a graduation gift. He put it into a savings account in which the interest rate was 5.5% per year.

a. How much did he have in his savings account after 5 years if he made no deposits or withdrawals?

b. After how many years will the amount in his savings account have doubled?

65. PROBABILITY Find the probability of rolling three 3s if a number cube is rolled three times.

A $\frac{1}{216}$ C $\frac{1}{6}$

B $\frac{1}{36}$ D $\frac{1}{4}$

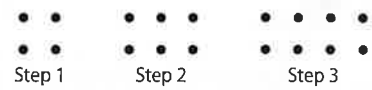
66. Use synthetic substitution to find $f(-2)$ for the function below.

$$f(x) = x^4 + 10x^2 + x + 8$$

F 62 H 30

G 38 J 8

67. SAT/ACT The pattern of dots below continues infinitely, with more dots being added at each step.



Which expression can be used to determine the number of dots in the nth step?

A $2n$ D $2(n + 2)$

B $n(n + 2)$ E $2(n + 1)$

C $n(n + 1)$

Spiral Review

Find the exact value of each expression. (Lesson 13-4)

68. $\cos 165°$

69. $\sin 22\frac{1}{2}°$

70. $\sin \frac{7\pi}{8}$

71. $\cos \frac{7\pi}{12}$

Verify that each equation is an identity. (Lesson 13-3)

72. $\sin (270° - \theta) = -\cos \theta$

73. $\cos (90° + \theta) = -\sin \theta$

74. $\cos (90° - \theta) = \sin \theta$

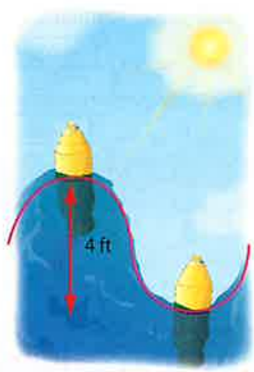
75. $\sin (90° - \theta) = \cos \theta$

76. WATER SAFETY A harbor buoy bobs up and down with the waves. The distance between the highest and lowest points is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds. (Lesson 12-7)

a. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level.

b. Draw a graph showing the height of the buoy as a function of time.

c. What is the height of the buoy after 12 seconds?



Find the first three terms of each arithmetic series described. (Lesson 10-2)

77. $a_1 = 17, a_n = 197, S_n = 2247$

78. $a_1 = -13, a_n = 427, S_n = 18{,}423$

79. $n = 31, a_n = 78, S_n = 1023$

80. $n = 19, a_n = 103, S_n = 1102$

Skills Review

Graph each rational function.

81. $f(x) = \frac{1}{(x + 3)^2}$

82. $f(x) = \frac{x + 4}{x - 1}$

83. $f(x) = \frac{x + 2}{x^2 - x - 6}$

CHAPTER 13

Study Guide and Review

Study Guide

KeyConcepts

Trigonometric Identities (Lessons 13-1, 13-2, and 13-5)

- Trigonometric identities describe the relationships between trigonometric functions.
- Trigonometric identities can be used to simplify, verify, and solve trigonometric equations and expressions.

Sum and Difference of Angles Identities (Lesson 13-3)

- For all values of A and B:

$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\tan (A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double-Angle and Half-Angle Identities (Lesson 13-4)

- Double-angle identities:

$\sin 2\theta = 2 \sin \theta \cos \theta$

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$\cos 2\theta = 1 - 2 \sin^2 \theta$

$\cos 2\theta = 2 \cos^2 \theta - 1$

$\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$

- Half-angle identities:

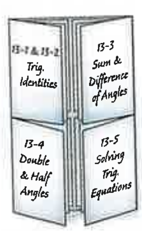
$\sin \dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 - \cos \theta}{2}}$

$\cos \dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 + \cos \theta}{2}}$

$\tan \dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 - \cos \theta}{1 + \cos \theta}}, \cos \theta \neq -1$

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.



KeyVocabulary

cofunction identity (p. 873)

negative angle identity (p. 873)

Pythagorean identity (p. 873)

quotient identity (p. 873)

reciprocal identity (p. 873)

trigonometric equation (p. 901)

trigonometric identity (p. 873)

VocabularyCheck

Choose the correct term to complete each sentence.

1. The ____________ can be used to find the sine or cosine of 75° if the sine and cosine of 90° and 15° are known.
2. The identities $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$ and $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$ are examples of ____________.
3. A ____________ is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.
4. The ____________ can be used to find sin 60° using 30° as a reference.
5. A ____________ is true for only certain values of the variable.
6. The ____________ identity can be used to find $\cos 22\frac{1}{2}°$.
7. The identities $\csc \theta = \dfrac{1}{\sin \theta}$ and $\sec \theta = \dfrac{1}{\cos \theta}$ are examples of ____________.
8. The ____________ can be used to find the sine or cosine of 120° if the sine and cosine of 90° and 30° are known.
9. $\cos^2 \theta + \sin^2 \theta = 1$ is an example of a ____________.

Lesson-by-Lesson Review

13-1 Trigonometric Identities

Find the value of each expression.

10. $\sin\theta$, if $\cos\theta = \frac{\sqrt{2}}{2}$ and $270° < \theta < 360°$

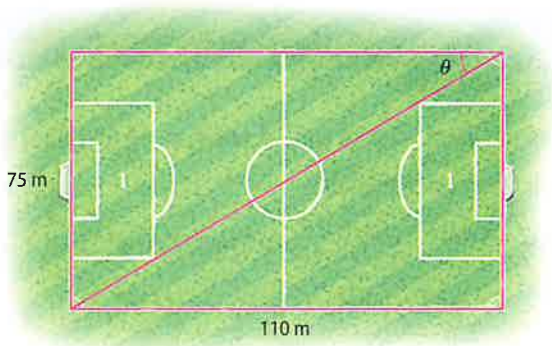
11. $\sec\theta$, if $\cot\theta = \frac{\sqrt{2}}{2}$ and $90° < \theta < 180°$

12. $\tan\theta$, if $\cot\theta = 2$ and $0° < \theta < 90°$

13. $\cos\theta$, if $\sin\theta = -\frac{3}{5}$ and $180° < \theta < 270°$

14. $\csc\theta$, if $\cot\theta = -\frac{4}{5}$ and $270° < \theta < 360°$

15. **SOCCER** For international matches, the maximum dimensions of a soccer field are 110 meters by 75 meters. Find $\sin\theta$.



Simplify each expression.

16. $1 - \tan\theta\sin\theta\cos\theta$

17. $\tan\theta\csc\theta$

18. $\sin\theta + \cos\theta\cot\theta$

19. $\cos\theta(1 + \tan^2\theta)$

Example 1

Find $\sin\theta$ if $\cos\theta = \frac{3}{4}$ and $0° < \theta < 90°$.

$\cos^2\theta + \sin^2\theta = 1$ — Trigonometric identity

$\sin^2\theta = 1 - \cos^2\theta$ — Subtract $\cos^2\theta$ from each side.

$\sin^2\theta = 1 - \left(\frac{3}{4}\right)^2$ — Substitute $\frac{3}{4}$ for $\cos\theta$.

$\sin^2\theta = 1 - \frac{9}{16}$ — Square $\frac{3}{4}$.

$\sin^2\theta = \frac{7}{16}$ — Subtract.

$\sin\theta = \pm\frac{\sqrt{7}}{4}$ — Take the square root of each side.

Because θ is in the first quadrant, $\sin\theta$ is positive.
Thus, $\sin\theta = \frac{\sqrt{7}}{4}$.

Example 2

Simplify $\cos\theta\sec\theta\cot\theta$.

$$\cos\theta\sec\theta\cot\theta = \cos\theta\left(\frac{1}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)$$
$$= \cot\theta$$

13-2 Verifying Trigonometric Identities

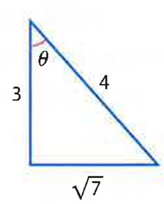
Verify that each of the following is an identity.

20. $\tan\theta\cos\theta + \cot\theta\sin\theta = \sin\theta + \cos\theta$

21. $\frac{\cos\theta}{\cot\theta} + \frac{\sin\theta}{\tan\theta} = \sin\theta + \cos\theta$

22. $\sec^2\theta - 1 = \frac{\sin^2\theta}{1 - \sin^2\theta}$

23. **GEOMETRY** The right triangle shown at the right is used in a special quilt. Use the measures of the sides of the triangle to show that $\tan^2\theta + 1 = \sec^2\theta$.



Example 3

Verify that $\frac{\cos\theta + 1}{\sin\theta} = \cot\theta + \csc\theta$ is an identity.

$\frac{\cos\theta + 1}{\sin\theta} \stackrel{?}{=} \cot\theta + \csc\theta$ — Original equation

$\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} \stackrel{?}{=} \cot\theta + \csc\theta$ — Simplify.

$\cot\theta + \csc\theta = \cot\theta + \csc\theta$ ✓ — Simplify.

13-3 Sum and Difference of Angles Identities

Find the exact value of each expression.

24. $\cos(-135°)$

25. $\cos 15°$

26. $\sin 210°$

27. $\sin 105°$

28. $\tan 75°$

29. $\cos 105°$

Verify that each of the following is an identity.

30. $\sin(\theta + 90) = \cos\theta$

31. $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$

32. $\tan(\theta - \pi) = \tan\theta$

Example 4

Find the exact value of sin 75°.

Use $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

$$\begin{aligned}\sin 75° &= \sin(30° + 45°)\\ &= \sin 30°\cos 45° + \cos 30°\sin 45°\\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \text{ or } \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

13-4 Double-Angle and Half-Angle Identities

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin\frac{\theta}{2}$, and $\cos\frac{\theta}{2}$ for each of the following.

33. $\cos\theta = \frac{4}{5}$; $0° < \theta < 90°$

34. $\sin\theta = -\frac{1}{4}$; $180° < \theta < 270°$

35. $\cos\theta = -\frac{2}{3}$; $\frac{\pi}{2} < \theta < \pi$

36. **BASEBALL** The infield of a baseball diamond is a square with side length 90 feet.

a. Find the length of the diagonal.

b. Write the ratio for sin 45° using the lengths of the baseball diamond.

c. Use the formula $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ to verify the ratio you wrote in part **b**.

Example 5

Find the exact value of $\sin\frac{\theta}{2}$ if $\cos\theta = -\frac{3}{5}$ and θ is in the second quadrant.

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} && \text{Half-angle identity}\\ &= \pm\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} && \cos\theta = -\frac{3}{5}\\ &= \pm\sqrt{\frac{\frac{8}{5}}{2}} && \text{Subtract.}\\ &= \pm\sqrt{\frac{4}{5}} && \text{Divide.}\\ &= \pm\frac{2\sqrt{5}}{5} && \text{Simplify.}\end{aligned}$$

Since θ is in the second quadrant, $\sin\frac{\theta}{2} = \frac{2\sqrt{5}}{5.}$

13-5 Solving Trigonometric Equations

Find all solutions of each equation for the given interval.

37. $2\cos\theta - 1 = 0$; $0° \le \theta < 360°$

38. $4\cos^2\theta - 1 = 0$; $0 \le \theta < 2\pi$

39. $\sin 2\theta + \cos\theta = 0$; $0° \le \theta < 360°$

40. $\sin^2\theta = 2\sin\theta + 3$; $0° \le \theta < 360°$

41. $4\cos^2\theta - 4\cos\theta + 1 = 0$; $0 \le \theta < 2\pi$

Example 6

Find all solutions of $\sin 2\theta - \cos\theta = 0$ if $0 \le \theta < 2\pi$.

$$\begin{aligned}\sin 2\theta - \cos\theta &= 0 && \text{Original equation}\\ 2\sin\theta\cos\theta - \cos\theta &= 0 && \text{Double-angle identity}\\ \cos\theta(2\sin\theta - 1) &= 0 && \text{Factor.}\end{aligned}$$

$\cos\theta = 0$ or $2\sin\theta - 1 = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\sin\theta = \frac{1}{2}$; $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Practice Test

1. **MULTIPLE CHOICE** Which expression is equivalent to $\sin\theta + \cos\theta \cot\theta$?
   - A $\cot\theta$
   - B $\tan\theta$
   - C $\sec\theta$
   - D $\csc\theta$

2. Verify that $\cos(30° - \theta) = \sin(60° + \theta)$ is an identity.

3. Verify that $\cos(\theta - \pi) = -\cos\theta$.

4. **MULTIPLE CHOICE** What is the exact value of $\sin\theta$, if $\cos\theta = -\frac{3}{5}$ and $90° < \theta < 180°$?
   - F $\frac{5}{3}$
   - G $\frac{\sqrt{34}}{8}$
   - H $-\frac{4}{5}$
   - J $\frac{4}{5}$

Find the value of each expression.

5. $\cot\theta$, if $\sec\theta = \frac{4}{3}$; $270° < \theta < 360°$

6. $\tan\theta$, if $\cos\theta = -\frac{1}{2}$; $90° < \theta < 180°$

7. $\sec\theta$, if $\csc\theta = -2$; $180° < \theta < 270°$

8. $\cot\theta$, if $\csc\theta = -\frac{5}{3}$; $270° < \theta < 360°$

9. $\sec\theta$, if $\sin\theta = \frac{1}{2}$; $0° \leq \theta < 90°$

Verify that each of the following is an identity.

10. $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$

11. $\frac{\cos^2\theta}{1 - \sin\theta} = \frac{\cos\theta}{\sec\theta - \tan\theta}$

12. $(\tan\theta + \cot\theta)^2 = \csc^2\theta\sec^2\theta$

13. $\frac{1 + \sec\theta}{\sec\theta} = \frac{\sin^2\theta}{1 - \cos\theta}$

14. $\frac{\sin\theta}{1 - \cos\theta} = \csc\theta + \cot\theta$

15. **MULTIPLE CHOICE** What is the exact value of $\tan\frac{\pi}{8}$?
   - A $\frac{\sqrt{2 - \sqrt{3}}}{2}$
   - B $\sqrt{2} - 1$
   - C $1 - \sqrt{2}$
   - D $-\frac{\sqrt{2 - \sqrt{3}}}{2}$

16. **HISTORY** Some researchers believe that the builders of ancient pyramids, such as the Great Pyramid of Khufu, may have tried to build the faces as equilateral triangles. Later they had to change to other types of triangles. Suppose a pyramid is built such that a face is an equilateral triangle of side length 18 feet.

   a. Find the height of the equilateral triangle.

   b. Use the formula $\sin 2\theta = 2\sin\theta\cos\theta$ and the measures of the equilateral triangle and its height to show that $\sin 2(30°) = \sin 60°$. Find the exact values.

Find the exact value of each expression.

17. $\cos(-225°)$

18. $\sin 480°$

19. $\cos 75°$

20. $\sin 165°$

21. **ROCKETS** A model rocket is launched with an initial velocity of 20 meters per second. The range of a projectile is given by the formula $R = \frac{v^2}{g}\sin 2\theta$, where R is the range, v is the initial velocity, g is acceleration due to gravity or 9.8 meters per second squared, and θ is the launch angle. What angle is needed in order for the rocket to reach a range of 25 meters?

Solve each equation for all values of θ if θ is measured in radians.

22. $2\cos^2\theta - 3\cos\theta - 2 = 0$

23. $2\sin 3\theta - 1 = 0$

Solve each equation for $0° \leq \theta \leq 360°$ if θ is measured in degrees.

24. $\cos 2\theta + \cos\theta = 2$

25. $\sin\theta\cos\theta - \frac{1}{2}\sin\theta = 0$

CHAPTER 13 Preparing for Standardized Tests

Simplify Expressions

Some standardized test questions will require you to use the properties of algebra to simplify expressions. Follow the steps below to help prepare to solve these kinds of problems.

Strategies for Simplifying Expressions

Step 1

Study the expression that you are being asked to simplify.

Ask yourself:

- Are there any mathematical operations I can apply to help simplify the expression?
- Are there any laws or identities I can apply to help simplify the expression?

Step 2

Solve the problem and check your solution.

- Use the order of operations.
- Combine terms and factor as appropriate.
- Apply laws and identities.

Step 3

Check your solution if time permits.

- Retrace the steps in your work to make sure you answered the question thoroughly and accurately.
- If needed, sometimes you can use your scientific calculator to help you check your solution. Evaluate the original expression and your answer for some value and make sure they are the same.

Standardized Test Example

Solve the problem below. Responses will be graded using the short-response scoring rubric shown.

Simplify the trigonometric expression shown below by writing it in terms of sin θ. Show your work to receive full credit.

$$\frac{\cos\theta}{\sec\theta + \tan\theta}$$

Scoring Rubric	
Criteria	**Score**
Full Credit: The answer is correct and a full explanation is provided that shows each step.	2
Partial Credit: • The answer is correct, but the explanation is incomplete. • The answer is incorrect, but the explanation is correct.	1
No Credit: Either an answer is not provided or the answer does not make sense.	0

Read the problem statement carefully. You are given a trigonometric expression and asked to simplify it by writing it in terms of sin θ. So, your final answer must contain only numbers and terms involving sin θ. Show your work to receive full credit.

Example of a 2-point response:

Use trigonometric identities to simplify the expression.

$$\frac{\cos\theta}{\sec\theta + \tan\theta} = \frac{\cos\theta}{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}} \quad \text{Definition of sec } \theta \text{ and tan } \theta$$

$$= \frac{\cos\theta}{\frac{1+\sin\theta}{\cos\theta}} \quad \text{Simplify the denominator.}$$

$$= \frac{\cos^2\theta}{1+\sin\theta} \quad \text{Simplify the complex fraction.}$$

$$= \frac{1-\sin^2\theta}{1+\sin\theta} \quad \text{Pythagorean identity}$$

$$= \frac{(1+\sin\theta)(1-\sin\theta)}{1+\sin\theta} \quad \text{Factor.}$$

$$= 1 - \sin\theta \quad \text{Simplify.}$$

The simplified expression is $1 - \sin\theta$.

The steps, calculations, and reasoning are clearly stated. The student also arrives at the correct answer. So this response is worth the full 2 points.

Exercises

Solve each problem. Show your work. Responses will be graded using the short-response scoring rubric given at the beginning of the lesson.

1. Simplify $\frac{\sec\theta}{\cot\theta + \tan\theta}$ by writing it in terms of sin θ.

2. What is $\frac{10a^{-3}}{29b^4} \div \frac{5a^{-5}}{16b^{-7}}$?

3. Write $\frac{y+1}{y-1} + \frac{y+2}{y-2} + \frac{y}{y^2 - 3y + 2}$ in simplest form.

4. Simplify $\frac{\cot^2\theta - \csc^2\theta}{\tan^2\theta - \sec^2\theta}$ by writing it as a constant.

5. Multiply $(-5 + 2i)(6 - i)(4 + 3i)$.

6. Simplify $(\cot\theta + 1)^2 - 2\cot\theta$ by writing it in terms of csc θ.

7. Express $\frac{4 - \sqrt{7}}{3 + \sqrt{7}}$ in simplest form.

CHAPTER 13

Standardized Test Practice

Cumulative, Chapters 1 through 13

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. The profit p that Selena's Shirt Store makes in a day can be represented by the inequality $10t + 200 < p < 15t + 250$, where t represents the number of shirts sold. If the store sold 45 shirts on Friday, which of the following is a reasonable amount that the store made?

A \$200 **B** \$625 **C** \$850 **D** \$950

2. Use a sum or difference of angles identity to find the exact value of cos 75°.

F $\frac{\sqrt{2}-\sqrt{6}}{4}$ **H** $\frac{\sqrt{6}-\sqrt{2}}{4}$

G $\frac{\sqrt{2}+\sqrt{6}}{2}$ **J** $\frac{\sqrt{6}+\sqrt{2}}{4}$

3. Use the table to determine the expression that best represents the degree measure of an interior angle of a regular polygon with n sides.

Polygon	Number of Sides	Angle Measure
triangle	3	60
quadrilateral	4	90
pentagon	5	108
hexagon	6	120
heptagon	7	128.5
octagon	8	135

A $(180 + n) \div n$

B $\frac{180}{n}$

C $[180(n - 2)] \div n$

D $30(n - 1)$

4. Which of the following best describes the graphs of $y = 3x - 5$ and $4y = 12x + 16$?

F The lines have the same y-intercept.

G The lines have the same x-intercept.

H The lines are perpendicular.

J The lines are parallel.

Test-Taking Tip

Question 2 You can check your answer using a scientific calculator. Find cos 75° and compare it to the value of your answer.

5. What is the product of $\begin{bmatrix} 5 & -2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$?

A $\begin{bmatrix} 11 \\ -1 \end{bmatrix}$ **C** $\begin{bmatrix} 5 & -10 \\ 0 & -6 \\ 6 & -15 \end{bmatrix}$

B $\begin{bmatrix} 11 & -1 \end{bmatrix}$ **D** undefined

6. Which quadratic equation has roots $\frac{1}{2}$ and $\frac{1}{3}$?

F $5x^2 - 5x - 2 = 0$

G $5x^2 - 5x + 1 = 0$

H $6x^2 + 5x - 1 = 0$

J $6x^2 - 5x + 1 = 0$

7. How can you express $\cos\theta \csc\theta \cot\theta$ in terms of $\sin\theta$?

A $\frac{1-\sin^2\theta}{\sin^2\theta}$ **C** $\frac{\sin^2\theta}{2}$

B $\frac{1+\sin^2\theta}{\sin^2\theta}$ **D** $\frac{1-\sin^2\theta}{\sin\theta}$

8. The area of a rectangle is $25a^4 - 16b^2$. Which factors could represent the length times width?

F $(5a^2 + 4b)(5a^2 + 4b)$ **H** $(5a - 4b)(5a - 4b)$

G $(5a^2 + 4b)(5a^2 - 4b)$ **J** $(5a + 4b)(5a - 4b)$

9. What is the domain of $f(x) = \sqrt{5x - 3}$?

A $\left\{x \,\middle|\, x > \frac{3}{5}\right\}$ **C** $\left\{x \,\middle|\, x \geq \frac{3}{5}\right\}$

B $\left\{x \,\middle|\, x > -\frac{3}{5}\right\}$ **D** $\left\{x \,\middle|\, x \geq -\frac{3}{5}\right\}$

10. If the equation $y = 3^x$ is graphed, which of the following values of x would produce a point closest to the x-axis?

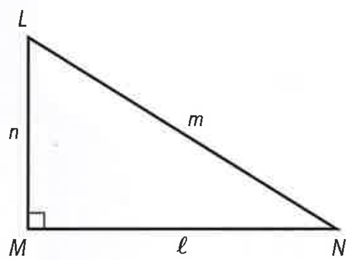
F $\frac{3}{4}$ **H** 0

G $\frac{1}{4}$ **J** $-\frac{3}{4}$

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. Use right triangle *LMN* at the right to show that $\sin 2N = \frac{2n\ell}{m^2}$.

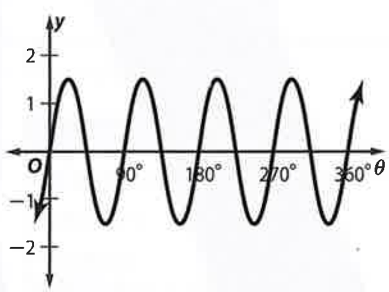


12. GRIDDED RESPONSE Solve the trigonometric equation below in the interval from 0 to 2π. Round your answer to the nearest hundredth if necessary.

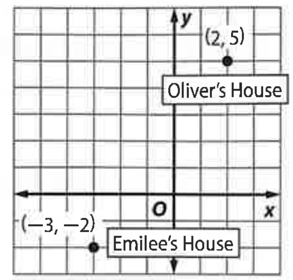
$$3 \cos \frac{t}{3} = 2$$

13. Kelly is designing a 12-inch by 12-inch scrapbook page. She cuts one picture that is 4 inches by 6 inches. She decides that she wants the next picture to be 75% as large as the first picture and the third picture to be 150% as large as the second picture. What are the approximate dimensions of the third picture?

14. Identify the amplitude and period of the function graphed below. Then write an equation for the function.



15. GRIDDED RESPONSE A coordinate grid is placed over a map. Emilee's house is located at (−3, −2), and Oliver's house is located at (2, 5). A side of each square represents one block. What is the approximate distance between Emilee's house and Oliver's house?



Extended Response

Record your answers on a sheet of paper. Show your work.

16. Kyla's annual salary is $50,000. Each year she gets a 6% raise.

a. To the nearest dollar, what will her salary be in four years?

b. To the nearest dollar, what will her salary be in 10 years?

17. COMPACT DISCS According to a recent survey, 91% of high school students do not buy compact discs. 8 random students are chosen.

a. Determine the probabilities associated with the number of students that do not buy compact discs by calculating the probability distribution.

b. What is the probability that at least 7 of the 8 students do not buy compact discs?

c. How many students should you expect to not buy compact discs?

Need ExtraHelp?

If you missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Go to Lesson...	1-5	13-3	2-4	3-1	3-6	4-3	13-1	5-2	6-3	7-1	13-4	13-5	8-5	12-8	9-1	10-2	11-4