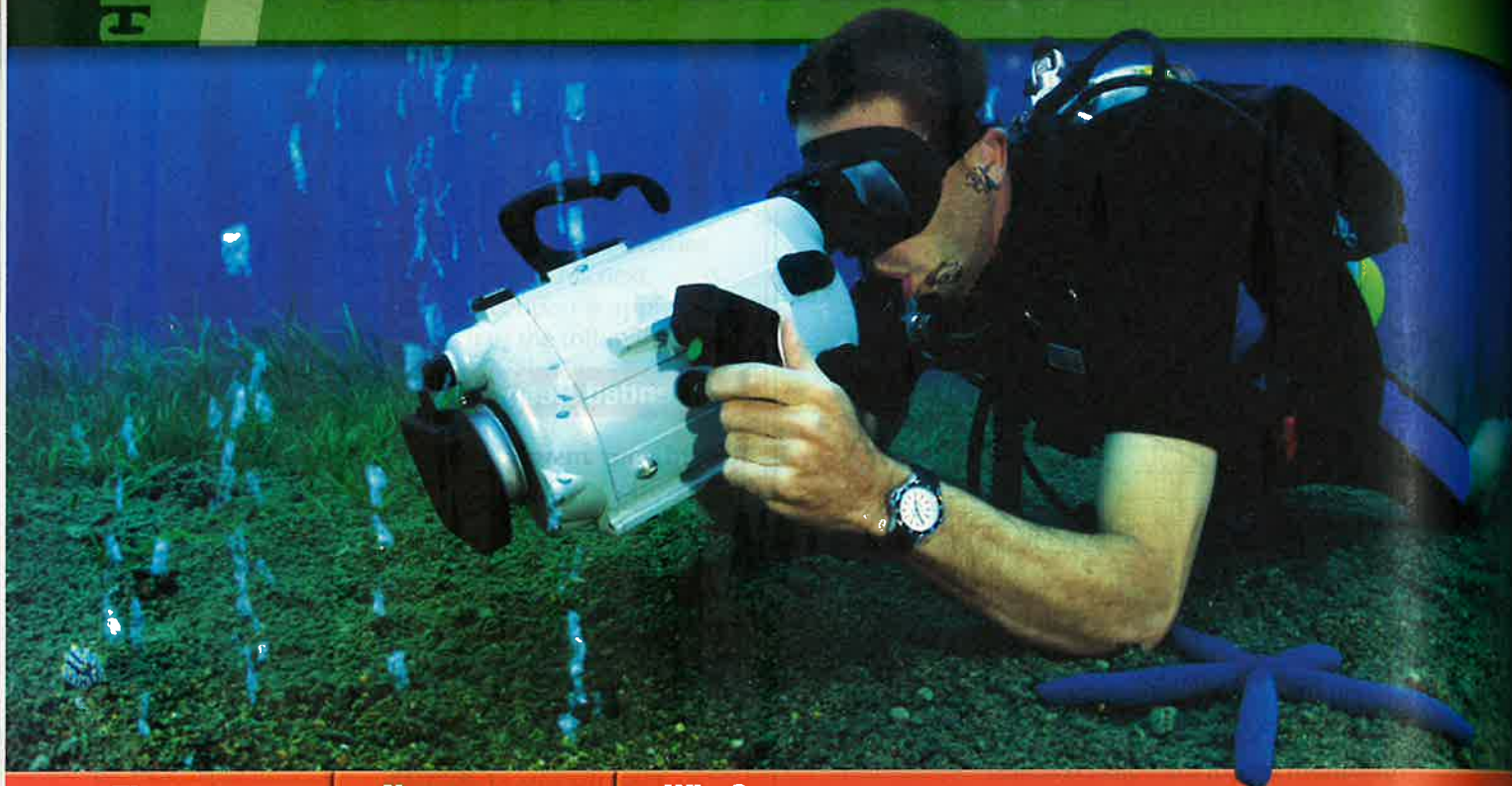


7 Exponential and Logarithmic Functions and Relations



Then

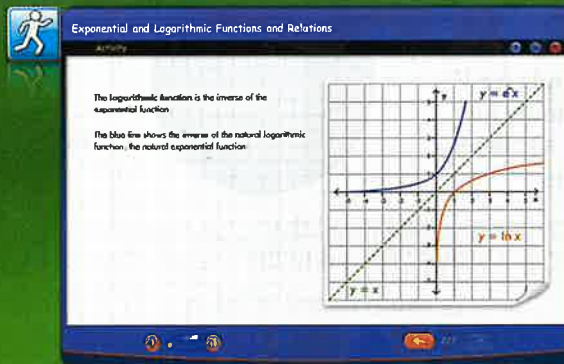
- You graphed functions and transformations of functions.

Now

- You will:
 - Graph exponential and logarithmic functions.
 - Solve exponential and logarithmic equations and inequalities.
 - Solve problems involving exponential growth and decay.

Why? ▲

- SCIENCE** Mathematics and science go hand in hand. Whether it is chemistry, biology, paleontology, zoology, or anthropology, you will need strong math skills. In this chapter, you will learn mathematical aspects of science such as computer viruses, populations of insects, bacteria growth, cell division, astronomy, tornados, and earthquakes.



connectED.mcgraw-hill.com Your Digital Math Portal

Animation



Vocabulary



eGlossary



Personal Tutor



Virtual Manipulatives



Graphing Calculator



Audio



Foldables



Self-Check Practice



Worksheets



Get Ready for the Chapter

Diagnose Readiness | You have two options for checking prerequisite skills.

1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck	QuickReview
<p>Simplify. Assume that no variable equals zero.</p> <ol style="list-style-type: none"> $a^4a^3a^5$ $(2xy^3z^2)^3$ $\frac{-24x^8y^5z}{16x^2y^8z^6}$ $\left(\frac{-8r^2n}{36n^3t}\right)^2$ <p>5. DENSITY The density of an object is equal to the mass divided by the volume. An object has a mass of 7.5×10^3 grams and a volume of 1.5×10^3 cubic centimeters. What is the density of the object?</p>	<p>Example 1</p> <p>Simplify $\frac{(a^3bc^2)^2}{a^4a^2b^2bc^5c^3}$. Assume that no variable equals zero.</p> $\frac{(a^3bc^2)^2}{a^4a^2b^2bc^5c^3}$ $= \frac{a^6b^2c^4}{a^6b^3c^8}$ $= \frac{1}{bc^4} \text{ or } b^{-1}c^{-4}$ <p>Simplify the numerator by using the Power of a Power Rule and the denominator by using the Product of Powers Rule.</p> <p>Simplify by using the Quotient of Powers Rule.</p>

Find the inverse of each function. Then graph the function and its inverse.

- | | |
|----------------------------|------------------------------|
| 6. $f(x) = 2x + 5$ | 7. $f(x) = x - 3$ |
| 8. $f(x) = -4x$ | 9. $f(x) = \frac{1}{4}x - 3$ |
| 10. $f(x) = \frac{x-1}{2}$ | 11. $y = \frac{1}{3}x + 4$ |

Determine whether each pair of functions are inverse functions.

- | | |
|--------------------------------------|--|
| 12. $f(x) = x - 6$
$g(x) = x + 6$ | 13. $f(x) = 2x + 5$
$g(x) = 2x - 5$ |
|--------------------------------------|--|
14. **FOOD** A pizzeria charges \$12 for a medium cheese pizza and \$2 for each additional topping. If $f(x) = 2x + 12$ represents the cost of a medium pizza with x toppings, find $f^{-1}(x)$ and explain its meaning.

Example 2

Find the inverse of $f(x) = 3x - 1$.

Step 1 Replace $f(x)$ with y in the original equation:
 $f(x) = 3x - 1 \rightarrow y = 3x - 1$.

Step 2 Interchange x and y : $x = 3y - 1$.

Step 3 Solve for y .

$$x = 3y - 1 \quad \text{Inverse}$$

$$x + 1 = 3y \quad \text{Add 1 to each side.}$$

$$\frac{x+1}{3} = y \quad \text{Divide each side by 3.}$$

$$\frac{1}{3}x + \frac{1}{3} = y \quad \text{Simplify.}$$

Step 4 Replace y with $f^{-1}(x)$.

$$y = \frac{1}{3}x + \frac{1}{3} \rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$$

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.



Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 7. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

FOLDABLES Study Organizer



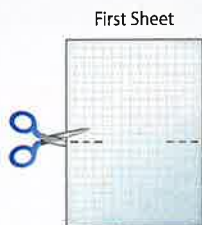
Exponential and Logarithmic Functions and Relations

Make this Foldable to help you organize your Chapter 7 notes about exponential and logarithmic functions. Begin with two sheets of grid paper.

- 1** Fold in half along the width.



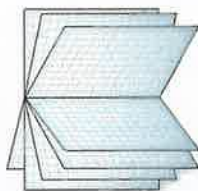
- 2** On the first sheet, cut 5 cm along the fold at the ends.



- 3** On the second sheet, cut in the center, stopping 5 cm from the ends.



- 4** Insert the first sheet through the second sheet and align the folds. Label the pages with lesson numbers.



New Vocabulary



English		Español
exponential function	p. 451	función exponencial
exponential growth	p. 451	crecimiento exponencial
asymptote	p. 451	asíntota
growth factor	p. 453	factor de crecimiento
exponential decay	p. 453	desintegración exponencial
decay factor	p. 454	factor de desintegración
exponential equation	p. 461	ecuación exponencial
compound interest	p. 462	interés compuesto
exponential inequality	p. 463	desigualdad exponencial
logarithm	p. 468	logaritmo
logarithmic function	p. 469	función logarítmica
logarithmic equation	p. 478	ecuación logarítmica
logarithmic inequality	p. 479	desigualdad logarítmica
common logarithm	p. 492	logaritmos comunes
Change of Base Formula	p. 494	fórmula del cambio de base
natural base, e	p. 501	e base natural
natural base	p. 501	base natural
exponential function		función exponencial
natural logarithm	p. 501	logaritmo natural

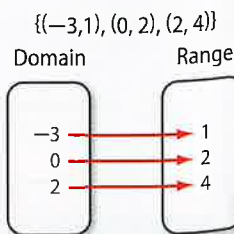
Review Vocabulary



domain **dominio** the set of all x -coordinates of the ordered pairs of a relation

function **función** a relation in which each element of the domain is paired with exactly one element in the range

range **rango** the set of all y -coordinates of the ordered pairs of a relation



7-1 Graphing Exponential Functions

Then

You graphed polynomial functions.

Now

- Graph exponential growth functions.
- Graph exponential decay functions.

Why?

Have you ever received an e-mail that tells you to forward it to 5 friends? If each of those 5 friends then forwards it to 5 of their friends, who each forward it to 5 of their friends, the number of people receiving the e-mail is growing exponentially.

The equation $y = 5^x$ can be used to represent this situation, where x is the number of rounds that the e-mail has been forwarded.



New Vocabulary

exponential function
exponential growth
asymptote
growth factor
exponential decay
decay factor

Common Core State Standards

Content Standards

F.IF.7.e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.IF.8.b Use the properties of exponents to interpret expressions for exponential functions.

Mathematical Practices

3 Construct viable arguments and critique the reasoning of others.

1 Exponential Growth A function like $y = 5^x$, where the base is a constant and the exponent is the independent variable, is an **exponential function**. One type of exponential function is exponential growth. An **exponential growth** function is a function of the form $f(x) = b^x$, where $b > 1$. The graph of an exponential function has an **asymptote**, which is a line that the graph of the function approaches.

Key Concept Parent Function of Exponential Growth Functions

Parent Functions: $f(x) = b^x, b > 1$

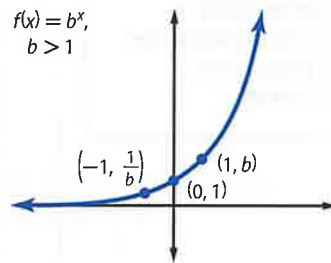
Type of graph: continuous, one-to-one, and increasing

Domain: all real numbers

Range: all positive real numbers

Asymptote: x -axis

Intercept: $(0, 1)$

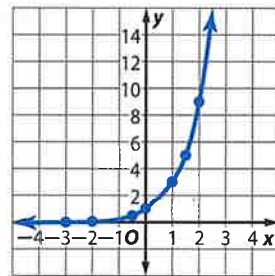


Example 1 Graph Exponential Growth Functions

Graph $y = 3^x$. State the domain and range.

Make a table of values. Then plot the points and sketch the graph.

x	-3	-2	$-\frac{1}{2}$	0
$y = 3^x$	$3^{-3} = \frac{1}{27}$	$3^{-2} = \frac{1}{9}$	$3^{-\frac{1}{2}} = \frac{\sqrt{3}}{3}$	$3^0 = 1$
x	1	$\frac{3}{2}$	2	
$y = 3^x$	$3^1 = 3$	$3^{\frac{3}{2}} = \sqrt{27}$	$3^2 = 9$	



The domain is all real numbers, and the range is all positive real numbers.

Guided Practice

1. Graph $y = 4^x$. State the domain and range.



The graph of $f(x) = b^x$ represents a parent graph of the exponential functions. The same techniques used to transform the graphs of other functions you have studied can be applied to the graphs of exponential functions.

KeyConcept Transformations of Exponential Functions

$$f(x) = ab^{x-h} + k$$

h – Horizontal Translation

h units right if h is positive
 $|h|$ units left if h is negative

k – Vertical Translation

k units up if k is positive
 $|k|$ units down if k is negative

a – Orientation and Shape

If $a < 0$, the graph is reflected in the x -axis.

If $|a| > 1$, the graph is stretched vertically.
 If $0 < |a| < 1$, the graph is compressed vertically.

StudyTip

CCSS Precision Remember that end behavior is the action of the graph as x approaches positive infinity or negative infinity. In Example 2a, as x approaches infinity, y approaches infinity. In Example 2b, as x approaches negative infinity, y approaches negative infinity.

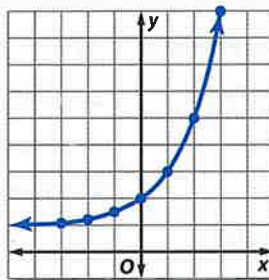
Example 2 Graph Transformations

Graph each function. State the domain and range.

a. $y = 2^x + 1$

The equation represents a translation of the graph of $y = 2^x$ one unit up.

x	$y = 2^x + 1$
-3	$2^{-3} + 1 = 1.125$
-2	$2^{-2} + 1 = 1.25$
-1	$2^{-1} + 1 = 1.5$
0	$2^0 + 1 = 2$
1	$2^1 + 1 = 3$
2	$2^2 + 1 = 5$
3	$2^3 + 1 = 9$



Domain = {all real numbers}; Range = $\{y \mid y > 1\}$

b. $y = -\frac{1}{2} \cdot 5^{x-2}$

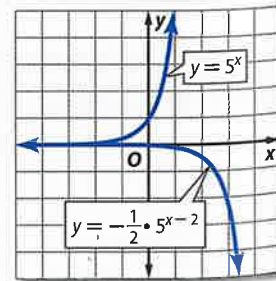
The equation represents a transformation of the graph of $y = 5^x$.

Graph $y = 5^x$ and transform the graph.

- $a = -\frac{1}{2}$: The graph is reflected in the x -axis and compressed vertically.
- $h = 2$: The graph is translated 2 units right.
- $k = 0$: The graph is not translated vertically.

Domain = {all real numbers}

Range = $\{y \mid y < 0\}$



GuidedPractice

2A. $y = 2^{x+3} - 5$

2B. $y = 0.1(6)^x - 3$



You can model exponential growth with a constant percent increase over specific time periods using the following function.

$$A(t) = a(1 + r)^t$$

The function can be used to find the amount $A(t)$ after t time periods, where a is the initial amount and r is the percent of increase per time period. Note that the base of the exponential expression, $1 + r$, is called the **growth factor**.

The exponential growth function is often used to model population growth.

Real-World Example 3 Graph Exponential Growth Functions

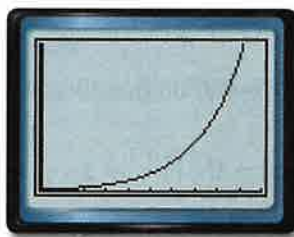


CENSUS The first U.S. census was conducted in 1790. At that time, the population was 3,929,214. Since then, the U.S. population has grown by approximately 2.03% annually. Draw a graph showing the population growth of the U.S. since 1790.

First, write an equation using $a = 3,929,214$, and $r = 0.0203$.

$$y = 3,929,214(1.0203)^t$$

Then graph the equation.



$[0, 250]$ scl: 25 by $[0, 400,000,000]$
scl: 40,000,000

Real-WorldLink

The U.S. Census Bureau's American Community Survey is mailed to approximately 1 out of every 480 households.

Source: Census Bureau

StudyTip

Interest The formula for simple interest, $i = prt$, illustrates linear growth over time. However, the formula for compound interest, $A(t) = a(1 + r)^t$, illustrates exponential growth over time. This is why investments with compound interest make more money.

Guided Practice

- FINANCIAL LITERACY** Teen spending is expected to grow 3.5% annually from \$79.7 billion in 2006. Draw a graph to show the spending growth.

2 Exponential Decay

The second type of exponential function is **exponential decay**.

Key Concept Parent Function of Exponential Decay Functions



Parent Functions: $f(x) = b^x, 0 < b < 1$

Model

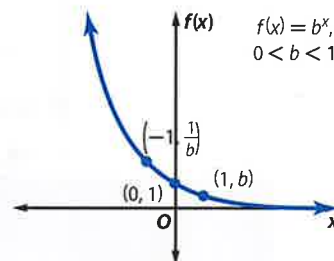
Type of graph: continuous, one-to-one, and decreasing

Domain: all real numbers

Range: positive real numbers

Asymptote: x -axis

Intercept: $(0, 1)$



The graphs of exponential decay functions can be transformed in the same manner as those of exponential growth.



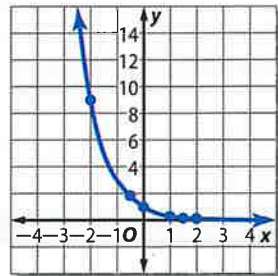
StudyTip
Exponential Decay Be sure not to confuse a dilation in which $|a| < 1$ with exponential decay in which $0 < b < 1$.

Example 4 Graph Exponential Decay Functions

Graph each function. State the domain and range.

a. $y = \left(\frac{1}{3}\right)^x$

x	$y = \left(\frac{1}{3}\right)^x$
-3	$\left(\frac{1}{3}\right)^{-3} = 27$
-2	$\left(\frac{1}{3}\right)^{-2} = 9$
$-\frac{1}{2}$	$\left(\frac{1}{3}\right)^{-\frac{1}{2}} = \sqrt{3}$
0	$\left(\frac{1}{3}\right)^0 = 1$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$
$\frac{3}{2}$	$\left(\frac{1}{3}\right)^{\frac{3}{2}} = \sqrt{\frac{1}{27}}$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$



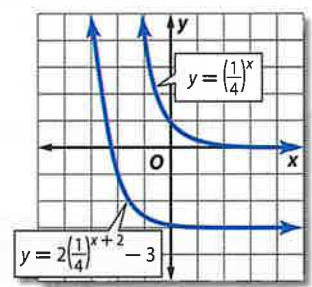
The domain is all real numbers, and the range is all positive real numbers.

b. $y = 2\left(\frac{1}{4}\right)^{x+2} - 3$

The equation represents a transformation of the graph of $y = \left(\frac{1}{4}\right)^x$.

Examine each parameter.

- $a = 2$: The graph is stretched vertically.
- $h = -2$: The graph is translated 2 units left.
- $k = -3$: The graph is translated 3 units down.



The domain is all real numbers, and the range is all real numbers greater than -3 .

Guided Practice

4A. $y = -3\left(\frac{2}{5}\right)^{x-4} + 2$

4B. $y = \frac{3}{8}\left(\frac{5}{6}\right)^{x-1} + 1$

Similar to exponential growth, you can model exponential decay with a constant percent of decrease over specific time periods using the following function.

$$A(t) = a(1 - r)^t$$

The base of the exponential expression, $1 - r$, is called the **decay factor**.

Real-World Example 5 Graph Exponential Decay Functions

TEA A cup of green tea contains 35 milligrams of caffeine. The average teen can eliminate approximately 12.5% of the caffeine from their system per hour.

- a. Draw a graph to represent the amount of caffeine remaining after drinking a cup of green tea.

$$y = a(1 - r)^t$$

$$= 35(1 - 0.125)^t$$

$$= 35(0.875)^t$$

Graph the equation.



[0, 10] scl: 1 by [0, 50] scl: 5

- b. Estimate the amount of caffeine in a teenager's body 3 hours after drinking a cup of green tea.

$$y = 35(0.875)^t \quad \text{Equation from part a}$$

$$= 35(0.875)^3 \quad \text{Replace } t \text{ with 3.}$$

$$\approx 23.45 \quad \text{Use a calculator.}$$

The caffeine in a teenager will be about 23.45 milligrams after 3 hours.

Guided Practice

5. A cup of black tea contains about 68 milligrams of caffeine. Draw a graph to represent the amount of caffeine remaining in the body of an average teen after drinking a cup of black tea. Estimate the amount of caffeine in the body 2 hours after drinking a cup of black tea.



Real-WorldLink

After water, tea is the most consumed beverage in the U.S. It can be found in over 80% of American households. Just over half the American population drinks tea daily.

Source: The Tea Association of the USA

Check Your Understanding = Step-by-Step Solutions begin on page R14.

Examples 1–2 Graph each function. State the domain and range.

1. $f(x) = 2^x$ 2. $f(x) = 5^x$ 3. $f(x) = 3^{x-2} + 4$
 4. $f(x) = 2^{x+1} + 3$ 5. $f(x) = 0.25(4)^x - 6$ 6. $f(x) = 3(2)^x + 8$

Example 3 7. **CCSS SENSE-MAKING** A virus spreads through a network of computers such that each minute, 25% more computers are infected. If the virus began at only one computer, graph the function for the first hour of the spread of the virus.

Example 4 Graph each function. State the domain and range.

8. $f(x) = 2\left(\frac{2}{3}\right)^{x-3} - 4$ 9. $f(x) = -\frac{1}{2}\left(\frac{3}{4}\right)^{x+1} + 5$
 10. $f(x) = -\frac{1}{3}\left(\frac{4}{5}\right)^{x-4} + 3$ 11. $f(x) = \frac{1}{8}\left(\frac{1}{4}\right)^{x+6} + 7$

Example 5 12. **FINANCIAL LITERACY** A new SUV depreciates in value each year by a factor of 15%. Draw a graph of the SUV's value for the first 20 years after the initial purchase.



Examples 1–2 Graph each function. State the domain and range.

13. $f(x) = 2(3)^x$

14. $f(x) = -2(4)^x$

15. $f(x) = 4^{x+1} - 5$

16. $f(x) = 3^{2x} + 1$

17. $f(x) = -0.4(3)^{x+2} + 4$

18. $f(x) = 1.5(2)^x + 6$

Example 3

- 19 SCIENCE** The population of a colony of beetles grows 30% each week for 10 weeks. If the initial population is 65 beetles, graph the function that represents the situation.

Example 4

Graph each function. State the domain and range.

20. $f(x) = -4\left(\frac{3}{5}\right)^{x+4} + 3$

21. $f(x) = 3\left(\frac{2}{5}\right)^{x-3} - 6$

22. $f(x) = \frac{1}{2}\left(\frac{1}{5}\right)^{x+5} + 8$

23. $f(x) = \frac{3}{4}\left(\frac{2}{3}\right)^{x+4} - 2$

24. $f(x) = -\frac{1}{2}\left(\frac{3}{8}\right)^{x+2} + 9$

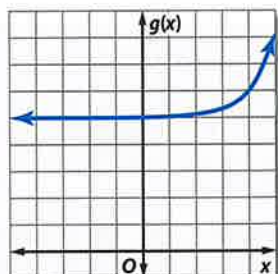
25. $f(x) = -\frac{5}{4}\left(\frac{4}{5}\right)^{x+4} + 2$

Example 5

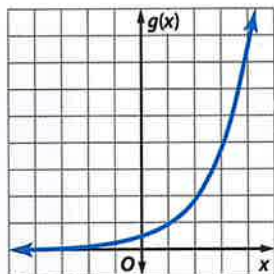
- 26. ATTENDANCE** The attendance for a basketball team declined at a rate of 5% per game throughout a losing season. Graph the function modeling the attendance if 15 home games were played and 23,500 people were at the first game.
- 27. PHONES** The function $P(x) = 2.28(0.9^x)$ can be used to model the number of pay phones in millions x years since 1999.
- Classify the function as either exponential *growth* or *decay*, and identify the growth or decay factor. Then graph the function.
 - Explain what the $P(x)$ -intercept and the asymptote represent in this situation.
- 28. HEALTH** Each day, 10% of a certain drug dissipates from the system.
- Classify the function representing this situation as either exponential *growth* or *decay*, and identify the growth or decay factor. Then graph the function.
 - How much of the original amount remains in the system after 9 days?
 - If a second dose should not be taken if more than 50% of the original amount is in the system, when should the label say it is safe to redose? Design the label and explain your reasoning.
- 29. CCSS REASONING** A sequence of numbers follows a pattern in which the next number is 125% of the previous number. The first number in the pattern is 18.
- Write the function that represents the situation.
 - Classify the function as either exponential *growth* or *decay*, and identify the growth or decay factor. Then graph the function for the first 10 numbers.
 - What is the value of the tenth number? Round to the nearest whole number.

For each graph, $f(x)$ is the parent function and $g(x)$ is a transformation of $f(x)$. Use the graph to determine the equation of $g(x)$.

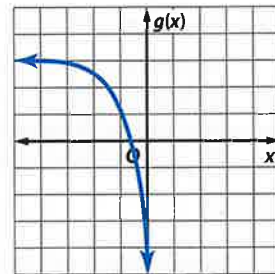
30. $f(x) = 3^x$




31. $f(x) = 2^x$



32. $f(x) = 4^x$



- 33**  **MULTIPLE REPRESENTATIONS** In this problem, you will use the tables below for exponential functions $f(x)$, $g(x)$, and $h(x)$.

x	-1	0	1	2	3	4	5
$f(x)$	2.5	2	1	-1	-5	-13	-29

x	-1	0	1	2	3	4	5
$g(x)$	5	11	23	47	95	191	383

x	-1	0	1	2	3	4	5
$h(x)$	3	2.5	2.25	2.125	2.0625	2.0313	2.0156

- Graphical** Graph the functions for $-1 \leq x \leq 5$ on separate graphs.
- Verbal** List any function with a negative coefficient. Explain your reasoning.
- Analytical** List any function with a graph that is translated to the left.
- Analytical** Determine which functions are growth models and which are decay models.

H.O.T. Problems Use Higher-Order Thinking Skills

- 34. REASONING** Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your reasoning.
- An exponential function of the form $y = ab^{x-h} + k$ has a y -intercept.
 - An exponential function of the form $y = ab^{x-h} + k$ has an x -intercept.
 - The function $f(x) = |b|^x$ is an exponential growth function if b is an integer.
- 35. CCSS CRITIQUE** Vince and Grady were asked to graph the following functions. Vince thinks they are the same, but Grady disagrees. Who is correct? Explain your reasoning.

x	y
0	2
1	1
2	0.5
3	0.25
4	0.125
5	0.0625
6	0.03125

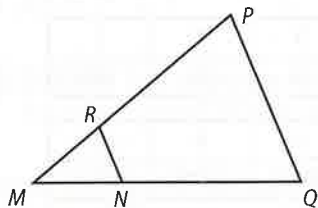
an exponential function
with rate of decay of $\frac{1}{2}$ and
an initial amount of 2

- 36. CHALLENGE** A substance decays 35% each day. After 8 days, there are 8 milligrams of the substance remaining. How many milligrams were there initially?
- 37. OPEN ENDED** Give an example of a value of b for which $f(x) = \left(\frac{8}{b}\right)^x$ represents exponential decay.
- 38. WRITING IN MATH** Write the procedure for transforming the graph of $g(x) = b^x$ to the graph of $f(x) = ab^{x-h} + k$. Justify each step.



Standardized Test Practice

39. **GRIDDED RESPONSE** In the figure, $\overline{PQ} \parallel \overline{RN}$, $QN = 12$, $MN = 6$, and $RN = 4$. What is the length of \overline{PQ} ?



40. Ivan has enough money to buy 12 used CDs. If the cost of each CD was \$0.20 less, Ivan could buy 2 more CDs. How much money does Ivan have to spend on CDs?
- A \$16.80 C \$15.80
 B \$16.40 D \$15.40

41. One hundred students will attend the fall dance if tickets cost \$30 each. For each \$5 increase in price, 10 fewer students will attend. What price will deliver the maximum dollar sales?

- F \$30
 G \$35
 H \$40
 J \$45

42. **SAT/ACT** Javier mows a lawn in 2 hours. Tonya mows the same lawn in 1.5 hours. About how many minutes will it take to mow the lawn if Javier and Tonya work together?

- A 28 minutes D 1.2 hours
 B 42 minutes E 1.4 hours
 C 51 minutes

Spiral Review

Solve each equation or inequality. (Lesson 6-7)

43. $\sqrt{y+5} = \sqrt{2y-3}$ 44. $\sqrt{y+1} + \sqrt{y-4} = 5$ 45. $10 - \sqrt{2x+7} \leq 3$
 46. $6 + \sqrt{3y+4} < 6$ 47. $\sqrt{d+3} + \sqrt{d+7} > 4$ 48. $\sqrt{2x+5} - \sqrt{9+x} > 0$

Simplify. (Lesson 6-6)

49. $\frac{1}{y^{\frac{2}{5}}}$ 50. $\frac{xy}{\sqrt[3]{z}}$ 51. $\frac{3x+4x^2}{x^{\frac{2}{3}}}$
 52. $\sqrt[6]{27x^3}$ 53. $\frac{\sqrt[4]{27}}{\sqrt[4]{3}}$ 54. $\frac{a^{\frac{1}{2}}}{6a^{\frac{1}{3}} \cdot a^{\frac{1}{4}}}$

55. **FOOTBALL** The path of a football thrown across a field is given by the equation $y = -0.005x^2 + x + 5$, where x represents the distance, in feet, the ball has traveled horizontally and y represents the height, in feet, of the ball above ground level. About how far has the ball traveled horizontally when it returns to ground level? (Lesson 4-6)
56. **COMMUNITY SERVICE** A drug awareness program is being presented at a theater that seats 300 people. Proceeds will be donated to a local drug information center. If every two adults must bring at least one student, what is the maximum amount of money that can be raised? (Lesson 3-3)



Skills Review

Simplify. Assume that no variable equals 0.

57. $f^{-7} \cdot f^4$ 58. $(3x^2)^3$ 59. $(2y)(4xy^3)$ 60. $\left(\frac{3}{5}cf\right)\left(\frac{4}{3}cd\right)^2$



EXPLORE 7-2 Graphing Technology Lab Solving Exponential Equations and Inequalities



You can use a TI-83/84 Plus graphing calculator to solve exponential equations by graphing or by using the table feature. To do this, you will write the equations as systems of equations.

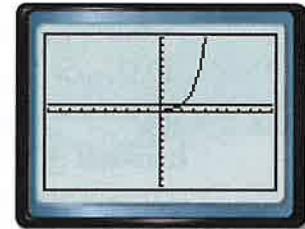
CCSS Common Core State Standards
Content Standards

A.REI.11 Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Activity 1

Solve $3^{x-4} = \frac{1}{9}$.

Step 1 Graph each side of the equation as a separate function. Enter 3^{x-4} as Y1. Be sure to include parentheses around the exponent. Enter $\frac{1}{9}$ as Y2. Then graph the two equations.

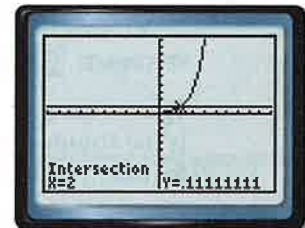


$[-10, 10]$ scl: 1 by $[-1, 1]$ scl: 0.1

Step 2 Use the **intersect** feature.

You can use the **intersect** feature on the **CALC** menu to approximate the ordered pair of the point at which the graphs cross.

The calculator screen shows that the x -coordinate of the point at which the curves cross is 2. Therefore, the solution of the equation is 2.



$[-10, 10]$ scl: 1 by $[-1, 1]$ scl: 0.1

Step 3 Use the **TABLE** feature.

You can also use the **TABLE** feature to locate the point at which the curves intersect.

The table displays x -values and corresponding y -values for each graph. Examine the table to find the x -value for which the y -values of the graphs are equal.

At $x = 2$, both functions have a y -value of $0.\bar{1}$ or $\frac{1}{9}$. Thus, the solution of the equation is 2.

X	Y1	Y2
-3	4.6E-4	.11111
-2	.00137	.11111
-1	.00412	.11111
0	.01238	.11111
1	.03704	.11111
2	.11111	.11111
3	.33333	.11111

X=2

CHECK Substitute 2 for x in the original equation.

$$3^{x-4} \stackrel{?}{=} \frac{1}{9} \quad \text{Original equation}$$

$$3^{2-4} \stackrel{?}{=} \frac{1}{9} \quad \text{Substitute 2 for } x.$$

$$3^{-2} \stackrel{?}{=} \frac{1}{9} \quad \text{Simplify.}$$

$$\frac{1}{9} = \frac{1}{9} \quad \checkmark \quad \text{The solution checks.}$$

A similar procedure can be used to solve exponential inequalities.

(continued on the next page)

Graphing Technology Lab

Solving Exponential Equations and Inequalities *Continued*


Activity 2

Solve $2^{x-2} \geq 0.5^{x-3}$.

Step 1 Enter the related inequalities.

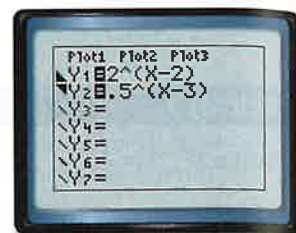
Rewrite the problem as a system of inequalities.

The first inequality is $2^{x-2} \geq y$ or $y \leq 2^{x-2}$. Since this inequality includes the *less than or equal to* symbol, shade below the curve.

First enter the boundary, and then use the arrow and **ENTER** keys to choose the shade below icon, .

The second inequality is $y \geq 0.5^{x-3}$. Shade above the curve since this inequality contains *greater than or equal to*.

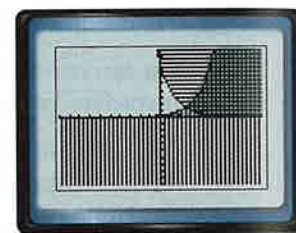
KEYSTROKES: **Y=** **◀** **◀** **ENTER** **ENTER** **ENTER** **▶** **▶** **2** **∧** **(**
X,T,θ,n **=** **2** **)** **ENTER** **◀** **◀** **ENTER** **ENTER** **▶**
▶ **.5** **∧** **(** **X,T,θ,n** **=** **3** **)**



Step 2 Graph the system.

KEYSTROKES: **GRAPH**

The x -values of the points in the region where the shadings overlap is the solution set of the original inequality. Using the **intersect** feature, you can conclude that the solution set is $\{x \mid x \geq 2.5\}$.



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

Step 3 Use the **TABLE** feature.

Verify using the **TABLE** feature. Set up the table to show x -values in increments of 0.5.

KEYSTROKES: **2nd** **[TBLSET]** **0** **ENTER** **.5** **ENTER** **2nd** **[TABLE]**

Notice that for x -values greater than $x = 2.5$, $Y1 > Y2$. This confirms that the solution of the inequality is $\{x \mid x \geq 2.5\}$.

X	Y1	Y2
0	.25	8
.5	.35355	6.6569
1	.5	5.6234
1.5	.70711	4.8284
2	1	4.1414
2.5	1.4142	3.6758
3	2	3.3436
3.5	2.8284	3.1117

Exercises

Solve each equation or inequality.

1. $9^{x-1} = \frac{1}{81}$

2. $4^{x+3} = 2^{5x}$

3. $5^{x-1} = 2^x$

4. $3.5^{x+2} = 1.75^{x+3}$

5. $-3x + 4 = -0.5^{2x+3}$

6. $6^{2-x} - 4 < -0.25^{x-2.5}$

7. $16^{x-1} > 2^{2x+2}$

8. $3^x - 4 \leq 5^{\frac{x}{2}}$

9. $5^{x+3} \leq 2^{x+4}$

10. **WRITING IN MATH** Explain why this technique of graphing a system of equations or inequalities works to solve exponential equations and inequalities.

7-2 Solving Exponential Equations and Inequalities

Then

- You graphed exponential functions.

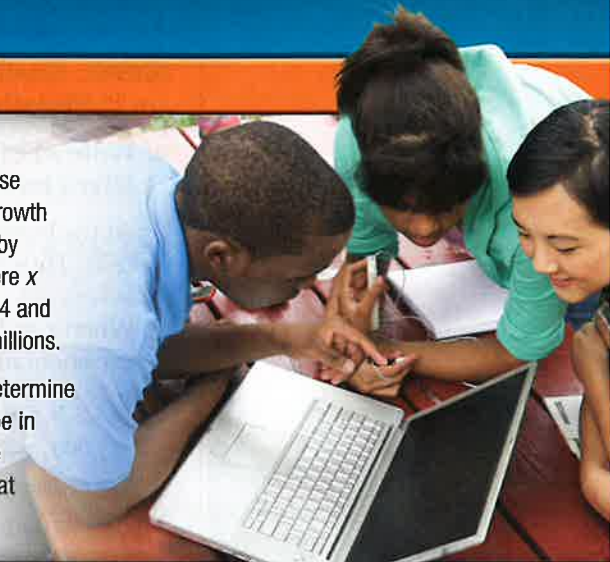
Now

- Solve exponential equations.
- Solve exponential inequalities.

Why?

- Membership on Internet social networking sites tends to increase exponentially. The membership growth of one Web site can be modeled by the equation $y = 2.2(1.37)^x$, where x is the number of years since 2004 and y is the number of members in millions.

You can use $y = 2.2(1.37)^x$ to determine how many members there will be in a given year, or to determine the year in which membership was at a certain level.



New Vocabulary

exponential equation
compound interest
exponential inequality



1 Solve Exponential Equations In an **exponential equation**, variables occur as exponents.



Key Concept Property of Equality for Exponential Functions

Words Let $b > 0$ and $b \neq 1$. Then $b^x = b^y$ if and only if $x = y$.

Example If $3^x = 3^5$, then $x = 5$. If $x = 5$, then $3^x = 3^5$.

The Property of Equality can be used to solve exponential equations.

Example 1 Solve Exponential Equations

Solve each equation.

a. $2^x = 8^3$

$$2^x = 8^3$$

Original equation

$$2^x = (2^3)^3$$

Rewrite 8 as 2^3 .

$$2^x = 2^9$$

Power of a Power

$$x = 9$$

Property of Equality for Exponential Functions

b. $9^{2x-1} = 3^{6x}$

$$9^{2x-1} = 3^{6x}$$

Original equation

$$(3^2)^{2x-1} = 3^{6x}$$

Rewrite 9 as 3^2 .

$$3^{4x-2} = 3^{6x}$$

Power of a Power

$$4x - 2 = 6x$$

Property of Equality for Exponential Functions

$$-2 = 2x$$

Subtract $4x$ from each side.

$$-1 = x$$

Divide each side by 2.

Guided Practice

1A. $4^{2n-1} = 64$

1B. $5^{5x} = 125^x + 2$



Common Core State Standards

Content Standards

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a , c , and d are numbers and the base b is 2, 10, or e ; evaluate the logarithm using technology.

Mathematical Practices

2 Reason abstractly and quantitatively.



You can use information about growth or decay to write the equation of an exponential function.



Real-World Example 2 Write an Exponential Function

SCIENCE Kristin starts an experiment with 7500 bacteria cells. After 4 hours, there are 23,000 cells.

- a. Write an exponential function that could be used to model the number of bacteria after x hours if the number of bacteria changes at the same rate.

At the beginning of the experiment, the time is 0 hours and there are 7500 bacteria cells. Thus, the y -intercept, and the value of a , is 7500.

When $x = 4$, the number of bacteria cells is 23,000. Substitute these values into an exponential function to determine the value of b .

$y = ab^x$	Exponential function
$23,000 = 7500 \cdot b^4$	Replace x with 4, y with 23,000, and a with 7500.
$3.067 \approx b^4$	Divide each side by 7500.
$\sqrt[4]{3.067} \approx b$	Take the 4th root of each side.
$1.323 \approx b$	Use a calculator.

An equation that models the number of bacteria is $y \approx 7500(1.323)^x$.

- b. How many bacteria cells can be expected in the sample after 12 hours?

$y \approx 7500(1.323)^x$	Modeling equation
$\approx 7500(1.323)^{12}$	Replace x with 12.
$\approx 215,665$	Use a calculator.

There will be approximately 215,665 bacteria cells after 12 hours.

Guided Practice

2. **RECYCLING** A manufacturer distributed 3.2 million aluminum cans in 2005.

- A. In 2010, the manufacturer distributed 420,000 cans made from the recycled cans it had previously distributed. Assuming that the recycling rate continues, write an equation to model the distribution each year of cans that are made from recycled aluminum.
- B. How many cans made from recycled aluminum can be expected in the year 2050?



Real-World Link

In 2008, the U.S. recycling rate for metals of 35% prevented the release of approximately 25 million metric tons of carbon into the air—roughly the amount emitted annually by 4.5 million cars.

Source: Environmental Protection Agency

Exponential functions are used in situations involving compound interest.

Compound interest is interest paid on the principal of an investment and any previously earned interest.

Key Concept Compound Interest

You can calculate compound interest using the following formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A is the amount in the account after t years, P is the principal amount invested, r is the annual interest rate, and n is the number of compounding periods each year.





Example 3 Compound Interest

An investment account pays 4.2% annual interest compounded monthly. If \$2500 is invested in this account, what will be the balance after 15 years?

Understand Find the total amount in the account after 15 years.

Plan Use the compound interest formula.

$$P = 2500, r = 0.042, n = 12, \text{ and } t = 15$$

$$\text{Solve } A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Compound Interest Formula}$$

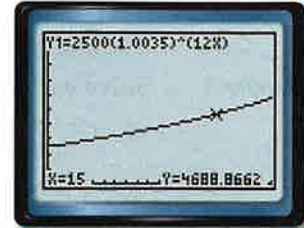
$$= 2500\left(1 + \frac{0.042}{12}\right)^{12 \cdot 15} \quad P = 2500, r = 0.042, n = 12, t = 15$$

$$\approx 4688.87 \quad \text{Use a calculator.}$$

Check Graph the corresponding equation

$y = 2500(1.0035)^{12t}$. Use **CALC: value** to find y when $x = 15$.

The y -value 4688.8662 is very close to 4688.87, so the answer is reasonable.



[0, 20] scl: 1 by [0, 10,000] scl: 1000

Watch Out!

Percents Remember to convert all percents to decimal form; 4.2% is 0.042.

Guided Practice

3. Find the account balance after 20 years if \$100 is placed in an account that pays 1.2% interest compounded twice a month.

2 Solve Exponential Inequalities

An **exponential inequality** is an inequality involving exponential functions.

Key Concept Property of Inequality for Exponential Functions

Words Let $b > 1$. Then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Example If $2^x > 2^6$, then $x > 6$. If $x > 6$, then $2^x > 2^6$.

This property also holds true for \leq and \geq .



Example 4 Solve Exponential Inequalities

Solve $16^{2x-3} < 8$.

$$16^{2x-3} < 8 \quad \text{Original inequality}$$

$$(2^4)^{2x-3} < 2^3 \quad \text{Rewrite 16 as } 2^4 \text{ and 8 as } 2^3.$$

$$2^{8x-12} < 2^3 \quad \text{Power of a Power}$$

$$8x - 12 < 3 \quad \text{Property of Inequality for Exponential Functions}$$

$$8x < 15 \quad \text{Add 12 to each side.}$$

$$x < \frac{15}{8} \quad \text{Divide each side by 8.}$$

Guided Practice

Solve each inequality.

4A. $3^{2x-1} \geq \frac{1}{243}$

4B. $2^{x+2} > \frac{1}{32}$



**Example 1** Solve each equation.

1. $3^{5x} = 27^{2x-4}$

2. $16^{2y-3} = 4^{y+1}$

3. $2^{6x} = 32^{x-2}$

4. $49^{x+5} = 7^{8x-6}$

Example 2

5. **SCIENCE** Mitosis is a process in which one cell divides into two. The *Escherichia coli* is one of the fastest growing bacteria. It can reproduce itself in 15 minutes.

- Write an exponential function to represent the number of cells c after t minutes.
- If you begin with one *Escherichia coli* cell, how many cells will there be in one hour?

Example 3

6. A certificate of deposit (CD) pays 2.25% annual interest compounded biweekly. If you deposit \$500 into this CD, what will the balance be after 6 years?

Example 4 Solve each inequality.

7. $4^{2x+6} \leq 64^{2x-4}$

8. $25^y - 3 \leq \left(\frac{1}{125}\right)^{y+2}$

Practice and Problem Solving

Extra Practice is on page R7.

Example 1 Solve each equation.

9. $8^{4x+2} = 64$

10. $5^{x-6} = 125$

11. $81^{a+2} = 3^{3a+1}$

12. $256^{b+2} = 4^{2-2b}$

13. $9^{3c+1} = 27^{3c-1}$

14. $8^{2y+4} = 16^{y+1}$

Example 2

15. **CCSS MODELING** In 2009, My-Lien received \$10,000 from her grandmother. Her parents invested all of the money, and by 2021, the amount will have grown to \$16,960.

- Write an exponential function that could be used to model the money y . Write the function in terms of x , the number of years since 2009.
- Assume that the amount of money continues to grow at the same rate. What would be the balance in the account in 2031?

Write an exponential function for the graph that passes through the given points.

16. (0, 6.4) and (3, 100)

17. (0, 256) and (4, 81)

18. (0, 128) and (5, 371,293)

19. (0, 144), and (4, 21,609)

Example 3

- Find the balance of an account after 7 years if \$700 is deposited into an account paying 4.3% interest compounded monthly.
- Determine how much is in a retirement account after 20 years if \$5000 was invested at 6.05% interest compounded weekly.
- A savings account offers 0.7% interest compounded bimonthly. If \$110 is deposited in this account, what will the balance be after 15 years?
- A college savings account pays 13.2% annual interest compounded semiannually. What is the balance of an account after 12 years if \$21,000 was initially deposited?

Example 4 Solve each inequality.

24. $625 \geq 5^{a+8}$

25. $10^{5b+2} > 1000$

26. $\left(\frac{1}{64}\right)^{c-2} < 32^{2c}$

27. $\left(\frac{1}{27}\right)^{2d-2} \leq 81^{d+4}$

28. $\left(\frac{1}{9}\right)^{3t+5} \geq \left(\frac{1}{243}\right)^{t-6}$

29. $\left(\frac{1}{36}\right)^{w+2} < \left(\frac{1}{216}\right)^{4w}$



30. **SCIENCE** A mug of hot chocolate is 90°C at time $t = 0$. It is surrounded by air at a constant temperature of 20°C . If stirred steadily, its temperature in Celsius after t minutes will be $y(t) = 20 + 70(1.071)^{-t}$.
- Find the temperature of the hot chocolate after 15 minutes.
 - Find the temperature of the hot chocolate after 30 minutes.
 - The optimum drinking temperature is 60°C . Will the mug of hot chocolate be at or below this temperature after 10 minutes?
31. **ANIMALS** Studies show that an animal will defend a territory, with area in square yards, that is directly proportional to the 1.31 power of the animal's weight in pounds.
- If a 45-pound beaver will defend 170 square yards, write an equation for the area a defended by a beaver weighing w pounds.
 - Scientists believe that thousands of years ago, the beaver's ancestors were 11 feet long and weighed 430 pounds. Use your equation to determine the area defended by these animals.

Solve each equation.

32. $\left(\frac{1}{2}\right)^{4x+1} = 8^{2x+1}$ 33. $\left(\frac{1}{5}\right)^{x-5} = 25^{3x+2}$ 34. $216 = \left(\frac{1}{6}\right)^{x+3}$


35. $\left(\frac{1}{8}\right)^{3x+4} = \left(\frac{1}{4}\right)^{-2x+4}$ 36. $\left(\frac{2}{3}\right)^{5x+1} = \left(\frac{27}{8}\right)^{x-4}$ 37. $\left(\frac{25}{81}\right)^{2x+1} = \left(\frac{729}{125}\right)^{-3x+1}$

38. **CCSS MODELING** In 1950, the world population was about 2.556 billion. By 1980, it had increased to about 4.458 billion.
- Write an exponential function of the form $y = ab^x$ that could be used to model the world population y in billions for 1950 to 1980. Write the equation in terms of x , the number of years since 1950. (Round the value of b to the nearest ten-thousandth.)
 - Suppose the population continued to grow at that rate. Estimate the population in 2000.
 - In 2000, the population of the world was about 6.08 billion. Compare your estimate to the actual population.
 - Use the equation you wrote in Part a to estimate the world population in the year 2020. How accurate do you think the estimate is? Explain your reasoning.
39. **TREES** The diameter of the base of a tree trunk in centimeters varies directly with the $\frac{3}{2}$ power of its height in meters.
- A young sequoia tree is 6 meters tall, and the diameter of its base is 19.1 centimeters. Use this information to write an equation for the diameter d of the base of a sequoia tree if its height is h meters high.
 - The General Sherman Tree in Sequoia National Park, California, is approximately 84 meters tall. Find the diameter of the General Sherman Tree at its base.

40. **FINANCIAL LITERACY** Mrs. Jackson has two different retirement investment plans from which to choose.
- Write equations for Option A and Option B given the minimum deposits.
 - Draw a graph to show the balances for each investment option after t years.
 - Explain whether Option A or Option B is the better investment choice.

Option A:	Option B:
6.5% annual rate compounded quarterly; minimum deposit \$5000	4.2% annual rate compounded monthly; minimum deposit \$5000
	PLUS
	2.3% annual rate compounded weekly; minimum deposit \$5000



41.  **MULTIPLE REPRESENTATIONS** In this problem, you will explore the rapid increase of an exponential function. A large sheet of paper is cut in half, and one of the resulting pieces is placed on top of the other. Then the pieces in the stack are cut in half and placed on top of each other. Suppose this procedure is repeated several times.
- Concrete** Perform this activity and count the number of sheets in the stack after the first cut. How many pieces will there be after the second cut? How many pieces after the third cut? How many pieces after the fourth cut?
 - Tabular** Record your results in a table.
 - Symbolic** Use the pattern in the table to write an equation for the number of pieces in the stack after x cuts.
 - Analytical** The thickness of ordinary paper is about 0.003 inch. Write an equation for the thickness of the stack of paper after x cuts.
 - Analytical** How thick will the stack of paper be after 30 cuts?

H.O.T. Problems Use Higher-Order Thinking Skills

42. **WRITING IN MATH** In a problem about compound interest, describe what happens as the compounding period becomes more frequent while the principal and overall time remain the same.
43. **ERROR ANALYSIS** Beth and Liz are solving $6^{x-3} > 36^{-x-1}$. Is either of them correct? Explain your reasoning.

Beth

$$6^{x-3} > 36^{-x-1}$$

$$6^{x-3} > (6^2)^{-x-1}$$

$$6^{x-3} > 6^{-2x-2}$$

$$x-3 > -2x-2$$

$$3x > 1$$

$$x > \frac{1}{3}$$

Liz

$$6^{x-3} > 36^{-x-1}$$


$$6^{x-3} > (6^2)^{-x-1}$$

$$6^{x-3} > 6^{-x+1}$$

$$x-3 > -x+1$$

$$2x > 4$$

$$x > 2$$

44. **CHALLENGE** Solve for x : $16^{18} + 16^{18} + 16^{18} + 16^{18} + 16^{18} = 4^x$.
45. **OPEN ENDED** What would be a more beneficial change to a 5-year loan at 8% interest compounded monthly: reducing the term to 4 years or reducing the interest rate to 6.5%?
46.  **ARGUMENTS** Determine whether the following statements are *sometimes*, *always*, or *never* true. Explain your reasoning.
- $2^x > -8^{20x}$ for all values of x .
 - The graph of an exponential growth equation is increasing.
 - The graph of an exponential decay equation is increasing.
47. **OPEN ENDED** Write an exponential inequality with a solution of $x \leq 2$.
48. **PROOF** Show that $27^{2x} \cdot 81^{x+1} = 3^{2x+2} \cdot 9^{4x+1}$.
49. **WRITING IN MATH** If you were given the initial and final amounts of a radioactive substance and the amount of time that passes, how would you determine the rate at which the amount was increasing or decreasing in order to write an equation?



Standardized Test Practice

50. $3 \times 10^{-4} =$

- A 0.003 C 0.00003
 B 0.0003 D 0.000003

51. Which of the following could *not* be a solution to $5 - 3x < -3$?

- F 2.5 H 3.5
 G 3 J 4

52. **GRIDDED RESPONSE** The three angles of a triangle are $3x$, $x + 10$, and $2x - 40$. Find the measure of the smallest angle in the triangle.

53. **SAT/ACT** Which of the following is equivalent to $(x)(x)(x)(x)$ for all x ?

- A $x + 4$ D $4x^2$
 B $4x$ E x^4
 C $2x^2$

Spiral Review

Graph each function. (Lesson 7-1)

54. $y = 2(3)^x$

55. $y = 5(2)^x$

56. $y = 4\left(\frac{1}{3}\right)^x$

Solve each equation. (Lesson 6-7)

57. $\sqrt{x+5} - 3 = 0$

58. $\sqrt{3t-5} - 3 = 4$

59. $\sqrt[4]{2x-1} = 2$

60. $\sqrt{x-6} - \sqrt{x} = 3$

61. $\sqrt[3]{5m+2} = 3$

62. $(6n-5)^{\frac{1}{3}} + 3 = -2$

63. $(5x+7)^{\frac{1}{5}} + 3 = 5$

64. $(3x-2)^{\frac{1}{5}} + 6 = 5$

65. $(7x-1)^{\frac{1}{3}} + 4 = 2$

66. **SALES** A salesperson earns \$10 per hour plus a 10% commission on sales. Write a function to describe the salesperson's income. If the salesperson wants to earn \$1000 in a 40-hour week, what should his sales be? (Lesson 6-2)

67. **STATE FAIR** A dairy makes three types of cheese—cheddar, Monterey Jack, and Swiss—and sells the cheese in three booths at the state fair. At the beginning of one day, the first booth received x pounds of each type of cheese. The second booth received y pounds of each type of cheese, and the third booth received z pounds of each type of cheese. By the end of the day, the dairy had sold 131 pounds of cheddar, 291 pounds of Monterey Jack, and 232 pounds of Swiss. The table below shows the percent of the cheese delivered in the morning that was sold at each booth. How many pounds of cheddar cheese did each booth receive in the morning? (Lesson 3-4)

Type	Booth 1	Booth 2	Booth 3
Cheddar	40%	30%	10%
Monterey Jack	40%	90%	80%
Swiss	30%	70%	70%

Skills Review

Find $[g \circ h](x)$ and $[h \circ g](x)$.

68. $h(x) = 2x - 1$
 $g(x) = 3x + 4$

69. $h(x) = x^2 + 2$
 $g(x) = x - 3$

70. $h(x) = x^2 + 1$
 $g(x) = -2x + 1$

71. $h(x) = -5x$
 $g(x) = 3x - 5$

72. $h(x) = x^3$
 $g(x) = x - 2$

73. $h(x) = x + 4$
 $g(x) = |x|$



Then

You found the inverse of a function.

Now

1 Evaluate logarithmic expressions.

2 Graph logarithmic functions.

Why?

Many scientists believe the extinction of the dinosaurs was caused by an asteroid striking Earth. Astronomers use the Palermo scale to classify objects near Earth based on the likelihood of impact. To make comparing several objects easier, the scale was developed using *logarithms*. The Palermo scale value of any object can be found using the equation $PS = \log_{10} R$, where R is the relative risk posed by the object.



New Vocabulary

logarithm
logarithmic function



Common Core State Standards

Content Standards

F.IF.7.e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

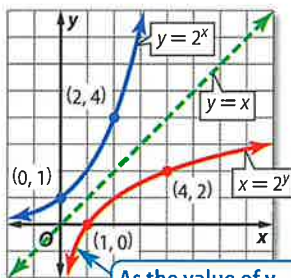
Mathematical Practices

6 Attend to precision.

1 Logarithmic Functions and Expressions Consider the exponential function $f(x) = 2^x$ and its inverse. Recall that you can graph an inverse function by interchanging the x - and y -values in the ordered pairs of the function.

$y = 2^x$	
x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$x = 2^y$	
x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



As the value of y decreases, the value of x approaches 0.

The inverse of $y = 2^x$ can be defined as $x = 2^y$. In general, the inverse of $y = b^x$ is $x = b^y$. In $x = b^y$, the variable y is called the **logarithm** of x . This is usually written as $y = \log_b x$, which is read *y equals log base b of x*.

KeyConcept Logarithm with Base b

Words Let b and x be positive numbers, $b \neq 1$. The *logarithm of x with base b* is denoted $\log_b x$ and is defined as the exponent y that makes the equation $b^y = x$ true.

Symbols Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that

$$\log_b x = y \text{ if and only if } b^y = x.$$

Example If $\log_3 27 = y$, then $3^y = 27$.

The definition of logarithms can be used to express logarithms in exponential form.

Example 1 Logarithmic to Exponential Form

Write each equation in exponential form.

a. $\log_2 8 = 3$

$$\log_2 8 = 3 \rightarrow 8 = 2^3$$

b. $\log_4 \frac{1}{256} = -4$

$$\log_4 \frac{1}{256} = -4 \rightarrow \frac{1}{256} = 4^{-4}$$

Guided Practice

1A. $\log_4 16 = 2$

1B. $\log_3 729 = 6$



The definition of logarithms can also be used to write exponential equations in logarithmic form.



Example 2 Exponential to Logarithmic Form

Write each equation in logarithmic form.

a. $15^3 = 3375$

$$15^3 = 3375 \rightarrow \log_{15} 3375 = 3$$

b. $4^{\frac{1}{2}} = 2$

$$4^{\frac{1}{2}} = 2 \rightarrow \log_4 2 = \frac{1}{2}$$

Guided Practice

2A. $4^3 = 64$

2B. $125^{\frac{1}{3}} = 5$

You can use the definition of a logarithm to evaluate a logarithmic expression.



Example 3 Evaluate Logarithmic Expressions

Evaluate $\log_{16} 4$.

$$\log_{16} 4 = y$$

Let the logarithm equal y .

$$4 = 16^y$$

Definition of logarithm

$$4^1 = (4^2)^y$$

$$16 = 4^2$$

$$4^1 = 4^{2y}$$

Power of a Power

$$1 = 2y$$

Property of Equality for Exponential Functions

$$\frac{1}{2} = y$$

Divide each side by 2.

Thus, $\log_{16} 4 = \frac{1}{2}$.

Guided Practice

Evaluate each expression.

3A. $\log_3 81$

3B. $\log_{\frac{1}{2}} 256$

Watch Out!

Logarithmic Base It is easy to get confused about which number is the base and which is the exponent in logarithmic equations. Consider highlighting each number as you solve to help organize your calculations.

2 Graphing Logarithmic Functions The function $y = \log_b x$, where $b \neq 1$, is called a **logarithmic function**. The graph of $f(x) = \log_b x$ represents a parent graph of the logarithmic functions.



Key Concept Parent Function of Logarithmic Functions

Parent function: $f(x) = \log_b x$

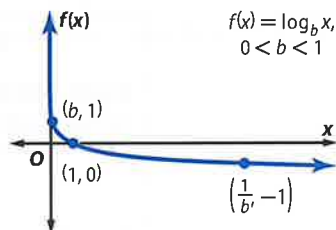
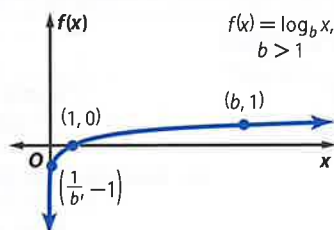
Domain: all positive real numbers

Asymptote: $f(x)$ -axis

Type of graph: continuous, one-to-one

Range: all real numbers

Intercept: $(1, 0)$



Example 4 Graph Logarithmic Functions

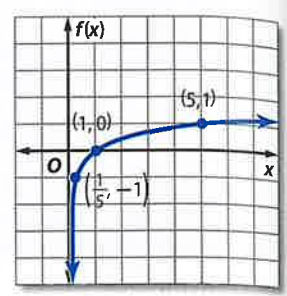
Graph each function.

a. $f(x) = \log_5 x$

Step 1 Identify the base.
 $b = 5$

Step 2 Determine points on the graph.
Because $5 > 1$, use the points $(\frac{1}{b}, -1)$, $(1, 0)$, and $(b, 1)$.

Step 3 Plot the points and sketch the graph.
 $(\frac{1}{b}, -1) \rightarrow (\frac{1}{5}, -1)$
 $(1, 0)$
 $(b, 1) \rightarrow (5, 1)$

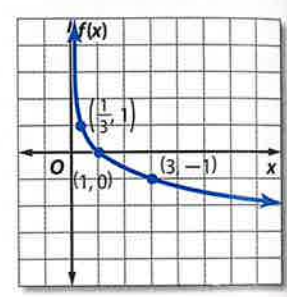


b. $f(x) = \log_{\frac{1}{3}} x$

Step 1 $b = \frac{1}{3}$

Step 2 $0 < \frac{1}{3} < 1$, so use the points $(\frac{1}{3}, 1)$, $(1, 0)$ and $(3, -1)$.

Step 3 Sketch the graph.



StudyTip

Zero Exponent Recall that for any $b \neq 0$, $b^0 = 1$. Therefore, $\log_2 0$ is undefined because $2^x \neq 0$ for any x -value.

GuidedPractice

4A. $f(x) = \log_2 x$

4B. $f(x) = \log_{\frac{1}{8}} x$

The same techniques used to transform the graphs of other functions you have studied can be applied to the graphs of logarithmic functions.

KeyConcept Transformations of Logarithmic Functions

$$f(x) = a \log_b (x - h) + k$$

h – Horizontal Translation	k – Vertical Translation
h units right if h is positive $ h $ units left if h is negative	k units up if k is positive $ k $ units down if k is negative
a – Orientation and Shape	
If $a < 0$, the graph is reflected across the x -axis.	If $ a > 1$, the graph is stretched vertically. If $0 < a < 1$, the graph is compressed vertically.



StudyTip

End Behavior In Example 5a, as x approaches infinity, $f(x)$ approaches infinity.

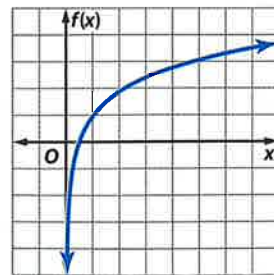
Example 5 Graph Logarithmic Functions

Graph each function.

a. $f(x) = 3 \log_{10} x + 1$

This represents a transformation of the graph of $f(x) = \log_{10} x$.

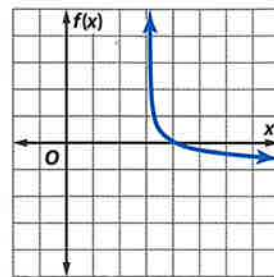
- $|a| = 3$: The graph stretches vertically
- $h = 0$: There is no horizontal shift.
- $k = 1$: The graph is translated 1 unit up.



b. $f(x) = \frac{1}{2} \log_{\frac{1}{4}} (x - 3)$

This is a transformation of the graph of $f(x) = \log_{\frac{1}{4}} x$.

- $|a| = \frac{1}{2}$: The graph is compressed vertically.
- $h = 3$: The graph is translated 3 units to the right.
- $k = 0$: There is no vertical shift.



GuidedPractice

Graph each function.

5A. $f(x) = 2 \log_3 (x - 2)$

5B. $f(x) = \frac{1}{4} \log_{\frac{1}{2}} (x + 1) - 5$

Real-World Example 6 Find Inverses of Exponential Functions



EARTHQUAKES The Richter scale measures earthquake intensity. The increase in intensity between each number is 10 times. For example, an earthquake with a rating of 7 is 10 times more intense than one measuring 6. The intensity of an earthquake can be modeled by $y = 10^{x-1}$, where x is the Richter scale rating.

a. Use the information at the left to find the intensity of the strongest recorded earthquake in the United States.

$$\begin{aligned} y &= 10^{x-1} && \text{Original equation} \\ &= 10^{9.2-1} && \text{Substitute 9.2 for } x. \\ &= 10^{8.2} && \text{Simplify.} \\ &= 158,489,319.2 && \text{Use a calculator.} \end{aligned}$$

b. Write an equation of the form $y = \log_{10} x + c$ for the inverse of the function.

$$\begin{aligned} y &= 10^{x-1} && \text{Original equation} \\ x &= 10^{y-1} && \text{Replace } x \text{ with } y, \text{ replace } y \text{ with } x, \text{ and solve for } y. \\ y - 1 &= \log_{10} x && \text{Definition of logarithm} \\ y &= \log_{10} x + 1 && \text{Add 1 to each side.} \end{aligned}$$

GuidedPractice

6. Write an equation for the inverse of the function $y = 0.5^x$.



Real-WorldLink

The largest recorded earthquake in the United States was a magnitude 9.2 that struck Prince William Sound, Alaska, on Good Friday, March 28, 1964.

Source: United States Geological Survey



Check Your Understanding



= Step-by-Step Solutions begin on page R14.



Example 1 Write each equation in exponential form.

1. $\log_8 512 = 3$

2. $\log_5 625 = 4$

Example 2 Write each equation in logarithmic form.

3. $11^3 = 1331$

4. $16^{\frac{3}{4}} = 8$

Example 3 Evaluate each expression.

5. $\log_{13} 169$

6. $\log_2 \frac{1}{128}$

7. $\log_6 1$

Examples 4-5 Graph each function.

8. $f(x) = \log_3 x$

9. $f(x) = \log_{\frac{1}{6}} x$

10. $f(x) = 4 \log_4 (x - 6)$

11. $f(x) = 2 \log_{\frac{1}{10}} x - 5$

Example 6 12. **SCIENCE** Use the information at the beginning of the lesson. The Palermo scale value of any object can be found using the equation $PS = \log_{10} R$, where R is the relative risk posed by the object. Write an equation in exponential form for the inverse of the function.

Practice and Problem Solving

Extra Practice is on page R7.

Example 1 Write each equation in exponential form.

13. $\log_2 16 = 4$

14. $\log_7 343 = 3$

15. $\log_9 \frac{1}{81} = -2$

16. $\log_3 \frac{1}{27} = -3$

17. $\log_{12} 144 = 2$

18. $\log_9 1 = 0$

Example 2 Write each equation in logarithmic form.

19. $9^{-1} = \frac{1}{9}$

20. $6^{-3} = \frac{1}{216}$

21. $2^8 = 256$

22. $4^6 = 4096$

23. $27^{\frac{2}{3}} = 9$

24. $25^{\frac{3}{2}} = 125$

Example 3 Evaluate each expression.

25. $\log_3 \frac{1}{9}$

26. $\log_4 \frac{1}{64}$

27. $\log_8 512$

28. $\log_6 216$

29. $\log_{27} 3$

30. $\log_{32} 2$

31. $\log_9 3$

32. $\log_{121} 11$

33. $\log_{\frac{1}{5}} 3125$

34. $\log_{\frac{1}{8}} 512$

35. $\log_{\frac{1}{3}} \frac{1}{81}$

36. $\log_{\frac{1}{6}} \frac{1}{216}$

Examples 4-5 **CCSS PRECISION** Graph each function.

37. $f(x) = \log_6 x$

38. $f(x) = \log_{\frac{1}{5}} x$

39. $f(x) = 4 \log_2 x + 6$

40. $f(x) = \log_{\frac{1}{9}} x$

41. $f(x) = \log_{10} x$

42. $f(x) = -3 \log_{\frac{1}{12}} x + 2$

43. $f(x) = 6 \log_{\frac{1}{8}} (x + 2)$

44. $f(x) = -8 \log_3 (x - 4)$

45. $f(x) = \log_{\frac{1}{4}} (x + 1) - 9$

46. $f(x) = \log_5 (x - 4) - 5$

47. $f(x) = \frac{1}{6} \log_8 (x - 3) + 4$

48. $f(x) = -\frac{1}{3} \log_{\frac{1}{6}} (x + 2) - 5$



Example 6

- 49. PHOTOGRAPHY** The formula $n = \log_2 \frac{1}{p}$ represents the change in the f-stop setting n to use in less light where p is the fraction of sunlight.
- Benito's camera is set up to take pictures in direct sunlight, but it is a cloudy day. If the amount of sunlight on a cloudy day is $\frac{1}{4}$ as bright as direct sunlight, how many f-stop settings should he move to accommodate less light?
 - Graph the function.
 - Use the graph in part b to predict what fraction of daylight Benito is accommodating if he moves down 3 f-stop settings. Is he allowing more or less light into the camera?
- 50. EDUCATION** To measure a student's retention of knowledge, the student is tested after a given amount of time. A student's score on an Algebra 2 test t months after the school year is over can be approximated by $y(t) = 85 - 6 \log_2(t + 1)$, where $y(t)$ is the student's score as a percent.
- What was the student's score at the time the school year ended ($t = 0$)?
 - What was the student's score after 3 months?
 - What was the student's score after 15 months?

Graph each function.

51. $f(x) = 4 \log_2(2x - 4) + 6$

52. $f(x) = -3 \log_{12}(4x + 3) + 2$

53. $f(x) = 15 \log_{14}(x + 1) - 9$

54. $f(x) = 10 \log_5(x - 4) - 5$

55. $f(x) = -\frac{1}{6} \log_8(x - 3) + 4$

56. $f(x) = -\frac{1}{3} \log_6(6x + 2) - 5$

- 57. CCSS MODELING** In general, the more money a company spends on advertising, the higher the sales. The amount of money in sales for a company, in thousands, can be modeled by the equation $S(a) = 10 + 20 \log_4(a + 1)$, where a is the amount of money spent on advertising in thousands, when $a \geq 0$.
- The value of $S(0) \approx 10$, which means that if \$10 is spent on advertising, \$10,000 is returned in sales. Find the values of $S(3)$, $S(15)$, and $S(63)$.
 - Interpret the meaning of each function value in the context of the problem.
 - Graph the function.
 - Use the graph in part c and your answers from part a to explain why the money spent in advertising becomes less "efficient" as it is used in larger amounts.
- 58. BIOLOGY** The generation time for bacteria is the time that it takes for the population to double. The generation time G for a specific type of bacteria can be found using experimental data and the formula $G = \frac{t}{3.3 \log_b f}$, where t is the time period, b is the number of bacteria at the beginning of the experiment, and f is the number of bacteria at the end of the experiment.
- The generation time for mycobacterium tuberculosis is 16 hours. How long will it take four of these bacteria to multiply into 1024 bacteria?
 - An experiment involving rats that had been exposed to salmonella showed that the generation time for the salmonella was 5 hours. After how long would 20 of these bacteria multiply into 8000?
 - E. coli* are fast growing bacteria. If 6 *E. coli* can grow to 1296 in 4.4 hours, what is the generation time of *E. coli*?



- 59 FINANCIAL LITERACY** Jacy has spent \$2000 on a credit card. The credit card company charges 24% interest, compounded monthly. The credit card company uses $\log_{\left(1 + \frac{0.24}{12}\right)} \frac{A}{2000} = 12t$ to determine how much time it will be until Jacy's debt reaches a certain amount, if A is the amount of debt after a period of time, and t is time in years.

- Graph the function for Jacy's debt.
- Approximately how long will it take Jacy's debt to double?
- Approximately how long will it be until Jacy's debt triples?

H.O.T. Problems Use Higher-Order Thinking Skills

- 60. ? WRITING IN MATH** What should you consider when using exponential and logarithmic models to make decisions?
- 61. CCSS ARGUMENTS** Consider $y = \log_b x$ in which b , x , and y are real numbers. Zero can be in the domain *sometimes*, *always* or *never*. Justify your answer.
- 62. ERROR ANALYSIS** Betsy says that the graphs of all logarithmic functions cross the y -axis at $(0, 1)$ because any number to the zero power equals 1. Tyrone disagrees. Is either of them correct? Explain your reasoning.
- 63. REASONING** Without using a calculator, compare $\log_7 51$, $\log_8 61$, and $\log_9 71$. Which of these is the greatest? Explain your reasoning.
- 64. OPEN ENDED** Write a logarithmic equation of the form $y = \log_b x$ for each of the following conditions.
- y is equal to 25.
 - y is negative.
 - y is between 0 and 1.
 - x is 1.
 - x is 0.
- 65. ERROR ANALYSIS** Elisa and Matthew are evaluating $\log_{\frac{1}{7}} 49$. Is either of them correct? Explain your reasoning.

Elisa

$$\log_{\frac{1}{7}} 49 = y$$

$$\frac{1^y}{7} = 49$$

$$(7^{-1})^y = 7^2$$

$$(7)^{-y} = 7^2$$

$$y = 2$$

Matthew

$$\log_{\frac{1}{7}} 49 = y$$

$$49^y = \frac{1}{7}$$

$$(7^2)^y = (7)^{-1}$$

$$7^{2y} = (7)^{-1}$$

$$2y = -1$$

$$y = -\frac{1}{2}$$

- 66. WRITING IN MATH** A transformation of $\log_{10} x$ is $g(x) = a \log_{10} (x - h) + k$. Explain the process of graphing this transformation.



Standardized Test Practice

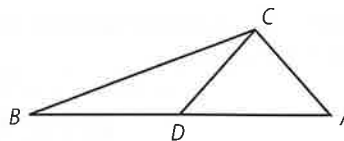
67. A rectangle is twice as long as it is wide. If the width of the rectangle is 3 inches, what is the area of the rectangle in square inches?

- A 9
- B 12
- C 15
- D 18

68. **SAT/ACT** Ichiro has some pizza. He sold 40% more slices than he ate. If he sold 70 slices of pizza, how many did he eat?

- F 25
- G 50
- H 75
- J 98
- K 100

69. **SHORT RESPONSE** In the figure $AB = BC$, $CD = BD$, and $m\angle CAD = 70^\circ$. What is the measure of angle ADC ?



70. If $6x - 3y = 30$ and $4x = 2 - y$ then find $x + y$.

- A -4
- B -2
- C 2
- D 4

Spiral Review

Solve each inequality. Check your solution. (Lesson 7-2)

71. $3^{n-2} > 27$

72. $2^{2n} \leq \frac{1}{16}$

73. $16^n < 8^{n+1}$

74. $32^{5p} + 2 \geq 16^{5p}$

Graph each function. (Lesson 7-1)

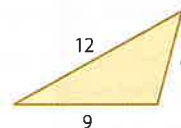
75. $y = -\left(\frac{1}{5}\right)^x$

76. $y = -2.5(5)^x$

77. $y = 30^{-x}$

78. $y = 0.2(5)^{-x}$

79. **GEOMETRY** The area of a triangle with sides of length a , b , and c is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. If the lengths of the sides of a triangle are 6, 9, and 12 feet, what is the area of the triangle expressed in radical form? (Lesson 6-5)



80. **GEOMETRY** The volume of a rectangular box can be written as $6x^3 + 31x^2 + 53x + 30$ when the height is $x + 2$. (Lesson 5-5)

- a. What are the width and length of the box?
- b. Will the ratio of the dimensions of the box always be the same regardless of the value of x ? Explain.

81. **AUTO MECHANICS** Shandra is inventory manager for a local repair shop. She orders 6 batteries, 5 cases of spark plugs, and two dozen pairs of wiper blades and pays \$830. She orders 3 batteries, 7 cases of spark plugs, and four dozen pairs of wiper blades and pays \$820. The batteries are \$22 less than twice the price of a dozen wiper blades. Use augmented matrices to determine what the cost of each item on her order is. (Lesson 3-8)

Skills Review

Solve each equation or inequality. Check your solution.

82. $9^x = \frac{1}{81}$

83. $2^{6x} = 4^{5x+2}$

84. $49^{3p+1} = 7^{2p-5}$

85. $9x^2 \leq 27x^{2-2}$





We can find exponential and logarithmic functions of best fit using a TI-83/84 Plus graphing calculator.

CCSS Common Core State Standards
Mathematical Practices
4 Model with mathematics.



Activity

The population per square mile in the United States has changed dramatically over a period of years. The table shows the number of people per square mile for several years.

- a. Use a graphing calculator to enter the data. Then draw a scatter plot that shows how the number of people per square mile is related to the year.

Step 1 Enter the year into L1 and the people per square mile into L2.

KEYSTROKES: See pages 94 and 95 to review how to enter lists.

Be sure to clear the $Y=$ list. Use the \blacktriangleright key to move the cursor from L1 to L2.

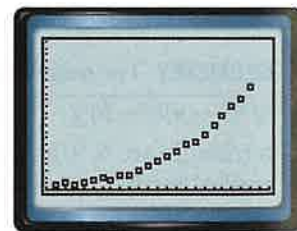
U.S. Population Density			
Year	People per square mile	Year	People per square mile
1790	4.5	1900	21.5
1800	6.1	1910	26.0
1810	4.3	1920	29.9
1820	5.5	1930	34.7
1830	7.4	1940	37.2
1840	9.8	1950	42.6
1850	7.9	1960	50.6
1860	10.6	1970	57.5
1870	10.9	1980	64.0
1880	14.2	1990	70.3
1890	17.8	2000	80.0

Source: Northeast-Midwest Institute

Step 2 Draw the scatter plot.

KEYSTROKES: See pages 94 and 95 to review how to graph a scatter plot.

Make sure that Plot 1 is on, the scatter plot is chosen, Xlist is L1, and Ylist is L2.



[1780, 2020] scl: 10 by [0, 115] scl: 5

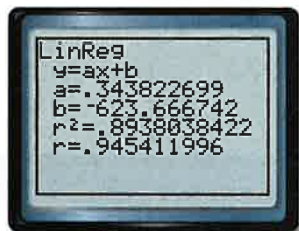
Step 3 Find a regression equation.

To find an equation that best fits the data, use the regression feature of the calculator. Examine various regressions to determine the best model.

Recall that the calculator returns the correlation coefficient r , which is used to indicate how well the model fits the data. The closer r is to 1 or -1 , the better the fit.

Linear regression

KEYSTROKES: $\boxed{\text{STAT}}$ \blacktriangleright 4 $\boxed{\text{ENTER}}$



Quadratic regression

KEYSTROKES: $\boxed{\text{STAT}}$ \blacktriangleright 5 $\boxed{\text{ENTER}}$



$$r^2 = 0.9974003374$$

$$r = \sqrt{0.9974003374}$$

$$r \approx 0.9986993228$$

Exponential regression

KEYSTROKES: **STAT** **▶** **0** **ENTER**



Compare the r -values.

Linear: 0.945411996

Exponential: 0.991887235

Power regression

KEYSTROKES: **STAT** **▶** **ALPHA** **[A]** **ENTER**

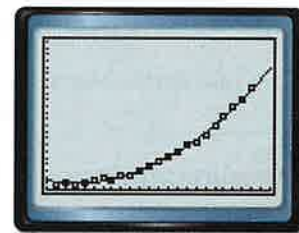


Quadratic: 0.9986993228

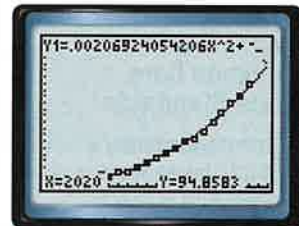
Power: 0.9917543535

The r -value of the quadratic regression is closest to 1, so it appears to model the data best. You can examine the equation visually by graphing the regression equation with the scatter plot.

KEYSTROKES: **STAT** **▶** **5** **ENTER** **Y=** **VAR** **5** **▶** **▶** **1** **GRAPH**



[1780, 2020] scl: 10 by [0, 115] scl: 5



[1780, 2020] scl: 10 by [0, 115] scl: 5

- b. If this trend continues, what will be the population per square mile in 2020?

To determine the population per square mile in 2020, find the value of y when $x = 2020$.

KEYSTROKES: **2nd** **[CALC]** **ENTER** 2020

If this trend continues, there will be approximately 94.9 people per square mile.

Exercises

For Exercises 1–5, Jewel deposited \$50 into an account, then forgot about it and made no further deposits or withdrawals. The table shows the account balance for several years.

- Use a graphing calculator to draw a scatter plot of the data.
- Calculate and graph a curve of fit using an exponential regression.
- Write the equation of best fit. What are the domain and range?
- Based on the model, what will the account balance be after 25 years?
- Is an exponential model the best fit for the data? Explain.
- YOUR TURN** Write a question that can be answered by examining the data of a logarithmic model. First choose a topic and then collect relevant data through Internet research or a survey. Next, make a scatter plot and find a regression equation for your data. Then answer your question.

Elapsed Time (years)	Balance
0	\$50.00
2	\$55.80
4	\$64.80
6	\$83.09
8	\$101.40
10	\$123.14
12	\$162.67

Solving Logarithmic Equations and Inequalities

Then

- You evaluated logarithmic expressions.

Now

- Solve logarithmic equations.
- Solve logarithmic inequalities.

Why?

- Each year the National Weather Service documents about 1000 tornado touchdowns in the United States. The intensity of a tornado is measured on the Fujita scale. Tornadoes are divided into six categories according to their wind speed, path length, path width, and damage caused.

F-Scale	Wind Speed (mph)	Type of Damage
F-0 Gale	40-72	chimneys, branches
F-1 Moderate	73-112	mobile homes overturned
F-2 Significant	113-157	roof torn off
F-3 Severe	158-206	tree uprooted
F-4 Devastating	207-260	homes leveled, cars thrown
F-5 Incredible	261-318	homes thrown
F-6 Inconceivable	319-379	level has never been achieved



New Vocabulary

logarithmic equation
logarithmic inequality



Common Core State Standards

Content Standards

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

Mathematical Practices

4 Model with mathematics.

- Solve Logarithmic Equations** A **logarithmic equation** contains one or more logarithms. You can use the definition of a logarithm to help you solve logarithmic equations.



Example 1 Solve a Logarithmic Equation

$$\text{Solve } \log_{36} x = \frac{3}{2}.$$

$$\log_{36} x = \frac{3}{2} \quad \text{Original equation}$$

$$x = 36^{\frac{3}{2}} \quad \text{Definition of logarithm}$$

$$x = (6^2)^{\frac{3}{2}} \quad 36 = 6^2$$

$$x = 6^3 \text{ or } 216 \quad \text{Power of a Power}$$

Guided Practice

Solve each equation.

1A. $\log_9 x = \frac{3}{2}$

1B. $\log_{16} x = \frac{5}{2}$

Use the following property to solve logarithmic equations that have logarithms with the same base on each side.

KeyConcept Property of Equality for Logarithmic Functions

Symbols If b is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.

Example If $\log_5 x = \log_5 8$, then $x = 8$. If $x = 8$, then $\log_5 x = \log_5 8$.



Standardized Test Example 2 Solve a Logarithmic Equation

Test-Taking Tip

Substitution To save time, you can substitute each answer choice in the original equation to find the one that results in a true statement.

Solve $\log_2(x^2 - 4) = \log_2 3x$.

A -2

B -1

C 2

D 4

Read the Test Item

You need to find x for the logarithmic equation.

Solve the Test Item

$$\log_2(x^2 - 4) = \log_2 3x$$

Original equation

$$x^2 - 4 = 3x$$

Property of Equality for Logarithmic Functions

$$x^2 - 3x - 4 = 0$$

Subtract $3x$ from each side.

$$(x - 4)(x + 1) = 0$$

Factor.

$$x - 4 = 0 \quad \text{or} \quad x + 1 = 0$$

Zero Product Property

$$x = 4 \quad \quad \quad x = -1$$

Solve each equation.

CHECK Substitute each value into the original equation.

$$x = 4$$

$$x = -1$$

$$\log_2(4^2 - 4) \stackrel{?}{=} \log_2 3(4)$$

$$\log_2[(-1)^2 - 4] \stackrel{?}{=} \log_2 3(-1)$$

$$\log_2 12 = \log_2 12 \quad \checkmark$$

$$\log_2(-3) \stackrel{?}{=} \log_2(-3) \quad \times$$

The domain of a logarithmic function cannot be 0, so $\log_2(-3)$ is undefined and -1 is an extraneous solution. The answer is D.

Guided Practice

2. Solve $\log_3(x^2 - 15) = \log_3 2x$.

F -3

G -1

H 5

J 15

2 Solve Logarithmic Inequalities A **logarithmic inequality** is an inequality that involves logarithms. The following property can be used to solve logarithmic inequalities.

Key Concept Property of Inequality for Logarithmic Functions

If $b > 1$, $x > 0$, and $\log_b x > y$, then $x > b^y$.

If $b > 1$, $x > 0$, and $\log_b x < y$, then $0 < x < b^y$.

This property also holds true for \leq and \geq .

Example 3 Solve a Logarithmic Inequality

Solve $\log_3 x > 4$.

$$\log_3 x > 4 \quad \text{Original inequality}$$

$$x > 3^4 \quad \text{Property of Inequality for Logarithmic Functions}$$

$$x > 81 \quad \text{Simplify.}$$

Guided Practice

Solve each inequality.

3A. $\log_4 x \geq 3$

3B. $\log_2 x < 4$

**Math History Link**

Zhang Heng (A.D. 78–139)

The earliest known seismograph was invented by Zhang Heng in China in 132 B.C. It was a large brass vessel with a heavy pendulum and several arms that tripped when an earthquake tremor was felt. This helped determine the direction of the quake.



The following property can be used to solve logarithmic inequalities that have logarithms with the same base on each side. Exclude from your solution set values that would result in taking the logarithm of a number less than or equal to zero in the original inequality.

 **Key Concept** Property of Inequality for Logarithmic Functions

Symbols If $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

Example If $\log_6 x > \log_6 35$, then $x > 35$

This property also holds true for \leq and \geq .

Example 4 Solve Inequalities with Logarithms on Each Side 

Solve $\log_4 (x + 3) > \log_4 (2x + 1)$.

$\log_4 (x + 3) > \log_4 (2x + 1)$ Original inequality

$$x + 3 > 2x + 1$$

Property of Inequality for Logarithmic Functions

$$2 > x$$

Subtract $x + 1$ from each side.

Exclude all values of x for which $x + 3 \leq 0$ or $2x + 1 \leq 0$. So, $x > -3$, $x > -\frac{1}{2}$, and $x < 2$. The solution set is $\left\{x \mid -\frac{1}{2} < x < 2\right\}$ or $\left(-\frac{1}{2}, 2\right)$.

Guided Practice

4. Solve $\log_5 (2x + 1) \leq \log_5 (x + 4)$. Check your solution.

Check Your Understanding

 = Step-by-Step Solutions begin on page R14. 

Example 1 Solve each equation.

1. $\log_8 x = \frac{4}{3}$

2. $\log_{16} x = \frac{3}{4}$

Example 2 3. **MULTIPLE CHOICE** Solve $\log_5 (x^2 - 10) = \log_5 3x$.

A 10

B 2

C 5

D 2, 5

Example 3 Solve each inequality.

4. $\log_5 x > 3$

5. $\log_8 x \leq -2$

6. $\log_4 (2x + 5) \leq \log_4 (4x - 3)$

7. $\log_8 (2x) > \log_8 (6x - 8)$

Practice and Problem Solving

Extra Practice is on page R7.

Examples 1–2  **STRUCTURE** Solve each equation.

8. $\log_{81} x = \frac{3}{4}$

9. $\log_{25} x = \frac{5}{2}$

10. $\log_8 \frac{1}{2} = x$

11. $\log_6 \frac{1}{36} = x$

12. $\log_x 32 = \frac{5}{2}$

13. $\log_x 27 = \frac{3}{2}$

14. $\log_3 (3x + 8) = \log_3 (x^2 + x)$

15. $\log_{12} (x^2 - 7) = \log_{12} (x + 5)$

16. $\log_6 (x^2 - 6x) = \log_6 (-8)$

17. $\log_9 (x^2 - 4x) = \log_9 (3x - 10)$

18. $\log_4 (2x^2 + 1) = \log_4 (10x - 7)$

19. $\log_7 (x^2 - 4) = \log_7 (-x + 2)$



- 37 SOUND** The relationship between the intensity of sound I and the number of decibels β is $\beta = 10 \log_{10} \left(\frac{I}{10^{-12}} \right)$, where I is the intensity of sound in watts per square meter.

- Find the number of decibels of a sound with an intensity of 1 watt per square meter.
- Find the number of decibels of sound with an intensity of 10^{-2} watts per square meter.
- The intensity of the sound of 1 watt per square meter is 100 times as much as the intensity of 10^{-2} watts per square meter. Why are the decibels of sound not 100 times as great?

H.O.T. Problems Use Higher-Order Thinking Skills

- 38. CCSS CRITIQUE** Ryan and Heather are solving $\log_3 x \geq -3$. Is either of them correct? Explain your reasoning.

Ryan

$$\log_3 x \geq -3$$

$$x \geq 3^{-3}$$

$$x \geq \frac{1}{27}$$

Heather

$$\log_3 x \geq -3$$

$$x \geq 3^{-3}$$

$$0 < x \leq \frac{1}{27}$$

- 39. CHALLENGE** Find $\log_3 27 + \log_9 27 + \log_{27} 27 + \log_{81} 27 + \log_{243} 27$.
- 40. REASONING** The Property of Inequality for Logarithmic Functions states that when $b > 1$, $\log_b x > \log_b y$ if and only if $x > y$. What is the case for when $0 < b < 1$? Explain your reasoning.
- 41. WRITING IN MATH** Explain how the domain and range of logarithmic functions are related to the domain and range of exponential functions.
- 42. OPEN ENDED** Give an example of a logarithmic equation that has no solution.
- 43. REASONING** Choose the appropriate term. Explain your reasoning. All logarithmic equations are of the form $y = \log_b x$.
- If the base of a logarithmic equation is greater than 1 and the value of x is between 0 and 1, then the value for y is (*less than, greater than, equal to*) 0.
 - If the base of a logarithmic equation is between 0 and 1 and the value of x is greater than 1, then the value of y is (*less than, greater than, equal to*) 0.
 - There is/are (*no, one, infinitely many*) solution(s) for b in the equation $y = \log_b 0$.
 - There is/are (*no, one, infinitely many*) solution(s) for b in the equation $y = \log_b 1$.
- 44. WRITING IN MATH** Explain why any logarithmic function of the form $y = \log_b x$ has an x -intercept of $(1, 0)$ and no y -intercept.



Standardized Test Practice

45. Find x if $\frac{6.4}{x} = \frac{4}{7}$.

- A 3.4
- B 9.4
- C 11.2
- D 44.8

46. The monthly precipitation in Houston for part of a year is shown.

Month	Precipitation (in.)
April	3.60
May	5.15
June	5.35
July	3.18
August	3.83

Find the median precipitation.

- F 4.25 in.
- G 4.22 in.
- H 3.83 in.
- J 3.60 in.

47. Clara received a 10% raise each year for 3 consecutive years. What was her salary after the three raises if her starting salary was \$12,000 per year?

- A \$14,520
- B \$15,972
- C \$16,248
- D \$16,410

48. **SAT/ACT** A vendor has 14 helium balloons for sale: 9 are yellow, 3 are red, and 2 are green. A balloon is selected at random and sold. If the balloon sold is yellow, what is the probability that the next balloon, selected at random, is also yellow?

- F $\frac{1}{9}$
- G $\frac{1}{8}$
- H $\frac{36}{91}$
- J $\frac{8}{13}$
- K $\frac{9}{14}$

Spiral Review

Evaluate each expression. (Lesson 7-3)

49. $\log_4 256$

50. $\log_2 \frac{1}{8}$

51. $\log_6 216$

52. $\log_3 27$

53. $\log_5 \frac{1}{125}$

54. $\log_7 2401$

Solve each equation or inequality. Check your solution. (Lesson 7-2)

55. $5^{2x+3} \leq 125$

56. $3^{3x-2} > 81$

57. $4^{4a+6} \leq 16^a$

58. $11^{2x+1} = 121^{3x}$

59. $3^{4x-7} = 27^{2x+3}$

60. $8^{x-4} \leq 2^{4-x}$

61. **SHIPPING** The height of a shipping cylinder is 4 feet more than the radius. If the volume of the cylinder is 5π cubic feet, how tall is it? Use the formula $V = \pi \cdot r^2 \cdot h$. (Lesson 5-8)

62. **NUMBER THEORY** Two complex conjugate numbers have a sum of 12 and a product of 40. Find the two numbers. (Lesson 4-4)

Skills Review

Simplify. Assume that no variable equals zero.

63. $x^5 \cdot x^3$

64. $a^2 \cdot a^6$

65. $(2p^2n)^3$

66. $(3b^3c^2)^2$

67. $\frac{x^4y^6}{xy^2}$

68. $\left(\frac{c^9}{d^7}\right)^0$



7 Mid-Chapter Quiz

Lessons 7-1 through 7-4

Graph each function. State the domain and range. (Lesson 7-1)

- $f(x) = 3(4)^x$
- $f(x) = -(2)^x + 5$
- $f(x) = -0.5(3)^{x+2} + 4$
- $f(x) = -3\left(\frac{2}{3}\right)^{x-1} + 8$

5. **SCIENCE** You are studying a bacteria population. The population originally started with 6000 bacteria cells. After 2 hours, there were 28,000 bacteria cells. (Lesson 7-1)
- Write an exponential function that could be used to model the number of bacteria after x hours if the number of bacteria changes at the same rate.
 - How many bacteria cells can be expected after 4 hours?

6. **MULTIPLE CHOICE** Which exponential function has a graph that passes through the points at $(0, 125)$ and $(3, 1000)$? (Lesson 7-1)

- A $f(x) = 125(3)^x$
 B $f(x) = 1000(3)^x$
 C $f(x) = 125(1000)^x$
 D $f(x) = 125(2)^x$

7. **POPULATION** In 1995, a certain city had a population of 45,000. It increased to 68,000 by 2007. (Lesson 7-2)

- What is an exponential function that could be used to model the population of this city x years after 1995?
- Use your model to estimate the population in 2020.

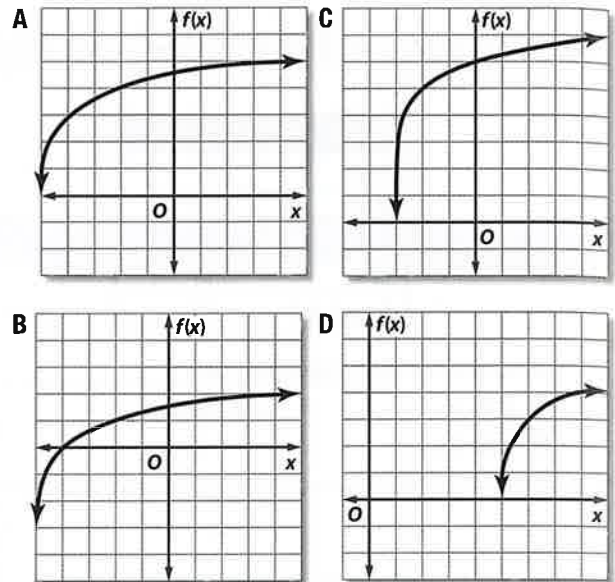
8. **MULTIPLE CHOICE** Find the value of x for $\log_3(x^2 + 2x) = \log_3(x + 2)$. (Lesson 7-3)

- F $x = -2, 1$
 G $x = -2$
 H $x = 1$
 J no solution

Graph each function. (Lesson 7-3)

- $f(x) = 3 \log_2(x - 1)$
- $f(x) = -4 \log_3(x - 2) + 5$

11. **MULTIPLE CHOICE** Which graph below is the graph of the function $f(x) = \log_3(x + 5) + 3$? (Lesson 7-3)



Evaluate each expression. (Lesson 7-3)

- $\log_4 32$
- $\log_5 5^{12}$
- $\log_{16} 4$
- Write $\log_9 729 = 3$ in exponential form. (Lesson 7-3)

Solve each equation or inequality. Check your solution. (Lessons 7-2 and 7-4)

- $3^x = 27^2$
- $4^{3x-1} = 16^x$
- $\frac{1}{9} = 243^{2x+1}$
- $16^{2x+3} < 64$
- $\left(\frac{1}{32}\right)^{x+3} \geq 16^{3x}$
- $\log_4 x = \frac{3}{2}$
- $\log_7(-x + 3) = \log_7(6x + 5)$
- $\log_2 x < -3$
- $\log_8(3x + 7) = \log_8(2x - 5)$

7-5 Properties of Logarithms

Then

You evaluated logarithmic expressions and solved logarithmic equations.

Now

- 1 Simplify and evaluate expressions using the properties of logarithms.
- 2 Solve logarithmic equations using the properties of logarithms.

Why?

The level of acidity in food is important to some consumers with sensitive stomachs. Most of the foods that we consume are more acidic than basic. The pH scale measures acidity; a low pH indicates an acidic solution, and a high pH indicates a basic solution. It is another example of a logarithmic scale based on powers of ten. Black coffee has a pH of 5, while neutral water has a pH of 7. Black coffee is one hundred times as acidic as neutral water, because $10^{7-5} = 10^2$ or 100.

Product	pH Level
lemon juice	2.1
sauerkraut	3.5
tomatoes	4.2
black coffee	5.0
milk	6.4
pure water	7.0
eggs	7.8
milk of magnesia	10.0



Common Core State Standards

Content Standards

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

Mathematical Practices

8 Look for and express regularity in repeated reasoning.

1 Properties of Logarithms Because logarithms are exponents, the properties of logarithms can be derived from the properties of exponents. The Product Property of Logarithms can be derived from the Product of Powers Property of Exponents.

Key Concept Product Property of Logarithms

Words The logarithm of a product is the sum of the logarithms of its factors.

Symbols For all positive numbers a , b , and x , where $x \neq 1$, $\log_x ab = \log_x a + \log_x b$.

Example $\log_2 [(5)(6)] = \log_2 5 + \log_2 6$

To show that this property is true, let $b^x = a$ and $b^y = c$. Then, using the definition of logarithm, $x = \log_b a$ and $y = \log_b c$.

$$b^x b^y = ac \quad \text{Substitution}$$

$$b^{x+y} = ac \quad \text{Product of Powers}$$

$$\log_b b^{x+y} = \log_b ac \quad \text{Property of Equality for Logarithmic Functions}$$

$$x + y = \log_b ac \quad \text{Inverse Property of Exponents and Logarithms}$$

$$\log_b a + \log_b c = \log_b ac \quad \text{Replace } x \text{ with } \log_b a \text{ and } y \text{ with } \log_b c.$$

You can use the Product Property of Logarithms to approximate logarithmic expressions.

Example 1 Use the Product Property



Use $\log_4 3 \approx 0.7925$ to approximate the value of $\log_4 192$.

$$\log_4 192 = \log_4 (4^3 \cdot 3) \quad \text{Replace 192 with } 64 \cdot 3 \text{ or } 4^3 \cdot 3.$$

$$= \log_4 4^3 + \log_4 3 \quad \text{Product Property}$$

$$= 3 + \log_4 3 \quad \text{Inverse Property of Exponents and Logarithms}$$

$$\approx 3 + 0.7925 \text{ or } 3.7925 \quad \text{Replace } \log_4 3 \text{ with } 0.7925.$$

Guided Practice

1. Use $\log_4 2 = 0.5$ to approximate the value of $\log_4 32$.



Recall that the quotient of powers is found by subtracting exponents. The property for the logarithm of a quotient is similar. Let $b^x = a$ and $b^y = c$. Then $\log_b a = x$ and $\log_b c = y$.

$$\frac{b^x}{b^y} = \frac{a}{c}$$

$$b^{x-y} = \frac{a}{c} \quad \text{Quotient Property}$$

$$\log_b b^{x-y} = \log_b \frac{a}{c} \quad \text{Property of Equality for Logarithmic Equations}$$

$$x - y = \log_b \frac{a}{c} \quad \text{Inverse Property of Exponents and Logarithms}$$

$$\log_b a - \log_b c = \log_b \frac{a}{c} \quad \text{Replace } x \text{ with } \log_b a \text{ and } y \text{ with } \log_b c.$$

Key Concept Quotient Property of Logarithms

Words	The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
Symbols	For all positive numbers a , b , and x , where $x \neq 1$, $\log_x \frac{a}{b} = \log_x a - \log_x b$.
Example	$\log_2 \frac{5}{6} = \log_2 5 - \log_2 6$



Real-WorldLink

Acid rain is more acidic than normal rain. Smoke and fumes from burning fossil fuels rise into the atmosphere and combine with the moisture in the air to form acid rain. Acid rain can be responsible for the erosion of statues, as in the photo above.

Real-World Example 2 Quotient Property

SCIENCE The pH of a substance is defined as the concentration of hydrogen ions $[H^+]$ in moles. It is given by the formula $\text{pH} = \log_{10} \frac{1}{H^+}$. Find the amount of hydrogen in a liter of acid rain that has a pH of 4.2.

Understand The formula for finding pH and the pH of the rain is given. You want to find the amount of hydrogen in a liter of this rain.

Plan Write the equation. Then, solve for $[H^+]$.

Solve $\text{pH} = \log_{10} \frac{1}{H^+}$ Original equation

$$4.2 = \log_{10} \frac{1}{H^+}$$
Substitute 4.2 for pH.

$$4.2 = \log_{10} 1 - \log_{10} H^+$$
Quotient Property

$$4.2 = 0 - \log_{10} H^+$$
 $\log_{10} 1 = 0$

$$4.2 = -\log_{10} H^+$$
Simplify.

$$-4.2 = \log_{10} H^+$$
Multiply each side by -1 .

$$10^{-4.2} = H^+$$
Definition of logarithm

There are $10^{-4.2}$, or about 0.000063, mole of hydrogen in a liter of this rain.

Check $4.2 = \log_{10} \frac{1}{H^+}$ pH = 4.2

$$4.2 \stackrel{?}{=} \log_{10} \frac{1}{10^{-4.2}}$$
 $H^+ = 10^{-4.2}$

$$4.2 \stackrel{?}{=} \log_{10} 1 - \log_{10} 10^{-4.2}$$
Quotient Property

$$4.2 \stackrel{?}{=} 0 - (-4.2)$$
Simplify.

$$4.2 = 4.2 \quad \checkmark$$

Guided Practice

2. **SOUND** The loudness L of a sound, measured in decibels, is given by $L = 10 \log_{10} R$, where R is the sound's relative intensity. Suppose one person talks with a relative intensity of 10^6 or 60 decibels. How much louder would 100 people be, talking at the same intensity?



Recall that the power of a power is found by multiplying exponents. The property for the logarithm of a power is similar.

Key Concept Power Property of Logarithms

Words	The logarithm of a power is the product of the logarithm and the exponent.
Symbols	For any real number p , and positive numbers m and b , where $b \neq 1$, $\log_b m^p = p \log_b m$.
Example	$\log_2 6^5 = 5 \log_2 6$

Study Tip

CCSS Tools You can check this answer by evaluating $2^{4.6438}$ on a calculator. The calculator should give a result of about 25, since $\log_2 25 \approx 4.6438$ means $2^{4.6438} \approx 25$.

Example 3 Power Property of Logarithms

Given $\log_2 5 \approx 2.3219$, approximate the value of $\log_2 25$.

$$\begin{aligned} \log_2 25 &= \log_2 5^2 && \text{Replace 25 with } 5^2. \\ &= 2 \log_2 5 && \text{Power Property} \\ &\approx 2(2.3219) \text{ or } 4.6438 && \text{Replace } \log_2 5 \text{ with } 2.3219. \end{aligned}$$

Guided Practice

3. Given $\log_3 7 \approx 1.7712$, approximate the value of $\log_3 49$.

2 Solve Logarithmic Equations

You can use the properties of logarithms to solve equations involving logarithms.

Example 4 Solve Equations Using Properties of Logarithms

Solve $\log_6 x + \log_6 (x - 9) = 2$.

$$\begin{aligned} \log_6 x + \log_6 (x - 9) &= 2 && \text{Original equation} \\ \log_6 x(x - 9) &= 2 && \text{Product Property} \\ x(x - 9) &= 6^2 && \text{Definition of logarithm} \\ x^2 - 9x - 36 &= 0 && \text{Subtract 36 from each side.} \\ (x - 12)(x + 3) &= 0 && \text{Factor.} \\ x - 12 = 0 \text{ or } x + 3 = 0 &&& \text{Zero Product Property} \\ x = 12 \quad x = -3 &&& \text{Solve each equation.} \end{aligned}$$

CHECK $\log_6 x + \log_6 (x - 9) = 2$	$\log_6 x + \log_6 (x - 9) = 2$
$\log_6 12 + \log_6 (12 - 9) \stackrel{?}{=} 2$	$\log_6 (-3) + \log_6 (-3 - 9) \stackrel{?}{=} 2$
$\log_6 12 + \log_6 3 \stackrel{?}{=} 2$	$\log_6 (-3) + \log_6 (-12) \stackrel{?}{=} 2$
$\log_6 (12 \cdot 3) \stackrel{?}{=} 2$	
$\log_6 36 \stackrel{?}{=} 2$	
$2 = 2 \checkmark$	

Because $\log_6 (-3)$ and $\log_6 (-12)$ are undefined, -3 is an extraneous solution.

The solution is $x = 12$.

Guided Practice

4A. $2 \log_7 x = \log_7 27 + \log_7 3$

4B. $\log_6 x + \log_6 (x + 5) = 2$





Examples 1–2 Use $\log_4 3 \approx 0.7925$ and $\log_4 5 \approx 1.1610$ to approximate the value of each expression.

1. $\log_4 18$
2. $\log_4 15$
3. $\log_4 \frac{5}{3}$
4. $\log_4 \frac{3}{4}$

Example 2

5. MOUNTAIN CLIMBING As elevation increases, the atmospheric air pressure decreases. The formula for pressure based on elevation is $a = 15,500(5 - \log_{10} P)$, where a is the altitude in meters and P is the pressure in pascals (1 psi \approx 6900 pascals). What is the air pressure at the summit in pascals for each mountain listed in the table at the right?

Mountain	Country	Height (m)
Everest	Nepal/Tibet	8850
Trisuli	India	7074
Bonete	Argentina/Chile	6872
McKinley	United States	6194
Logan	Canada	5959

Example 3

Given $\log_3 5 \approx 1.465$ and $\log_5 7 \approx 1.2091$, approximate the value of each expression.

6. $\log_3 25$
7. $\log_5 49$

Example 4

Solve each equation. Check your solutions.

8. $\log_4 48 - \log_4 n = \log_4 6$
9. $\log_3 2x + \log_3 7 = \log_3 28$
10. $3 \log_2 x = \log_2 8$
11. $\log_{10} a + \log_{10} (a - 6) = 2$

Practice and Problem Solving

Extra Practice is on page R7.

Examples 1–2 Use $\log_4 2 = 0.5$, $\log_4 3 \approx 0.7925$, and $\log_4 5 \approx 1.1610$ to approximate the value of each expression.

12. $\log_4 30$
13. $\log_4 20$
14. $\log_4 \frac{2}{3}$
15. $\log_4 \frac{4}{3}$
16. $\log_4 9$
17. $\log_4 8$

Example 2

18. SCIENCE The magnitude M of an earthquake is measured on the Richter scale using the formula $M = \log_{10} x$, where x is the intensity of the seismic wave causing the ground motion. In 2007, an earthquake near San Francisco registered approximately 5.6 on the Richter scale. The famous San Francisco earthquake of 1906 measured 8.3 in magnitude.

- a. How many times as intense was the 1906 earthquake as the 2007 earthquake?
- b. Richter himself classified the 1906 earthquake as having a magnitude of 8.3. More recent research indicates it was most likely a 7.9. How many times as great was the intensity of Richter's measure of the earthquake?

Example 3

Given $\log_6 8 \approx 1.1606$ and $\log_7 9 \approx 1.1292$, approximate the value of each expression.

19. $\log_6 48$
20. $\log_7 81$
21. $\log_6 512$
22. $\log_7 729$

Example 4

CCSS PERSEVERANCE Solve each equation. Check your solutions.

23. $\log_3 56 - \log_3 n = \log_3 7$
24. $\log_2 (4x) + \log_2 5 = \log_2 40$
25. $5 \log_2 x = \log_2 32$
26. $\log_{10} a + \log_{10} (a + 21) = 2$



27. PROBABILITY In the 1930s, Dr. Frank Benford demonstrated a way to determine whether a set of numbers has been randomly chosen or manually chosen. If the sets of numbers were not randomly chosen, then the Benford formula, $P = \log_{10} \left(1 + \frac{1}{d}\right)$, predicts the probability of a digit d being the first digit of the set. For example, there is a 4.6% probability that the first digit is 9.

- Rewrite the formula to solve for the digit if given the probability.
- Find the digit that has a 9.7% probability of being selected.
- Find the probability that the first digit is 1 ($\log_{10} 2 \approx 0.30103$).

Use $\log_5 3 \approx 0.6826$ and $\log_5 4 \approx 0.8614$ to approximate the value of each expression.

- | | |
|--------------------------|--------------------------|
| 28. $\log_5 40$ | 29. $\log_5 30$ |
| 30. $\log_5 \frac{3}{4}$ | 31. $\log_5 \frac{4}{3}$ |
| 32. $\log_5 9$ | 33. $\log_5 16$ |
| 34. $\log_5 12$ | 35. $\log_5 27$ |

Solve each equation. Check your solutions.

- | | |
|--|--|
| 36. $\log_3 6 + \log_3 x = \log_3 12$ | 37. $\log_4 a + \log_4 8 = \log_4 24$ |
| 38. $\log_{10} 18 - \log_{10} 3x = \log_{10} 2$ | 39. $\log_7 100 - \log_7 (y + 5) = \log_7 10$ |
| 40. $\log_2 n = \frac{1}{3} \log_2 27 + \log_2 36$ | 41. $3 \log_{10} 8 - \frac{1}{2} \log_{10} 36 = \log_{10} x$ |

Solve for n .

- | | |
|---|---|
| 42. $\log_a 6n - 3 \log_a x = \log_a x$ | 43. $2 \log_b 16 + 6 \log_b n = \log_b (x - 2)$ |
|---|---|

Solve each equation. Check your solutions.

- | | |
|---|---|
| 44. $\log_{10} z + \log_{10} (z + 9) = 1$ | 45. $\log_3 (a^2 + 3) + \log_3 3 = 3$ |
| 46. $\log_2 (15b - 15) - \log_2 (-b^2 + 1) = 1$ | 47. $\log_4 (2y + 2) - \log_4 (y - 2) = 1$ |
| 48. $\log_6 0.1 + 2 \log_6 x = \log_6 2 + \log_6 5$ | 49. $\log_7 64 - \log_7 \frac{8}{3} + \log_7 2 = \log_7 4p$ |

50. CCSS REASONING The humpback whale is an endangered species. Suppose there are 5000 humpback whales in existence today, and the population decreases at a rate of 4% per year.

- Write a logarithmic function for the time in years based upon population.
- After how long will the population drop below 1000? Round your answer to the nearest year.

State whether each equation is *true* or *false*.

- | | |
|--|---|
| 51. $\log_8 (x - 3) = \log_8 x - \log_8 3$ | 52. $\log_5 22x = \log_5 22 + \log_5 x$ |
| 53. $\log_{10} 19k = 19 \log_{10} k$ | 54. $\log_2 y^5 = 5 \log_2 y$ |
| 55. $\log_7 \frac{x}{3} = \log_7 x - \log_7 3$ | 56. $\log_4 (z + 2) = \log_4 z + \log_4 2$ |
| 57. $\log_8 p^4 = (\log_8 p)^4$ | 58. $\log_9 \frac{x^2 y^3}{z^4} = 2 \log_9 x + 3 \log_9 y - 4 \log_9 z$ |

59. PARADE An equation for loudness L , in decibels, is $L = 10 \log_{10} R$, where R is the relative intensity of the sound.

- Solve $120 = 10 \log_{10} R$ to find the relative intensity of the Macy's Thanksgiving Day Parade with a loudness of 120 decibels depending on how close you are.
- Some parents with young children want the decibel level lowered to 80. How many times less intense would this be? In other words, find the ratio of their intensities.



60. **FINANCIAL LITERACY** The average American carries a credit card debt of approximately \$8600 with an annual percentage rate (APR) of 18.3%. The formula $m = \frac{b\left(\frac{r}{n}\right)}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$

can be used to compute the monthly payment m that is necessary to pay off a credit card balance b in a given number of years t , where r is the annual percentage rate and n is the number of payments per year.

- a. What monthly payment should be made in order to pay off the debt in exactly three years? What is the total amount paid?

- b. The equation $t = \frac{\log\left(1 - \frac{br}{mn}\right)}{-n \log\left(1 + \frac{r}{n}\right)}$ can be used to

calculate the number of years necessary for a given payment schedule. Copy and complete the table.

- c. Graph the information in the table from part b.

- d. If you could only afford to pay \$100 a month, will you be able to pay off the debt? If so, how long will it take? If not, why not?

- e. What is the minimum monthly payment that will work toward paying off the debt?

Payment (m)	Years (t)
\$50	
\$100	
\$150	
\$200	
\$250	
\$300	

H.O.T. Problems Use Higher-Order Thinking Skills

61. **OPEN ENDED** Write a logarithmic expression for each condition. Then write the expanded expression.

- a. a product and a quotient
 b. a product and a power
 c. a product, a quotient, and a power

62. **CCSS ARGUMENTS** Use the properties of exponents to prove the Power Property of Logarithms.

63. **WRITING IN MATH** Explain why the following are true.

- a. $\log_b 1 = 0$ b. $\log_b b = 1$ c. $\log_b b^x = x$

64. **CHALLENGE** Simplify $\log_{\sqrt{a}}(a^2)$ to find an exact numerical value.

65. **WHICH ONE DOESN'T BELONG?** Find the expression that does not belong. Explain.

$$\log_b 24 = \log_b 2 + \log_b 12$$

$$\log_b 24 = \log_b 20 + \log_b 4$$

$$\log_b 24 = \log_b 8 + \log_b 3$$

$$\log_b 24 = \log_b 4 + \log_b 6$$

66. **REASONING** Use the properties of logarithms to prove that $\log_a \frac{1}{x} = -\log_a x$.

67. **CHALLENGE** Simplify $x^{3 \log_x 2 - \log_x 5}$ to find an exact numerical value.

68. **WRITING IN MATH** Explain how the properties of exponents and logarithms are related. Include examples like the one shown at the beginning of the lesson illustrating the Product Property, but with the Quotient Property and Power Property of Logarithms.



Standardized Test Practice

69. Find the mode of the data.

22, 11, 12, 23, 7, 6, 17, 15, 21, 19

- A 11 C 16
B 15 D There is no mode.

70. **SAT/ACT** What is the effect on the graph of $y = 4x^2$ when the equation is changed to $y = 2x^2$?

- F The graph is rotated 90 degrees about the origin.
G The graph is narrower.
H The graph is wider.
J The graph of $y = 2x^2$ is a reflection of the graph $y = 4x^2$ across the x -axis.
K The graph is unchanged.

71. **SHORT RESPONSE** In $y = 6.5(1.07)^x$, x represents the number of years since 2000, and y represents the approximate number of millions of Americans 7 years of age and older who went camping two or more times that year. Describe how the number of millions of Americans who go camping is changing over time.

72. What are the x -intercepts of the graph of $y = 4x^2 - 3x - 1$?

- A $-\frac{1}{4}$ and $\frac{1}{4}$ C -1 and 1
B -1 and $\frac{1}{4}$ D 1 and $-\frac{1}{4}$

Spiral Review

Solve each equation. Check your solutions. (Lesson 7-4)

73. $\log_5(3x - 1) = \log_5(2x^2)$

74. $\log_{10}(x^2 + 1) = 1$

75. $\log_{10}(x^2 - 10x) = \log_{10}(-21)$

Evaluate each expression. (Lesson 7-3)

76. $\log_{10} 0.001$

77. $\log_4 16^x$

78. $\log_3 27^x$

79. **ELECTRICITY** The amount of current in amperes I that an appliance uses can be calculated using the formula $I = \left(\frac{P}{R}\right)^{\frac{1}{2}}$, where P is the power in watts and R is the resistance in ohms. How much current does an appliance use if $P = 120$ watts and $R = 3$ ohms? Round to the nearest tenth. (Lesson 6-6)

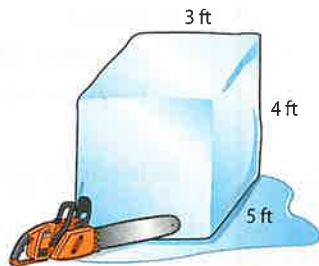
Determine whether each pair of functions are inverse functions. Write *yes* or *no*. (Lesson 6-2)

80. $f(x) = x + 73$
 $g(x) = x - 73$

81. $g(x) = 7x - 11$
 $h(x) = \frac{1}{7}x + 11$

82. **SCULPTING** Antonio is preparing to make an ice sculpture. He has a block of ice that he wants to reduce in size by shaving off the same amount from the length, width, and height. He wants to reduce the volume of the ice block to 24 cubic feet. (Lesson 5-7)

- a. Write a polynomial equation to model this situation.
b. How much should he take from each dimension?



Skills Review

Solve each equation or inequality. Check your solution.

83. $3^{4x} = 3^{3-x}$

84. $3^{2n} \leq \frac{1}{9}$

85. $3^{5x} \cdot 81^{1-x} = 9^{x-3}$

86. $49^x = 7^{x^2-15}$

87. $\log_2(x+6) > 5$

88. $\log_5(4x-1) = \log_5(3x+2)$



Then

- You simplified expressions and solved equations using properties of logarithms.

Now

- Solve exponential equations and inequalities using common logarithms.
- Evaluate logarithmic expressions using the Change of Base Formula.

Why?

- Seismologists use the Richter scale to measure the strength or magnitude of earthquakes. The magnitude of an earthquake is determined using the logarithm of the amplitude of waves recorded by seismographs.

The logarithmic scale used by the Richter scale is based on the powers of 10. For example, a magnitude 6.4 earthquake can be represented by $6.4 = \log_{10} x$.

Richter Number	Intensity
1	10^1 micro
2	10^2 minor
3	10^3 minor
4	10^4 light
5	10^5 moderate
6	10^6 strong
7	10^7 major
8	10^8 great



New Vocabulary
common logarithm
Change of Base Formula



Common Core State Standards

Content Standards
A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

Mathematical Practices
4 Model with mathematics.

1 Common Logarithms You have seen that the base 10 logarithm function, $y = \log_{10} x$, is used in many applications. Base 10 logarithms are called **common logarithms**. Common logarithms are usually written without the subscript 10.

$$\log_{10} x = \log x, x > 0$$

Most scientific calculators have a **LOG** key for evaluating common logarithms.



Example 1 Find Common Logarithms

Use a calculator to evaluate each expression to the nearest ten-thousandth.

a. $\log 5$

KEYSTROKES: **LOG** 5 **ENTER** .6989700043

$$\log 5 \approx 0.6990$$

b. $\log 0.3$

KEYSTROKES: **LOG** 0.3 **ENTER** -.5228787453

$$\log 0.3 \approx -0.5229$$

Guided Practice

1A. $\log 7$

1B. $\log 0.5$

The common logarithms of numbers that differ by integral powers of ten are closely related. Remember that a logarithm is an exponent. For example, in the equation $y = \log x$, y is the power to which 10 is raised to obtain the value of x .

$$\begin{array}{llll} \log x = y & \rightarrow & \text{means} & \rightarrow & 10^y = x \\ \log 1 = 0 & & \text{since} & & 10^0 = 1 \\ \log 10 = 1 & & \text{since} & & 10^1 = 10 \\ \log 10^m = m & & \text{since} & & 10^m = 10^m \end{array}$$



Common logarithms are used in the measurement of sound. Soft recorded music is about 36 decibels (dB).

Real-World Example 2 Solve Logarithmic Equations



ROCK CONCERT The loudness L , in decibels, of a sound is $L = 10 \log \frac{I}{m}$, where I is the intensity of the sound and m is the minimum intensity of sound detectable by the human ear. Residents living several miles from a concert venue can hear the music at an intensity of 66.6 decibels. How many times the minimum intensity of sound detectable by the human ear was this sound, if m is defined to be 1?

$$L = 10 \log \frac{I}{m} \quad \text{Original equation}$$

$$66.6 = 10 \log \frac{I}{1} \quad \text{Replace } L \text{ with } 66.6 \text{ and } m \text{ with } 1.$$

$$6.66 = \log I \quad \text{Divide each side by } 10 \text{ and simplify.}$$

$$I = 10^{6.66} \quad \text{Exponential form}$$

$$I \approx 4,570,882 \quad \text{Use a calculator.}$$

The sound heard by the residents was approximately 4,570,000 times the minimum intensity of sound detectable by the human ear.

Guided Practice

2. **EARTHQUAKES** The amount of energy E in ergs that an earthquake releases is related to its Richter scale magnitude M by the equation $\log E = 11.8 + 1.5M$. Use the equation to find the amount of energy released by the 2004 Sumatran earthquake, which measured 9.0 on the Richter scale and led to a tsunami.

If both sides of an exponential equation cannot easily be written as powers of the same base, you can solve by taking the logarithm of each side.

Example 3 Solve Exponential Equations Using Logarithms



Solve $4^x = 19$. Round to the nearest ten-thousandth.

$$4^x = 19 \quad \text{Original equation}$$

$$\log 4^x = \log 19 \quad \text{Property of Equality for Logarithmic Functions}$$

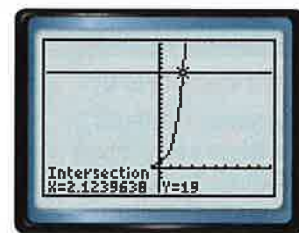
$$x \log 4 = \log 19 \quad \text{Power Property of Logarithms}$$

$$x = \frac{\log 19}{\log 4} \quad \text{Divide each side by } \log 4.$$

$$x \approx 2.1240 \quad \text{Use a calculator.}$$

The solution is approximately 2.1240.

CHECK You can check this answer graphically by using a graphing calculator. Graph the line $y = 4^x$ and the line $y = 19$. Then use the **CALC** menu to find the intersection of the two graphs. The intersection is very close to the answer that was obtained algebraically. ✓



$[-10, 10]$ scl: 1 by $[-5, 25]$ scl: 1

Guided Practice

3A. $3^x = 15$

3B. $6^x = 42$

Real-WorldLink

Acoustical Engineer

Acoustical engineers are concerned with reducing unwanted sounds, noise control, and making useful sounds. Examples of useful sounds are ultrasound, sonar, and sound reproduction. Employment in this field requires a minimum of a bachelor's degree.



The same strategies that are used to solve exponential equations can be used to solve exponential inequalities.

StudyTip

Solving Inequalities

Remember that the direction of an inequality must be switched if each side is multiplied or divided by a negative number. Since $5 \log 3 - \log 7 > 0$, the inequality does not change.

Example 4 Solve Exponential Inequalities Using Logarithms

Solve $3^{5y} < 7^{y-2}$. Round to the nearest ten-thousandth.

$3^{5y} < 7^{y-2}$	Original inequality
$\log 3^{5y} < \log 7^{y-2}$	Property of Inequality for Logarithmic Functions
$5y \log 3 < (y-2) \log 7$	Power Property of Logarithms
$5y \log 3 < y \log 7 - 2 \log 7$	Distributive Property
$5y \log 3 - y \log 7 < -2 \log 7$	Subtract $y \log 7$ from each side.
$y(5 \log 3 - \log 7) < -2 \log 7$	Distributive Property
$y < \frac{-2 \log 7}{5 \log 3 - \log 7}$	Divide each side by $5 \log 3 - \log 7$.
$\{y \mid y < -1.0972\}$	Use a calculator.

CHECK Test $y = -2$.

$3^{5y} < 7^{y-2}$	Original inequality
$3^{5(-2)} \geq 7^{(-2)-2}$	Replace y with -2 .
$3^{-10} \geq 7^{-4}$	Simplify.
$\frac{1}{59,049} < \frac{1}{2401}$ ✓	Negative Exponent Property

GuidedPractice

Solve each inequality. Round to the nearest ten-thousandth.

4A. $3^{2x} \geq 6^{x+1}$

4B. $4^y < 5^{2y+1}$

2 Change of Base Formula

The **Change of Base Formula** allows you to write equivalent logarithmic expressions that have different bases.

KeyConcept Change of Base Formula

Symbols For all positive numbers a , b , and n , where $a \neq 1$ and $b \neq 1$,

$$\log_a n = \frac{\log_b n}{\log_b a}$$

← log base b of original number
← log base b of old base

Example $\log_3 11 = \frac{\log_{10} 11}{\log_{10} 3}$

To prove this formula, let $\log_a n = x$.

$a^x = n$	Definition of logarithm
$\log_b a^x = \log_b n$	Property of Equality for Logarithmic Functions
$x \log_b a = \log_b n$	Power Property of Logarithms
$x = \frac{\log_b n}{\log_b a}$	Divide each side by $\log_b a$.
$\log_a n = \frac{\log_b n}{\log_b a}$	Replace x with $\log_a n$.



Math HistoryLink

John Napier (1550–1617) John Napier was a Scottish mathematician and theologian who began the use of logarithms to aid in calculations. He is also known for popularizing the use of the decimal point.



The Change of Base Formula makes it possible to evaluate a logarithmic expression of any base by translating the expression into one that involves common logarithms.



Example 5 Change of Base Formula

Express $\log_3 20$ in terms of common logarithms. Then round to the nearest ten-thousandth.

$$\log_3 20 = \frac{\log_{10} 20}{\log_{10} 3} \quad \text{Change of Base Formula}$$
$$\approx 2.7268 \quad \text{Use a calculator.}$$

Guided Practice

5. Express $\log_6 8$ in terms of common logarithms. Then round to the nearest ten-thousandth.

Check Your Understanding



= Step-by-Step Solutions begin on page R14.



Example 1 Use a calculator to evaluate each expression to the nearest ten-thousandth.

1. $\log 5$ 2. $\log 21$ 3. $\log 0.4$ 4. $\log 0.7$

Example 2 5. **SCIENCE** The amount of energy E in ergs that an earthquake releases is related to its Richter scale magnitude M by the equation $\log E = 11.8 + 1.5M$. Use the equation to find the amount of energy released by the 1960 Chilean earthquake, which measured 8.5 on the Richter scale.

Example 3 Solve each equation. Round to the nearest ten-thousandth.

6. $6^x = 40$ 7. $2.1^{a+2} = 8.25$ 8. $7^{x^2} = 20.42$ 9. $11^{b-3} = 5^b$

Example 4 Solve each inequality. Round to the nearest ten-thousandth.

10. $5^{4n} > 33$ 11. $6^{p-1} \leq 4^p$

Example 5 Express each logarithm in terms of common logarithms. Then approximate its value to the nearest ten-thousandth.

12. $\log_3 7$ 13. $\log_4 23$ 14. $\log_9 13$ 15. $\log_2 5$

Practice and Problem Solving

Extra Practice is on page R7.

Example 1 Use a calculator to evaluate each expression to the nearest ten-thousandth.

16. $\log 3$ 17. $\log 11$ 18. $\log 3.2$
19. $\log 8.2$ 20. $\log 0.9$ 21. $\log 0.04$

Example 2 22. **CCSS SENSE-MAKING** Loretta had a new muffler installed on her car. The noise level of the engine dropped from 85 decibels to 73 decibels.

- a. How many times the minimum intensity of sound detectable by the human ear was the car with the old muffler, if m is defined to be 1?
b. How many times the minimum intensity of sound detectable by the human ear is the car with the new muffler? Find the percent of decrease of the intensity of the sound with the new muffler.



Example 3 Solve each equation. Round to the nearest ten-thousandth.

23. $8^x = 40$

24. $5^x = 55$

25. $2.9^{a-4} = 8.1$

26. $9^{b-1} = 7^b$

27. $13^{x^2} = 33.3$

28. $15^{x^2} = 110$

Example 4 Solve each inequality. Round to the nearest ten-thousandth.

29. $6^{3n} > 36$

30. $2^{4x} \leq 20$

31. $3^{y-1} \leq 4^y$

32. $5^{p-2} \geq 2^p$

Example 5 Express each logarithm in terms of common logarithms. Then approximate its value to the nearest ten-thousandth.

33. $\log_7 18$

34. $\log_5 31$

35. $\log_2 16$

36. $\log_4 9$

37. $\log_3 11$

38. $\log_6 33$

39 PETS The number n of pet owners in thousands after t years can be modeled by $n = 35[\log_4(t + 2)]$. Let $t = 0$ represent 2000. Use the Change of Base Formula to answer the following questions.

- a. How many pet owners were there in 2010?
- b. How long until there are 80,000 pet owners? When will this occur?

40. CCSS PRECISION Five years ago the grizzly bear population in a certain national park was 325. Today it is 450. Studies show that the park can support a population of 750.

- a. What is the average annual rate of growth in the population if the grizzly bears reproduce once a year?
- b. How many years will it take to reach the maximum population if the population growth continues at the same average rate?

Solve each equation or inequality. Round to the nearest ten-thousandth.

41. $3^x = 40$

42. $5^{3p} = 15$

43. $4^{n+2} = 14.5$

44. $8^z - 4 = 6.3$

45. $7.4^{n-3} = 32.5$

46. $3.1^{y-5} = 9.2$

47. $5^x \geq 42$

48. $9^{2a} < 120$

49. $3^{4x} \leq 72$

50. $7^{2n} > 52^{4n+3}$

51. $6^p \leq 13^{5-p}$

52. $2^{y+3} \geq 8^{3y}$

Express each logarithm in terms of common logarithms. Then approximate its value to the nearest ten-thousandth.

53. $\log_4 12$

54. $\log_3 21$

55. $\log_8 2$

56. $\log_6 7$

57. $\log_5 (2.7)^2$

58. $\log_7 \sqrt{5}$

59. MUSIC A musical cent is a unit in a logarithmic scale of relative pitch or intervals. One octave is equal to 1200 cents. The formula $n = 1200\left(\log_2 \frac{a}{b}\right)$ can be used to determine the difference in cents between two notes with frequencies a and b .

- a. Find the interval in cents when the frequency changes from 443 Hertz (Hz) to 415 Hz.
- b. If the interval is 55 cents and the beginning frequency is 225 Hz, find the final frequency.



Solve each equation. Round to the nearest ten-thousandth.

60. $10^{x^2} = 60$

61. $4^{x^2-3} = 16$

62. $9^{6y-2} = 3^{3y+1}$

63. $8^{2x-4} = 4^{x+1}$

64. $16^x = \sqrt{4^{x+3}}$

65. $2^y = \sqrt{3^{y-1}}$

66. **ENVIRONMENTAL SCIENCE** An environmental engineer is testing drinking water wells in coastal communities for pollution, specifically unsafe levels of arsenic. The safe standard for arsenic is 0.025 parts per million (ppm). Also, the pH of the arsenic level should be less than 9.5. The formula for hydrogen ion concentration is $\text{pH} = -\log H$. (*Hint*: 1 kilogram of water occupies approximately 1 liter. 1 ppm = 1 mg/kg.)

- Suppose the hydrogen ion concentration of a well is 1.25×10^{-11} . Should the environmental engineer be worried about too high an arsenic content?
- The environmental engineer finds 1 milligram of arsenic in a 3 liter sample, is the well safe?
- What is the hydrogen ion concentration that meets the troublesome pH level of 9.5?

67. **MULTIPLE REPRESENTATIONS** In this problem, you will solve the exponential equation $4^x = 13$.

- Tabular** Enter the function $y = 4^x$ into a graphing calculator, create a table of values for the function, and scroll through the table to find x when $y = 13$.
- Graphical** Graph $y = 4^x$ and $y = 13$ on the same screen. Use the **intersect** feature to find the point of intersection.
- Numerical** Solve the equation algebraically. Do all of the methods produce the same result? Explain why or why not.

H.O.T. Problems Use Higher-Order Thinking Skills

68. **CCSS CRITIQUE** Sam and Rosamaria are solving $4^{3p} = 10$. Is either of them correct? Explain your reasoning.

Sam

$$4^{3p} = 10$$

$$\log 4^{3p} = \log 10$$

$$p \log 4 = \log 10$$

$$p = \frac{\log 10}{\log 4}$$

Rosamaria

$$4^{3p} = 10$$

$$\log 4^{3p} = \log 10$$

$$3p \log 4 = \log 10$$

$$p = \frac{\log 10}{3 \log 4}$$

69. **CHALLENGE** Solve $\log_{\sqrt{a}} 3 = \log_a x$ for x and explain each step.

70. **REASONING** Write $\frac{\log_5 9}{\log_5 3}$ as a single logarithm.

71. **PROOF** Find the values of $\log_3 27$ and $\log_{27} 3$. Make and prove a conjecture about the relationship between $\log_a b$ and $\log_b a$.

72. **WRITING IN MATH** Explain how exponents and logarithms are related. Include examples like how to solve a logarithmic equation using exponents and how to solve an exponential equation using logarithms.



Standardized Test Practice

73. Which expression represents $f[g(x)]$ if $f(x) = x^2 + 4x + 3$ and $g(x) = x - 5$?

- A $x^2 + 4x - 2$
- B $x^2 - 6x + 8$
- C $x^2 - 9x + 23$
- D $x^2 - 14x + 6$

74. **EXTENDED RESPONSE** Colleen rented 3 documentaries, 2 video games, and 2 movies. The charge was \$16.29. The next week, she rented 1 documentary, 3 video games, and 4 movies for a total charge of \$19.84. The third week she rented 2 documentaries, 1 video game, and 1 movie for a total charge of \$9.14.

- a. Write a system of equations to determine the cost to rent each item.
- b. What is the cost to rent each item?

75. **GEOMETRY** If the surface area of a cube is increased by a factor of 9, what is the change in the length of the sides of the cube?

- F The length is 2 times the original length.
- G The length is 3 times the original length.
- H The length is 6 times the original length.
- J The length is 9 times the original length.

76. **SAT/ACT** Which of the following *most* accurately describes the translation of the graph $y = (x + 4)^2 - 3$ to the graph of $y = (x - 1)^2 + 3$?

- A down 1 and to the right 3
- B down 6 and to the left 5
- C up 1 and to the left 3
- D up 1 and to the right 3
- E up 6 and to the right 5

Spiral Review

Solve each equation. Check your solutions. (Lesson 7-5)

77. $\log_5 7 + \frac{1}{2}\log_5 4 = \log_5 x$

78. $2 \log_2 x - \log_2 (x + 3) = 2$

79. $\log_6 48 - \log_6 \frac{16}{5} + \log_6 5 = \log_6 5x$

80. $\log_{10} a + \log_{10} (a + 21) = 2$

Solve each equation or inequality. (Lesson 7-4)

81. $\log_4 x = \frac{1}{2}$

82. $\log_{81} 729 = x$

83. $\log_8 (x^2 + x) = \log_8 12$

84. $\log_8 (3y - 1) < \log_8 (y + 5)$

85. **SAILING** The area of a triangular sail is $16x^4 - 60x^3 - 28x^2 + 56x - 32$ square meters. The base of the triangle is $x - 4$ meters. What is the height of the sail? (Lesson 5-2)

86. **HOME REPAIR** Mr. Turner is getting new locks installed. The locksmith charges \$85 for the service call, \$25 for each door, and \$30 for each lock. (Lesson 2-4)

- a. Write an equation that represents the cost for x number of doors.
- b. Mr. Turner wants the front, side, back, and garage door locks changed. How much will this cost?

Skills Review

Write an equivalent exponential equation.

87. $\log_2 5 = x$

88. $\log_4 x = 3$

89. $\log_5 25 = 2$

90. $\log_7 10 = x$

91. $\log_6 x = 4$

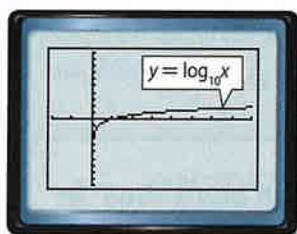
92. $\log_4 64 = 3$



7-6 Graphing Technology Lab Solving Logarithmic Equations and Inequalities



You have solved logarithmic equations algebraically. You can also solve logarithmic equations by graphing or by using a table. The TI-83/84 Plus has $y = \log_{10} x$ as a built-in function. Enter $\boxed{Y=}$ $\boxed{\text{LOG}}$ $\boxed{X,T,\theta,n}$ $\boxed{\text{GRAPH}}$ to view this graph. To graph logarithmic functions with bases other than 10, you must use the Change of Base Formula, $\log_a n = \frac{\log_b n}{\log_b a}$.



$[-2, 8]$ scl: 1 by $[-10, 10]$ scl: 1

CCSS Common Core State Standards Content Standards

A.REI.11 Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Activity 1

Solve $\log_2(6x - 8) = \log_3(20x + 1)$.

Step 1 Graph each side of the equation.

Graph each side of the equation as a separate function. Enter $\log_2(6x - 8)$ as Y_1 and $\log_3(20x + 1)$ as Y_2 . Then graph the two equations.

KEYSTROKES: $\boxed{Y=}$ $\boxed{\text{LOG}}$ $\boxed{6}$ $\boxed{X,T,\theta,n}$ $\boxed{-}$ $\boxed{8}$ $\boxed{)}$ $\boxed{\div}$ $\boxed{\text{LOG}}$ $\boxed{2}$ $\boxed{)}$
 $\boxed{\text{ENTER}}$ $\boxed{\text{LOG}}$ $\boxed{20}$ $\boxed{X,T,\theta,n}$ $\boxed{+}$ $\boxed{1}$ $\boxed{)}$ $\boxed{\div}$ $\boxed{\text{LOG}}$ $\boxed{3}$ $\boxed{)}$ $\boxed{\text{GRAPH}}$

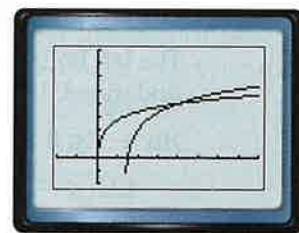
Step 2 Use the intersect feature.

Use the **intersect** feature on the **CALC** menu to approximate the ordered pair of the point at which the curves intersect.

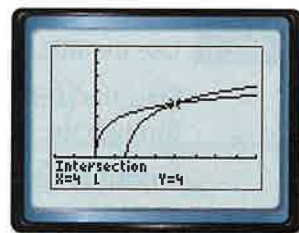
The calculator screen shows that the x -coordinate of the point at which the curves intersect is 4. Therefore, the solution of the equation is 4.

Step 3 Use the TABLE feature.

Examine the table to find the x -value for which the y -values for the graphs are equal. At $x = 4$, both functions have a y -value of 4. Thus, the solution of the equation is 4.



$[-2, 8]$ scl: 1 by $[-2, 8]$ scl: 1



$[-2, 8]$ scl: 1 by $[-2, 8]$ scl: 1

X	Y1	Y2
0	ERR	0
1	ERR	2.7712
2	3.3802	3.3802
3	3.2219	3.7419
4	4	4
5	4.4584	4.2009
6	4.8074	4.2653

You can use a similar procedure to solve logarithmic inequalities using a graphing calculator.

(continued on the next page)

Graphing Technology Lab

Solving Logarithmic Equations and Inequalities *Continued*


Activity 2

Solve $\log_4(10x + 1) < \log_5(16 + 6x)$.

Step 1 Enter the inequalities.

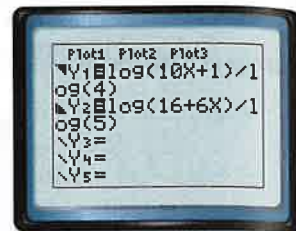
Rewrite the problem as a system of inequalities.

The first inequality is $\log_4(10x + 1) < y$ or $y > \log_4(10x + 1)$. Since this inequality includes the *greater than* symbol, shade above the curve.

First enter the boundary and then use the arrow and **ENTER** keys to choose the shade above icon, .

The second inequality is $y < \log_5(16 + 6x)$. Shade below the curve since this inequality contains *less than*.

KEYSTROKES: **Y=** **←** **←** **ENTER** **ENTER** **→** **→** **LOG** 10 **X,T,θ,n** **+** 1 **)** **÷**
LOG 4 **)** **ENTER** **←** **←** **ENTER** **ENTER** **ENTER** **→** **→** **LOG** 16
+ 6 **X,T,θ,n** **)** **÷** **LOG** 5 **)**



Step 2 Graph the system.

KEYSTROKES: **GRAPH**

The left boundary of the solution set is where the first inequality is undefined. It is undefined for $10x + 1 \leq 0$.

$$10x + 1 \leq 0$$

$$10x \leq -1$$

$$x \leq -\frac{1}{10}$$

Use the calculator's **intersect** feature to find the right boundary. You can conclude that the solution set is $\{x \mid -0.1 < x < 1.5\}$.



$[-2, 4]$ scl: 1 by $[-2, 4]$ scl: 1

Step 3 Use the **TABLE** feature to check your solution.

Start the table at -0.1 and show x -values in increments of 0.1 . Scroll through the table.

KEYSTROKES: **2nd** **[TBLSET]** -0.1 **ENTER** 0.1 **ENTER** **2nd** **[TABLE]**

The table confirms the solution of the inequality is $\{x \mid -0.1 < x < 1.5\}$.

X	Y1	Y2
-0.1	1.7287	1.8206
0.1	1.7525	1.8373
0.2	1.8502	1.9536
0.3	1.8037	1.9684
0.4	1.9534	1.9849
0.5	2	2
0.6	2.0437	2.0147

Exercises

Solve each equation or inequality. Check your solution.

1. $\log_2(3x + 2) = \log_3(12x + 3)$

2. $\log_6(7x + 1) = \log_4(4x - 4)$

3. $\log_2 3x = \log_3(2x + 2)$

4. $\log_{10}(1 - x) = \log_5(2x + 5)$

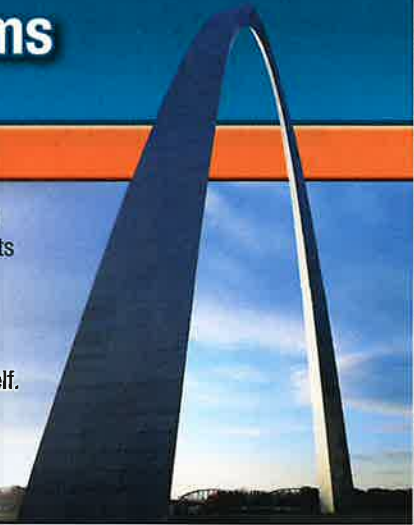
5. $\log_4(9x + 1) > \log_3(18x - 1)$

6. $\log_3(3x - 5) \geq \log_3(x + 7)$

7. $\log_5(2x + 1) < \log_4(3x - 2)$

8. $\log_2 2x \leq \log_4(x + 3)$

7-7 Base e and Natural Logarithms



Then

- You worked with common logarithms.

Now

- Evaluate expressions involving the natural base and natural logarithm.
- Solve exponential equations and inequalities using natural logarithms.

Why?

- The St. Louis Gateway Arch in Missouri is in the form of an inverted catenary curve. A catenary curve directs the force of its weight along itself, so that:
 - if a rope or chain is hanging, it is pulled into that shape, and,
 - if a catenary is standing upright, it can support itself.
 The equation for the catenary curve involves e , a special number that appears throughout mathematics and science.



New Vocabulary

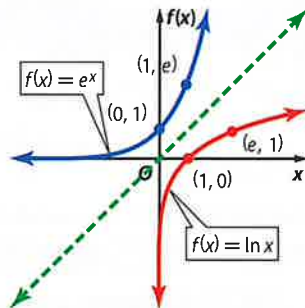
natural base, e
 natural base
 exponential function
 natural logarithm

1 Base e and Natural Logarithms Like π and $\sqrt{2}$, the number e is an irrational number. The value of e is 2.71828... It is referred to as the **natural base, e** . An exponential function with base e is called a **natural base exponential function**.

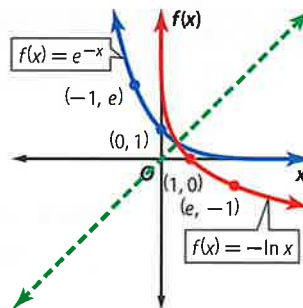
KeyConcept Natural Base Functions

The function $f(x) = e^x$ is used to model continuous exponential growth.
 The function $f(x) = e^{-x}$ is used to model continuous exponential decay.

The inverse of a natural base exponential function is called the **natural logarithm**. This logarithm can be written as $\log_e x$, but is more often abbreviated as $\ln x$.



Exponential Growth



Exponential Decay

You can write an equivalent base e exponential equation for a natural logarithmic equation by using the fact that $\ln x = \log_e x$.

$$\ln 4 = x \rightarrow \log_e 4 = x \rightarrow e^x = 4$$

Example 1 Write Equivalent Expressions

Write each exponential equation in logarithmic form.

a. $e^x = 8$

$$e^x = 8 \rightarrow \log_e 8 = x$$

$$\ln 8 = x$$

b. $e^5 = x$

$$e^5 = x \rightarrow \log_e x = 5$$

$$\ln x = 5$$

Guided Practice

1A. $e^x = 9$

1B. $e^7 = x$



You can also write an equivalent natural logarithm equation for a natural base e exponential equation.

$$e^x = 12 \rightarrow \log_e 12 = x \rightarrow \ln 12 = x$$



Example 2 Write Equivalent Expressions

Write each logarithmic equation in exponential form.

a. $\ln x \approx 0.7741$

$$\ln x \approx 0.7741 \rightarrow \log_e x = 0.7741$$

$$x \approx e^{0.7741}$$

b. $\ln 10 = x$

$$\ln 10 = x \rightarrow \log_e 10 = x$$

$$10 = e^x$$

Guided Practice

2A. $\ln x \approx 2.1438$

2B. $\ln 18 = x$

The properties of logarithms you learned in Lesson 7-5 also apply to the natural logarithms. The logarithmic expressions below can be simplified into a single logarithmic term.



Example 3 Simplify Expressions with e and the Natural Log

Write each expression as a single logarithm.

a. $3 \ln 10 - \ln 8$

$$3 \ln 10 - \ln 8 = \ln 10^3 - \ln 8 \quad \text{Power Property of Logarithms}$$

$$= \ln \frac{10^3}{8} \quad \text{Quotient Property of Logarithms}$$

$$= \ln 125 \quad \text{Simplify.}$$

$$= \ln 5^3 \quad 5^3 = 125$$

$$= 3 \ln 5 \quad \text{Power Property of Logarithms}$$

CHECK Use a calculator to verify the solution.

KEYSTROKES: 3 **LN** 10 **)** **-** **LN** 8 **)** **ENTER** 4.828313737

KEYSTROKES: 3 **LN** 5 **)** **ENTER** 4.828313737 ✓

b. $\ln 40 + 2 \ln \frac{1}{2} + \ln x$

$$\ln 40 + 2 \ln \frac{1}{2} + \ln x = \ln 40 + \ln \frac{1}{4} + \ln x \quad \text{Power Property of Logarithms}$$

$$= \ln \left(40 \cdot \frac{1}{4} \cdot x \right) \quad \text{Product Property of Logarithms}$$

$$= \ln 10x \quad \text{Simplify.}$$

Guided Practice

3A. $6 \ln 8 - 2 \ln 4$

3B. $2 \ln 5 + 4 \ln 2 + \ln 5y$

StudyTip

Simplifying When you simplify logarithmic expressions, verify that the logarithm contains no operations and no powers.

Because the natural base and natural log are inverse functions, they can be used to *undo* or eliminate each other.

$$e^{\ln x} = x$$

$$\ln e^x = x$$



2 Equations and Inequalities with e and \ln Equations and inequalities involving base e are easier to solve by using natural logarithms rather than by using common logarithms, because $\ln e = 1$.



Example 4 Solve Base e Equations

Solve $4e^{-2x} - 5 = 3$. Round to the nearest ten-thousandth.

$4e^{-2x} - 5 = 3$	Original equation
$4e^{-2x} = 8$	Add 5 to each side.
$e^{-2x} = 2$	Divide each side by 4.
$\ln e^{-2x} = \ln 2$	Property of Equality for Logarithms
$-2x = \ln 2$	$\ln e^x = x$
$x = \frac{\ln 2}{-2}$	Divide each side by -2 .
$x \approx -0.3466$	Use a calculator.

KEYSTROKES: $\boxed{\text{LN}} \ 2 \ \boxed{)} \ \boxed{\div} \ -2 \ \boxed{\text{ENTER}} \ -.34657359$

Guided Practice

Solve each equation. Round to the nearest ten-thousandth.

4A. $3e^{4x} - 12 = 15$

4B. $4e^{-x} + 8 = 17$

StudyTip

Calculators Most calculators have an e^x and LN key for evaluating natural base and natural log expressions.

Just like the natural logarithm can be used to eliminate e^x , the natural base exponential function can eliminate $\ln x$.



Example 5 Solve Natural Log Equations and Inequalities

Solve each equation or inequality. Round to the nearest ten-thousandth.

a. $3 \ln 4x = 24$

$3 \ln 4x = 24$	Original equation
$\ln 4x = 8$	Divide each side by 3.
$e^{\ln 4x} = e^8$	Property of Equality for Exponential Functions
$4x = e^8$	$e^{\ln x} = x$
$x = \frac{e^8}{4}$	Divide each side by 4.
$x \approx 745.2395$	Use a calculator.

b. $\ln (x - 8)^4 < 4$

$\ln (x - 8)^4 < 4$	Original equation
$e^{\ln (x - 8)^4} < e^4$	Write each side using exponents and base e .
$(x - 8)^4 < e^4$	$e^{\ln x} = x$
$x - 8 < e$	Property of Equality for Exponential Functions
$x < e + 8$	Add 8 to each side.
$x < 10.7183$	Use a calculator.

Guided Practice

Solve each equation or inequality. Round to the nearest ten-thousandth.

5A. $5 \ln 6x = 8$

5B. $\ln (2x - 3)^3 > 6$



Interest that is compounded continuously can be found using e .

KeyConcept Continuously Compounded Interest

Calculate continuously compounded interest using the following formula:

$$A = Pe^{rt},$$

where A is the amount in the account after t years, P is the principal amount invested, and r is the annual interest rate.



Real-WorldLink

The average cost of tuition, room, and board at four-year public colleges is about \$12,804 per year.

Source: National Center for Education Statistics

Real-World Example 6 Solve Base e Inequalities

FINANCIAL LITERACY When Angelina was born, her grandparents deposited \$3000 into a college savings account paying 4% interest compounded continuously.

- a. Assuming there are no deposits or withdrawals from the account, what will the balance be after 10 years?

$$\begin{aligned} A &= Pe^{rt} && \text{Continuous Compounding Formula} \\ &= 3000e^{(0.04)(10)} && P = 3000, r = 0.04, \text{ and } t = 10 \\ &= 3000e^{0.4} && \text{Simplify.} \\ &\approx 4475.47 && \text{Use a calculator.} \end{aligned}$$

The balance will be \$4475.47.

- b. How long will it take the balance to reach at least \$10,000?

$$\begin{aligned} A &< Pe^{rt} && \text{Continuous Compounding Formula} \\ 10,000 &< 3000e^{(0.04)t} && P = 3000, r = 0.04, \text{ and } A = 10,000 \\ \frac{10}{3} &< e^{0.04t} && \text{Divide each side by 3000.} \\ \ln \frac{10}{3} &< \ln e^{0.04t} && \text{Property of Equality of Logarithms} \\ \ln \frac{10}{3} &< 0.04t && \ln e^x = x \\ \frac{\ln \frac{10}{3}}{0.04} &< t && \text{Divide each side by 0.04.} \\ 30.099 &< t && \text{Use a calculator.} \end{aligned}$$

It will take about 30 years to reach at least \$10,000.

- c. If her grandparents want Angelina to have \$10,000 after 18 years, how much would they need to invest?

$$\begin{aligned} 10,000 &= Pe^{(0.04)18} && A = 10,000, r = 0.04, \text{ and } t = 18 \\ \frac{10,000}{e^{0.72}} &= P && \text{Divide each side by } e^{0.72}. \\ 4867.52 &\approx P && \text{Use a calculator.} \end{aligned}$$

They need to invest \$4867.52.

Guided Practice

6. Use the information in Example 6 to answer the following.
- If they invested \$8000 at 3.75% interest compounded continuously, how much money would be in the account in 30 years?
 - If they could only deposit \$10,000 in the account above, at what rate would the account need to grow in order for Angelina to have \$30,000 in 18 years?
 - If Angelina's grandparents found an account that paid 5% compounded continuously and wanted her to have \$30,000 after 18 years, how much would they need to deposit?

StudyTip

CCSS Reasoning In order to avoid any errors due to rounding, do not round until the very end of your calculations.



Check Your Understanding

 = Step-by-Step Solutions begin on page R14.



Examples 1–2 Write an equivalent exponential or logarithmic function.

- | | |
|---------------|-----------------|
| 1. $e^x = 30$ | 2. $\ln x = 42$ |
| 3. $e^3 = x$ | 4. $\ln 18 = x$ |

Example 3 Write each as a single logarithm.

- | | |
|------------------------|------------------------|
| 5. $3 \ln 2 + 2 \ln 4$ | 6. $5 \ln 3 - 2 \ln 9$ |
| 7. $3 \ln 6 + 2 \ln 9$ | 8. $3 \ln 5 + 4 \ln x$ |

Example 4 Solve each equation. Round to the nearest ten-thousandth.

- | | |
|------------------------|-----------------------|
| 9. $5e^x - 24 = 16$ | 10. $-3e^x + 9 = 4$ |
| 11. $3e^{-3x} + 4 = 6$ | 12. $2e^{-x} - 3 = 8$ |

Example 5 Solve each equation or inequality. Round to the nearest ten-thousandth.

- | | |
|----------------------|-----------------------|
| 13. $\ln 3x = 8$ | 14. $-4 \ln 2x = -26$ |
| 15. $\ln(x+5)^2 < 6$ | 16. $\ln(x-2)^3 > 15$ |
| 17. $e^x > 29$ | 18. $5 + e^{-x} > 14$ |

Example 6 19. **SCIENCE** A virus is spreading through a computer network according to the formula $v(t) = 30e^{0.1t}$, where v is the number of computers infected and t is the time in minutes. How long will it take the virus to infect 10,000 computers?

Practice and Problem Solving

Extra Practice is on page R7.

Examples 1–2 Write an equivalent exponential or logarithmic function.

- | | | | |
|-------------------|---------------------|--------------------|-------------------|
| 20. $e^{-x} = 8$ | 21. $e^{-5x} = 0.1$ | 22. $\ln 0.25 = x$ | 23. $\ln 5.4 = x$ |
| 24. $e^{x-3} = 2$ | 25. $\ln(x+4) = 36$ | 26. $e^{-2} = x^6$ | 27. $\ln e^x = 7$ |

Example 3 Write each as a single logarithm.

- | | | |
|-----------------------------------|----------------------------|---|
| 28. $\ln 125 - 2 \ln 5$ | 29. $3 \ln 10 + 2 \ln 100$ | 30. $4 \ln \frac{1}{3} - 6 \ln \frac{1}{9}$ |
| 31. $7 \ln \frac{1}{2} + 5 \ln 2$ | 32. $8 \ln x - 4 \ln 5$ | 33. $3 \ln x^2 + 4 \ln 3$ |

Example 4 Solve each equation. Round to the nearest ten-thousandth.

- | | | |
|-------------------------|-----------------------|------------------------|
| 34. $6e^x - 3 = 35$ | 35. $4e^x + 2 = 180$ | 36. $3e^{2x} - 5 = -4$ |
| 37. $-2e^{3x} + 19 = 3$ | 38. $6e^{4x} + 7 = 4$ | 39. $-4e^{-x} + 9 = 2$ |

Examples 5–6 40. **CCSS SENSE-MAKING** The value of a certain car depreciates according to $v(t) = 18500e^{-0.186t}$, where t is the number of years after the car is purchased new.

- What will the car be worth in 18 months?
- When will the car be worth half of its original value?
- When will the car be worth less than \$1000?

Solve each inequality. Round to the nearest ten-thousandth.

- | | | |
|----------------------|----------------------|--------------------------------|
| 41. $e^x \leq 8.7$ | 42. $e^x \geq 42.1$ | 43. $\ln(3x+4)^3 > 10$ |
| 44. $4 \ln x^2 < 72$ | 45. $\ln(8x^4) > 24$ | 46. $-2[\ln(x-6)^{-1}] \leq 6$ |



47. FINANCIAL LITERACY Use the formula for continuously compounded interest.

- If you deposited \$800 in an account paying 4.5% interest compounded continuously, how much money would be in the account in 5 years?
- How long would it take you to double your money?
- If you want to double your money in 9 years, what rate would you need?
- If you want to open an account that pays 4.75% interest compounded continuously and have \$10,000 in the account 12 years after your deposit, how much would you need to deposit?

Write the expression as a sum or difference of logarithms or multiples of logarithms.

48. $\ln 12x^2$ 49. $\ln \frac{16}{125}$ 50. $\ln \sqrt[5]{x^3}$ 51. $\ln xy^4z^{-3}$

Use the natural logarithm to solve each equation.

52. $8^x = 24$ 53. $3^x = 0.4$ 54. $2^{3x} = 18$ 55. $5^{2x} = 38$

56. CCSS MODELING Newton's Law of Cooling, which can be used to determine how fast an object will cool in given surroundings, is represented by $T(t) = T_s + (T_0 - T_s)e^{-kt}$, where T_0 is the initial temperature of the object, T_s is the temperature of the surroundings, t is the time in minutes, and k is a constant value that depends on the type of object.

- If a cup of coffee with an initial temperature of 180° is placed in a room with a temperature of 70° and the coffee cools to 140° after 10 minutes, find the value of k .
- Use this value of k to determine the temperature of the coffee after 20 minutes.
- When will the temperature of the coffee reach 75° ?

57. MULTIPLE REPRESENTATIONS In this problem, you will use $f(x) = e^x$ and $g(x) = \ln x$.

- Graphical** Graph both functions and their axis of symmetry, $y = x$, for $-5 \leq x \leq 5$. Then graph $a(x) = e^{-x}$ on the same graph.
- Analytical** The graphs of $a(x)$ and $f(x)$ are reflections along which axis? What function would be a reflection of $f(x)$ along the other axis?
- Graphical** Determine the two functions that are reflections of $g(x)$. Graph these new functions.
- Verbal** We know that $f(x)$ and $g(x)$ are inverses. Are any of the other functions that we have graphed inverses as well? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

- CHALLENGE** Solve $4^x - 2^{x+1} = 15$ for x .
- PROOF** Prove $\ln ab = \ln a + \ln b$ for natural logarithms.
- REASONING** Determine whether $x > \ln x$ is *sometimes*, *always*, or *never* true. Explain your reasoning.
- OPEN ENDED** Express the value 3 using e^x and the natural log.
- WRITING IN MATH** Explain how the natural log can be used to solve a natural base exponential function.



Standardized Test Practice

63. Given the function $y = 2.34x + 11.33$, which statement best describes the effect of moving the graph down two units?
- A The x -intercept decreases.
 - B The y -intercept decreases.
 - C The x -intercept remains the same.
 - D The y -intercept remains the same.
64. **GRIDDED RESPONSE** Aidan sells wooden picture frames over the Internet. He purchases materials for \$85 and pays \$19.95 for his website. If he charges \$15 each, how many frames will he need to sell in order to make a profit of at least \$270?
65. Solve $|2x - 5| = 17$.
- F $-6, -11$
 - G $-6, 11$
 - H $6, -11$
 - J $6, 11$
66. A local pet store sells rabbit food. The cost of two 5-pound bags is \$7.99. The total cost c of purchasing n bags can be found by—
- A multiplying n by c .
 - B multiplying n by 5.
 - C multiplying n by the cost of 1 bag.
 - D dividing n by c .

Spiral Review

Solve each equation or inequality. Round to the nearest ten-thousandth. (Lesson 7-6)

67. $2^x = 53$

68. $2.3x^2 = 66.6$

69. $3^{4x-7} < 4^{2x+3}$

70. $6^{3y} = 8^{y-1}$

71. $12^{x-5} \geq 9.32$

72. $2.1^{x-5} = 9.32$

73. **SOUND** Use the formula $L = 10 \log_{10} R$, where L is the loudness of a sound and R is the sound's relative intensity. Suppose the sound of one alarm clock is 80 decibels. Find out how much louder 10 alarm clocks would be than one alarm clock. (Lesson 7-5)

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. (Lesson 5-6)

74. $x^3 + 5x^2 + 8x + 4; x + 1$

75. $x^3 + 4x^2 + 7x + 6; x + 2$

76. **CRAFTS** Mrs. Hall is selling crocheted items. She sells large afghans for \$60, baby blankets for \$40, doilies for \$25, and pot holders for \$5. She takes the following number of items to the fair: 12 afghans, 25 baby blankets, 45 doilies, and 50 pot holders. (Lesson 3-6)
- a. Write an inventory matrix for the number of each item and a cost matrix for the price of each item.
 - b. Suppose Mrs. Hall sells all of the items. Find her total income as a matrix.

Skills Review

Solve each equation.

77. $2^{3x+5} = 128$

78. $5^{n-3} = \frac{1}{25}$

79. $\left(\frac{1}{9}\right)^m = 81^{m+4}$

80. $\left(\frac{1}{7}\right)^{y-3} = 343$

81. $10^{x-1} = 100^{2x-3}$

82. $36^{2p} = 216^{p-1}$



7-8 Spreadsheet Lab

Compound Interest



You can use a spreadsheet to organize and display data. A spreadsheet is an easy way to track the amount of interest earned over a period of time.

CCSS Common Core State Standards
Mathematical Practices

5 Use appropriate tools strategically.

Compound interest is earned not only on the original amount, but also on any interest that has been added to the principal.



Activity

Find the total amount of money after 5 years if you deposit \$100 at 7% compounded annually.

Step 1 Label your columns as shown. The period is one year. Enter the starting values and the rate.

Step 2 Each row will be generated using formulas. Enter the formulas as shown.

	A	B	C	D	E
1	End of Period	Principal	Interest	Balance	Rate per Period
2	0			\$100.00	7%
3	=A2+1	=D2	=B3*\$E\$2	=C3+D2	

Step 3 Use the FILL DOWN function to fill 4 additional rows.

	A	B	C	D	E
1	End of Period	Principal	Interest	Balance	Rate per Period
2	0			\$100.00	7%
3	1	\$100.00	\$7.00	\$107.00	
4	2	\$107.00	\$7.49	\$114.49	
5	3	\$114.49	\$8.01	\$122.50	
6	4	\$122.50	\$8.58	\$131.08	
7	5	\$131.08	\$9.18	\$140.26	

If you deposit \$100 at 7% annual interest for 5 years, you will have \$140.26 at the end of the 5 years.

Exercises

Find the total balance for each situation.

- deposit \$500 for 7 years at 5%
- deposit \$1000 for 5 years at 6%
- deposit \$200 for 2 years at 10%
- deposit \$800 for 3 years at 8%
- borrow \$10,000 for 5 years at 5.05%
- borrow \$25,000 for 30 years at 8%

LESSON 7-8 Using Exponential and Logarithmic Functions

Then

Now

Why?

You used exponential growth and decay formulas.

1 Use logarithms to solve problems involving exponential growth and decay.

2 Use logarithms to solve problems involving logistic growth.

The ancient footprints of Acahualinca, discovered in Managua, Nicaragua, are believed to be the oldest human footprints in the world. Using carbon dating, scientists estimate that these footprints are 6000 years old.

New Vocabulary
 rate of continuous growth
 rate of continuous decay
 logistic growth model

Common Core State Standards

Content Standards

F.IF.8.b Use the properties of exponents to interpret expressions for exponential functions.

F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a , c , and d are numbers and the base b is 2, 10, or e ; evaluate the logarithm using technology.

Mathematical Practices

1 Make sense of problems and persevere in solving them.

1 Exponential Growth and Decay Scientists and researchers frequently use alternate forms of the growth and decay formulas that you learned in Lesson 7-1.

Key Concept Exponential Growth and Decay

Exponential Growth	Exponential Decay
Exponential growth can be modeled by the function $f(x) = ae^{kt}$ where a is the initial value, t is time in years, and k is a constant representing the rate of continuous growth .	Exponential decay can be modeled by the function $f(x) = ae^{-kt}$ where a is the initial value, t is time in years, and k is a constant representing the rate of continuous decay .

Real-World Example 1 Exponential Decay

SCIENCE The half-life of a radioactive substance is the time it takes for half of the atoms of the substance to disintegrate. The half-life of Carbon-14 is 5730 years. Determine the value of k and the equation of decay for Carbon-14.

If a is the initial amount of the substance, then the amount y that remains after 5730 years can be represented by $\frac{1}{2}a$ or $0.5a$.

$$y = ae^{-kt} \quad \text{Exponential Decay Formula}$$

$$0.5a = ae^{-k(5730)} \quad y = 0.5a \text{ and } t = 5730$$

$$0.5 = e^{-5730k} \quad \text{Divide each side by } a.$$

$$\ln 0.5 = \ln e^{-5730k} \quad \text{Property of Equality for Logarithmic Functions}$$

$$\ln 0.5 = -5730k \quad \ln e^x = x$$

$$\frac{\ln 0.5}{-5730} = k \quad \text{Divide each side by } -5730.$$

$$0.00012 \approx k \quad \text{Use a calculator.}$$

Thus, the equation for the decay of Carbon-14 is $y = ae^{-0.00012t}$.

Guided Practice

1. The half-life of Plutonium-239 is 24,000 years. Determine the value of k .



Real-WorldLink

The oldest modern human fossil, found in Ethiopia, is approximately 160,000 years old.

Source: National Public Radio

StudyTip

Carbon Dating When given a percent or fraction of decay, use an original amount of 1 for a .

Now that the value of k for Carbon-14 is known, it can be used to date fossils.

Real-World Example 2 Carbon Dating



SCIENCE A paleontologist examining the bones of a prehistoric animal estimates that they contain 2% as much Carbon-14 as they would have contained when the animal was alive.

a. How long ago did the animal live?

Understand The formula for the decay of Carbon-14 is $y = ae^{-0.00012t}$. You want to find out how long ago the animal lived.

Plan Let a be the initial amount of Carbon-14 in the animal's body. The amount y that remains after t years is 2% of a or $0.02a$.

Solve	$y = ae^{-0.00012t}$	Formula for the decay of Carbon-14
	$0.02a = ae^{-0.00012t}$	$y = 0.02a$
	$0.02 = e^{-0.00012t}$	Divide each side by a .
	$\ln 0.02 = \ln e^{-0.00012t}$	Property of Equality for Logarithmic Functions
	$\ln 0.02 = -0.00012t$	$\ln e^x = x$
	$\frac{\ln 0.02}{-0.00012} = t$	Divide each side by -0.00012 .
	$32,600 \approx t$	Use a calculator.

The animal lived about 32,600 years ago.

Check Use the formula to find the amount of a sample remaining after 32,600 years. Use an original amount of 1.

$y = ae^{-0.00012t}$	Original equation
$= 1e^{-0.00012(32,600)}$	$a = 1$ and $t = 32,600$
≈ 0.02 or 2% ✓	Use a calculator.

b. If prior research points to the animal being around 20,000 years old, how much Carbon-14 should be in the animal?

$y = ae^{-0.00012t}$	Formula for the decay of Carbon-14
$= 1e^{-0.00012(20,000)}$	$a = 1$ and $t = 20,000$
$= e^{-2.4}$	Simplify.
$= 0.09$ or 9%	Use a calculator.

GuidedPractice

2. Use the information in Example 2 to answer the following questions.

- A specimen that originally contained 42 milligrams of Carbon-14 now contains 8 milligrams. How old is the fossil?
- A woolly mammoth specimen was thought to be about 12,000 years old. How much Carbon-14 should be in the animal?

The exponential growth equation $y = ae^{kt}$ is identical to the continuously compounded interest formula you learned in Lesson 7-7.

Continuous Compounding

$$A = Pe^{rt}$$

P = initial amount
 A = amount at time t
 r = interest rate

Population Growth

$$y = ae^{kt}$$

a = initial population
 y = population at time t
 k = rate of continuous growth



Real-World Example 3 Continuous Exponential Growth

Problem-Solving Tip

Use a Formula When dealing with population, it is almost always necessary to use an exponential growth or decay formula.

POPULATION In 2007, the population of the state of Georgia was 9.36 million people. In 2000, it was 8.18 million.

a. Determine the value of k , Georgia's relative rate of growth.

$$y = ae^{kt} \quad \text{Formula for continuous exponential growth}$$

$$9.36 = 8.18e^{k(7)} \quad y = 9.36, a = 8.18, \text{ and } t = 2007 - 2000 \text{ or } 7$$

$$\frac{9.36}{8.18} = e^{7k} \quad \text{Divide each side by 8.18.}$$

$$\ln \frac{9.36}{8.18} = \ln e^{7k} \quad \text{Property of Equality for Logarithmic Functions}$$

$$\ln \frac{9.36}{8.18} = 7k \quad \ln e^x = x$$

$$\frac{\ln \frac{9.36}{8.18}}{7} = k \quad \text{Divide each side by 7.}$$

$$0.01925 = k \quad \text{Use a calculator.}$$

Georgia's relative rate of growth is about 0.01925 or about 2%.

b. When will Georgia's population reach 12 million people?

$$y = ae^{kt} \quad \text{Formula for continuous exponential growth}$$

$$12 = 8.18e^{0.01925t} \quad y = 12, a = 8.18, \text{ and } k = 0.01925$$

$$1.4670 = e^{0.01925t} \quad \text{Divide each side by 8.18.}$$

$$\ln 1.4670 = \ln e^{0.01925t} \quad \text{Property of Equality for Logarithmic Functions}$$

$$\ln 1.4670 = 0.01925t \quad \ln e^x = x$$

$$\frac{\ln 1.4670}{0.01925} = t \quad \text{Divide each side by 0.01925.}$$

$$19.907 \approx t \quad \text{Use a calculator.}$$

Georgia's population will reach 12 million people by 2020.

c. Michigan's population in 2000 was 9.9 million and can be modeled by $y = 9.9e^{0.0028t}$. Determine when Georgia's population will surpass Michigan's.

$$8.18e^{0.01925t} > 9.9e^{0.0028t} \quad \text{Formula for exponential growth}$$

$$\ln 8.18e^{0.01925t} > \ln 9.9e^{0.0028t} \quad \text{Property of Inequality for Logarithms}$$

$$\ln 8.18 + \ln e^{0.01925t} > \ln 9.9 + \ln e^{0.0028t} \quad \text{Product Property of Logarithms}$$

$$\ln 8.18 + 0.01925t > \ln 9.9 + 0.0028t \quad \ln e^x = x$$

$$0.01645t > \ln 9.9 - \ln 8.18 \quad \text{Subtract } (0.0028t + \ln 8.18) \text{ from each side.}$$

$$t > \frac{\ln 9.9 - \ln 8.18}{0.01645} \quad \text{Divide each side by 0.01645.}$$

$$t > 11.6 \quad \text{Use a calculator.}$$

Georgia's population will surpass Michigan's during 2012.

Guided Practice

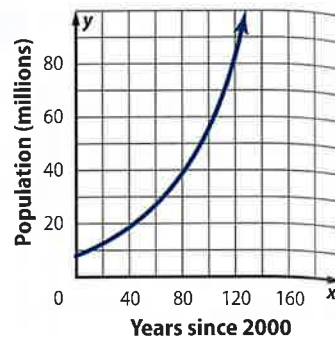
3. **BIOLOGY** A type of bacteria is growing exponentially according to the model $y = 1000e^{kt}$, where t is the time in minutes.

A. If there are 1000 cells initially and 1650 cells after 40 minutes, find the value of k for the bacteria.

B. Suppose a second type of bacteria is growing exponentially according to the model $y = 50e^{0.0432t}$. Determine how long it will be before the number of cells of this bacteria exceed the number of cells in the other bacteria.



2 Logistic Growth Refer to the equation representing Georgia's population in Example 3. According to the graph at the right, Georgia's population will be about one billion by the year 2130. Does this seem logical?

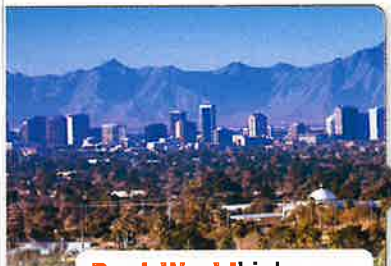


Populations cannot grow infinitely large. There are limitations, such as food supplies, war, living space, diseases, available resources, and so on.

Exponential growth is unrestricted, meaning it will increase without bound. A **logistic growth model**, however, represents growth that has a limiting factor. Logistic models are the most accurate models for representing population growth.

KeyConcept Logistic Growth Function

Let a , b , and c be positive constants where $b < 1$. The logistic growth function is represented by $f(t) = \frac{c}{1 + ae^{-bt}}$, where t represents time.



Real-WorldLink

Phoenix is the fifth largest city in the country and has a population of 1.5 million.

Real-World Example 4 Logistic Growth

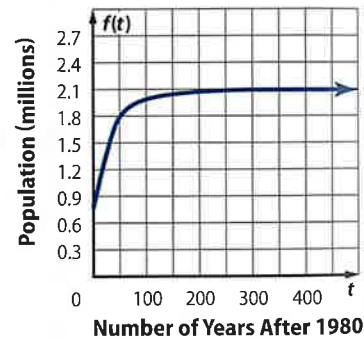
The population of Phoenix, Arizona, in millions can be modeled by the logistic function $f(t) = \frac{2.0666}{1 + 1.66e^{-0.048t}}$, where t is the number of years after 1980.

a. Graph the function for $0 \leq t \leq 500$.

b. What is the horizontal asymptote?
The horizontal asymptote is at $y = 2.0666$.

c. Will the population of Phoenix increase indefinitely? If not, what will be their maximum population?
No. The population will reach a maximum of a little less than 2.0666 million people.

d. According to the function, when will the population of Phoenix reach 1.8 million people?
The graph indicates the population will reach 1.8 million people at $t \approx 50$. Replacing $f(t)$ with 1.8 and solving for t in the equation yields $t = 50.35$ years. So, the population of Phoenix will reach 1.8 million people by 2031.



StudyTip

Intersections To determine where the graph intersects 1.8 on the calculator, graph $y = 1.8$ on the same graph and select *intersection* in the CALC menu.

GuidedPractice

4. The population of a certain species of fish in a lake after t years can be modeled by the function $P(t) = \frac{1880}{1 + 1.42e^{-0.037t}}$, where $t \geq 0$.
 - A. Graph the function for $0 \leq t \leq 500$.
 - B. What is the horizontal asymptote?
 - C. What is the maximum population of the fish in the lake?
 - D. When will the population reach 1875?





- Examples 1–2** 1. **PALEONTOLOGY** The half-life of Potassium-40 is about 1.25 billion years.
- Determine the value of k and the equation of decay for Potassium-40.
 - A specimen currently contains 36 milligrams of Potassium-40. How long will it take the specimen to decay to only 15 milligrams of Potassium-40?
 - How many milligrams of Potassium-40 will be left after 300 million years?
 - How long will it take Potassium-40 to decay to one eighth of its original amount?

- Example 3** 2. **SCIENCE** A certain food is dropped on the floor and is growing bacteria exponentially according to the model $y = 2e^{kt}$, where t is the time in seconds.
- If there are 2 cells initially and 8 cells after 20 seconds, find the value of k for the bacteria.
 - The “5-second rule” says that if a person who drops food on the floor eats it within 5 seconds, there will be no harm. How much bacteria is on the food after 5 seconds?
 - Would you eat food that had been on the floor for 5 seconds? Why or why not? Do you think that the information you obtained in this exercise is reasonable? Explain.

- Example 4** 3. **ZOOLOGY** Suppose the red fox population in a restricted habitat follows the function $P(t) = \frac{16,500}{1 + 18e^{-0.085t}}$, where t represents the time in years.
- Graph the function for $0 \leq t \leq 200$.
 - What is the horizontal asymptote?
 - What is the maximum population?
 - When does the population reach 16,450?

Practice and Problem Solving

Extra Practice is on page R7.

- Examples 1–2** 4. **CCSS PERSEVERANCE** The half-life of Rubidium-87 is about 48.8 billion years.
- Determine the value of k and the equation of decay for Rubidium-87.
 - A specimen currently contains 50 milligrams of Rubidium-87. How long will it take the specimen to decay to only 18 milligrams of Rubidium-87?
 - How many milligrams of Rubidium-87 will be left after 800 million years?
 - How long will it take Rubidium-87 to decay to one-sixteenth its original amount?

- Example 3** 5. **BIOLOGY** A certain bacteria is growing exponentially according to the model $y = 80e^{kt}$, where t is the time in minutes.
- If there are 80 cells initially and 675 cells after 30 minutes, find the value of k for the bacteria.
 - When will the bacteria reach a population of 6000 cells?
 - If a second type of bacteria is growing exponentially according to the model $y = 35e^{0.0978t}$, determine how long it will be before the number of cells of this bacteria exceed the number of cells in the other bacteria.

- Example 4** 6. **FORESTRY** The population of trees in a certain forest follows the function $f(t) = \frac{18000}{1 + 16e^{-0.084t}}$, where t is the time in years.
- Graph the function for $0 \leq t \leq 100$.
 - When does the population reach 17500 trees?



- 7 PALEONTOLOGY** A paleontologist finds a human bone and determines that the Carbon-14 found in the bone is 85% of that found in living bone tissue. How old is the bone?
- 8. ANTHROPOLOGY** An anthropologist has determined that a newly discovered human bone is 8000 years old. How much of the original amount of Carbon-14 is in the bone?
- 9. RADIOACTIVE DECAY** 100 milligrams of Uranium-238 are stored in a container. If Uranium-238 has a half-life of about 4.47 billion years, after how many years will only 10 milligrams be present?
- 10. POPULATION GROWTH** The population of the state of Oregon has grown from 3.4 million in 2000 to 3.7 million in 2006.
- Write an exponential growth equation of the form $y = ae^{kt}$ for Oregon, where t is the number of years after 2000.
 - Use your equation to predict the population of Oregon in 2020.
 - According to the equation, when will Oregon reach 6 million people?
- 11. HALF-LIFE** A substance decays 99.9% of its total mass after 200 years. Determine the half-life of the substance.
- 12. LOGISTIC GROWTH** The population in millions of the state of Ohio after 1900 can be modeled by $P(t) = \frac{12.95}{1 + 2.4e^{-kt}}$, where t is the number of years after 1900 and k is a constant.
- If Ohio had a population of 10 million in 1970, find the value of k .
 - According to the equation, when will the population of Ohio reach 12 million?
- 13. MULTIPLE REPRESENTATIONS** In this problem, you will explore population growth. The population growth of a country follows the exponential function $f(t) = 8e^{0.075t}$ or the logistic function $g(t) = \frac{400}{1 + 16e^{-0.025t}}$. The population is measured in millions and t is time in years.
- Graphical** Graph both functions for $0 \leq t \leq 100$.
 - Analytical** Determine the intersection of the graphs. What is the significance of this intersection?
 - Analytical** Which function is a more accurate estimate of the country's population 100 years from now? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

- 14. OPEN ENDED** Give an example of a quantity that grows or decays at a fixed rate. Write a real-world problem involving the rate and solve by using logarithms.
- 15. CHALLENGE** Solve $\frac{120,000}{1 + 48e^{-0.015t}} = 24e^{0.055t}$ for t .
- 16. CCSS ARGUMENTS** Explain mathematically why $f(t) = \frac{c}{1 + 60e^{-0.5t}}$ approaches, but never reaches the value of c as $t \rightarrow +\infty$.
- 17. OPEN ENDED** Give an example of a quantity that grows logistically and has limitations to growth. Explain why the quantity grows in this manner.
- 18. WRITING IN MATH** How are exponential, continuous exponential, and logistic functions used to model different real-world situations?



Standardized Test Practice

19. Kareem is making a circle graph showing the favorite ice cream flavors of customers at his store. The table summarizes the data. What central angle should Kareem use for the section representing chocolate?

Flavor	Customers
chocolate	35
vanilla	42
strawberry	7
mint chip	12
butter pecan	4

- A 35° C 126°
 B 63° D 150°
20. **PROBABILITY** Lydia has 6 books on her bookshelf. Two are literature books, one is a science book, two are math books, and one is a dictionary. What is the probability that she randomly chooses a science book and the dictionary?
- F $\frac{1}{3}$ H $\frac{1}{12}$
 G $\frac{1}{4}$ J $\frac{1}{15}$

21. **SAT/ACT** Peter has made a game for his daughter's birthday party. The playing board is a circle divided evenly into 8 sectors. If the circle has a radius of 18 inches, what is the approximate area of one of the sectors?

- A 4 in^2 D 127 in^2
 B 14 in^2 E 254 in^2
 C 32 in^2

22. **STATISTICS** In a survey of 90 physical trainers, 15 said they went for a run at least 5 times per week. Of that group, 5 said they also swim during the week, and at least 25% of all trainers run and swim every week. Which conclusion is valid based on the information given?

- F The report is accurate because 15 out of 90 is 25%.
 G The report is accurate because 5 out of 15 is 33%, which is at least 25%.
 H The report is inaccurate because 5 out of 90 is only 5.6%.
 J The report is inaccurate because no one knows if swimming is really exercising.

Spiral Review

Write an equivalent exponential or logarithmic equation. (Lesson 7-7)

23. $e^7 = y$

24. $e^{2n-4} = 36$

25. $\ln 5 + 4 \ln x = 9$

26. **EARTHQUAKES** The table shows the magnitudes of some major earthquakes. (Hint: A magnitude x earthquake has an intensity of 10^x .) (Lessons 7-5 and 7-6)

Year	Location	Magnitude
1963	Yugoslavia	6.0
1970	Peru	7.8
1988	Armenia	7.0
2004	Morocco	6.4
2007	Indonesia	8.4
2010	Haiti	7.0

- a. For which two earthquakes was the intensity of one 10 times that of the other? For which two was the intensity of one 100 times that of the other?
- b. What would be the magnitude of an earthquake that is 1000 times as intense as the 1963 earthquake in Yugoslavia?
- c. Suppose you know that $\log_7 2 \approx 0.3562$ and $\log_7 3 \approx 0.5646$. Describe two different methods that you could use to approximate $\log_7 2.5$. (You may use a calculator, of course.) Then describe how you can check your result.

Skills Review

Solve each equation. Write in simplest form.

27. $\frac{8}{5}x = \frac{4}{15}$

28. $\frac{27}{14}n = \frac{6}{7}$

29. $\frac{3}{10} = \frac{12}{25}a$

30. $\frac{6}{7} = 9p$

31. $\frac{9}{8}b = 18$

32. $\frac{6}{7}y = \frac{3}{4}$

33. $\frac{1}{3}z = \frac{5}{6}$

34. $\frac{2}{3}q = 7$





In this lab, you will explore the type of equation that models the change in the temperature of water as it cools under various conditions.

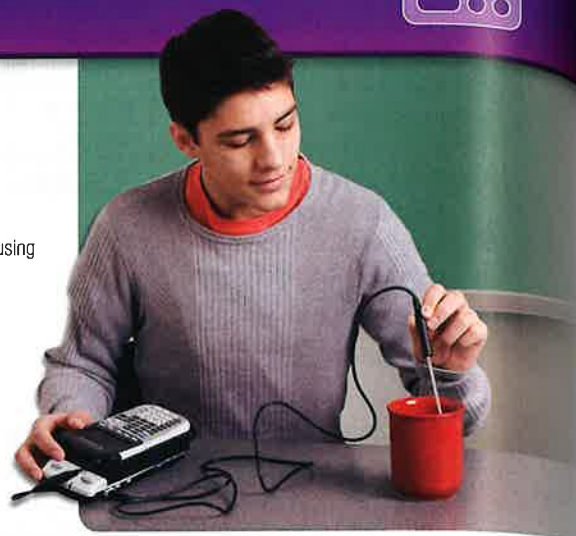


**Common Core State Standards
Content Standards**

F.BF.1.b Combine standard function types using arithmetic operations.

Set Up the Lab

- Collect a variety of containers, such as a foam cup, a ceramic coffee mug, and an insulated cup.
- Boil water or collect hot water from a tap.
- Choose a container to test and fill with hot water. Place the temperature probe in the cup.
- Connect the temperature probe to your data collection device.



Activity

- Step 1** Program the device to collect 20 or more samples in 1 minute intervals.
- Step 2** Wait a few seconds for the probe to warm to the temperature of the water.
- Step 3** Press the button to begin collecting data.

Analyze the Results

1. When the data collection is complete, graph the data in a scatter plot. Use time as the independent variable and temperature as the dependent variable. Write a sentence that describes the points on the graph.
2. Use the **STAT** menu to find an equation to model the data you collected. Try linear, quadratic, and exponential models. Which model appears to fit the data best? Explain.
3. Would you expect the temperature of the water to drop below the temperature of the room? Explain your reasoning.
4. Use the data collection device to find the temperature of the air in the room. Graph the function $y = t$, where t is the temperature of the room, along with the scatter plot and the model equation. Describe the relationship among the graphs. What is the meaning of the relationship in the context of the experiment?

Make a Conjecture

5. Do you think the results of the experiment would change if you used an insulated container for the water? What part of the function will change, the constant or the rate of decay? Repeat the experiment to verify your conjecture.
6. How might the results of the experiment change if you added ice to the water? What part of the function will change, the constant or the rate of decay? Repeat the experiment to verify your conjecture.

Study Guide and Review

Study Guide

Key Concepts

Exponential Functions (Lessons 7-1 and 7-2)

- An exponential function is in the form $y = ab^x$, where $a \neq 0$, $b > 0$ and $b \neq 1$.
- Property of Equality for Exponential Functions: If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.
- Property of Inequality for Exponential Functions: If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Logarithms and Logarithmic Functions

(Lessons 7-3 through 7-6)

- Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.
- The logarithm of a product is the sum of the logarithms of its factors.
- The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
- The logarithm of a power is the product of the logarithm and the exponent.
- The Change of Base Formula: $\log_a n = \frac{\log_b n}{\log_b a}$

Base e and Natural Logarithms (Lesson 7-7)

- Since the natural base function and the natural logarithmic function are inverses, these two can be used to “undo” each other.

Using Exponential and Logarithmic Functions (Lesson 7-8)

- Exponential growth can be modeled by the function $f(x) = ae^{kt}$, where k is a constant representing the rate of continuous growth.
- Exponential decay can be modeled by the function $f(x) = ae^{-kt}$, where k is a constant representing the rate of continuous decay.

FOLDABLES Study Organizer

Be sure the Key Concepts are noted in your Foldable.



Key Vocabulary



- | | |
|---------------------------------|--|
| asymptote (p. 451) | logarithm (p. 468) |
| Change of Base Formula (p. 494) | logarithmic equation (p. 478) |
| common logarithm (p. 492) | logarithmic function (p. 469) |
| compound interest (p. 462) | logarithmic inequality (p. 479) |
| decay factor (p. 454) | logistic growth model (p. 512) |
| exponential decay (p. 453) | natural base, e (p. 501) |
| exponential equation (p. 461) | natural base exponential function (p. 501) |
| exponential function (p. 451) | natural logarithm (p. 501) |
| exponential growth (p. 451) | rate of continuous decay (p. 509) |
| exponential inequality (p. 463) | rate of continuous growth (p. 509) |
| growth factor (p. 453) | |

Vocabulary Check

Choose a word or term from the list above that best completes each statement or phrase.

1. A function of the form $f(x) = b^x$ where $b > 1$ is a(n) _____ function.
2. In $x = b^y$, the variable y is called the _____ of x .
3. Base 10 logarithms are called _____.
4. $A(n)$ _____ is an equation in which variables occur as exponents.
5. The _____ allows you to write equivalent logarithmic expressions that have different bases.
6. The base of the exponential function, $A(t) = a(1 - r)^t$, $1 - r$ is called the _____.
7. The function $y = \log_b x$, where $b > 0$ and $b \neq 1$, is called a(n) _____.
8. An exponential function with base e is called the _____.
9. The logarithm with base e is called the _____.
10. The number e is referred to as the _____.



Lesson-by-Lesson Review

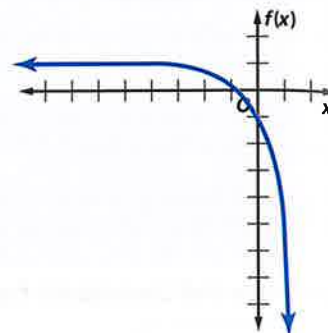
7-1 Graphing Exponential Functions

Graph each function. State the domain and range.

11. $f(x) = 3^x$ 12. $f(x) = -5(2)^x$
 13. $f(x) = 3(4)^x - 6$ 14. $f(x) = 3^{2x} + 5$
 15. $f(x) = 3\left(\frac{1}{4}\right)^{x+3} - 1$ 16. $f(x) = \frac{3}{5}\left(\frac{2}{3}\right)^{x-2} + 3$
 17. **POPULATION** A city with a population of 120,000 decreases at a rate of 3% annually.
 a. Write the function that represents this situation.
 b. What will the population be in 10 years?

Example 1

Graph $f(x) = -2(3)^x + 1$.
 State the domain and range.



The domain is all real numbers, and the range is all real numbers less than 1.

7-2 Solving Exponential Equations and Inequalities

Solve each equation or inequality.

18. $16^x = \frac{1}{64}$ 19. $3^{4x} = 9^{3x+7}$
 20. $64^{3n} = 8^{2n-3}$ 21. $8^3 - 3y = 256^{4y}$
 22. $9^{x-2} > \left(\frac{1}{81}\right)^{x+2}$ 23. $27^{3x} \leq 9^{2x-1}$
 24. **BACTERIA** A bacteria population started with 5000 bacteria. After 8 hours there were 28,000 in the sample.
 a. Write an exponential function that could be used to model the number of bacteria after x hours if the number of bacteria changes at the same rate.
 b. How many bacteria can be expected in the sample after 32 hours?

Example 2

Solve $4^{3x} = 32^{x-1}$ for x .

$$4^{3x} = 32^{x-1} \quad \text{Original equation}$$

$$(2^2)^{3x} = (2^5)^{x-1} \quad \text{Rewrite so each side has the same base.}$$

$$2^{6x} = 2^{5x-5} \quad \text{Power of a Power}$$

$$6x = 5x - 5 \quad \text{Property of Equality for Exponential Functions}$$

$$x = -5 \quad \text{Subtract } 5x \text{ from each side.}$$

The solution is -5 .

7-3 Logarithms and Logarithmic Functions

25. Write $\log_2 \frac{1}{16} = -4$ in exponential form.
 26. Write $10^2 = 100$ in logarithmic form.

Evaluate each expression.

27. $\log_4 256$ 28. $\log_2 \frac{1}{8}$

Graph each function.

29. $f(x) = 2 \log_{10} x + 4$ 30. $f(x) = \frac{1}{6} \log_{\frac{1}{3}} (x - 2)$

Example 3

Evaluate $\log_2 64$.

$$\log_2 64 = y \quad \text{Let the logarithm equal } y.$$

$$64 = 2^y \quad \text{Definition of logarithm}$$

$$2^6 = 2^y \quad 64 = 2^6$$

$$6 = y \quad \text{Property of Equality for Exponential Functions}$$

7-4 Solving Logarithmic Equations and Inequalities

Solve each equation or inequality.

31. $\log_4 x = \frac{3}{2}$ 32. $\log_2 \frac{1}{64} = x$

33. $\log_4 x < 3$ 34. $\log_5 x < -3$

35. $\log_9 (3x - 1) = \log_9 (4x)$

36. $\log_2 (x^2 - 18) = \log_2 (-3x)$

37. $\log_3 (3x + 4) \leq \log_3 (x - 2)$

38. **EARTHQUAKE** The magnitude of an earthquake is measured on a logarithmic scale called the Richter scale. The magnitude M is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion. How many times as great is the amplitude caused by an earthquake with a Richter scale rating of 10 as an aftershock with a Richter scale rating of 7?

Example 4

Solve $\log_{27} x < \frac{2}{3}$.

$\log_{27} x < \frac{2}{3}$ Original inequality

$x < 27^{\frac{2}{3}}$ Logarithmic to Exponential Inequality

$x < 9$ Simplify.

Example 5

Solve $\log_5 (p^2 - 2) = \log_5 p$.

$\log_5 (p^2 - 2) = \log_5 p$ Original equation

$p^2 - 2 = p$ Property of Equality

$p^2 - p - 2 = 0$ Subtract p from each side.

$(p - 2)(p + 1) = 0$ Factor.

$p - 2 = 0$ or $p + 1 = 0$ Zero Product Property

$p = 2$ $p = -1$ Solve each equation.

The solution is $p = 2$, since $\log_5 p$ is undefined for $p = -1$.

7-5 Properties of Logarithms

Use $\log_5 16 \approx 1.7227$ and $\log_5 2 \approx 0.4307$ to approximate the value of each expression.

39. $\log_5 8$ 40. $\log_5 64$

41. $\log_5 4$ 42. $\log_5 \frac{1}{8}$

43. $\log_5 \frac{1}{2}$

Solve each equation. Check your solution.

44. $\log_5 x - \log_5 2 = \log_5 15$

45. $3 \log_4 a = \log_4 27$

46. $2 \log_3 x + \log_3 3 = \log_3 36$

47. $\log_4 n + \log_4 (n - 4) = \log_4 5$

48. **SOUND** Use the formula $L = 10 \log_{10} R$, where L is the loudness of a sound and R is the sound's relative intensity, to find out how much louder 20 people talking would be than one person talking. Suppose the sound of one person talking has a relative intensity of 80 decibels.

Example 6

Use $\log_5 16 \approx 1.7227$ and $\log_5 2 \approx 0.4307$ to approximate $\log_5 32$.

$\log_5 32 = \log_5 (16 \cdot 2)$ Replace 32 with 16.

$= \log_5 16 + \log_5 2$ Product Property

$\approx 1.7227 + 0.4307$ Use a calculator.

≈ 2.1534

Example 7

Solve $\log_3 3x + \log_3 4 = \log_3 36$.

$\log_3 3x + \log_3 4 = \log_3 36$ Original equation

$\log_3 3x(4) = \log_3 36$ Product Property

$3x(4) = 36$ Definition of logarithm

$12x = 36$ Multiply.

$x = 3$ Divide each side by 12.

7-6 Common Logarithms

Solve each equation or inequality. Round to the nearest ten-thousandth.

49. $3^x = 15$ 50. $6^{x^2} = 28$
51. $8^{m+1} = 30$ 52. $12^{r-1} = 7r$
53. $3^{5n} > 24$ 54. $5^{x+2} \leq 3^x$
55. **SAVINGS** You deposited \$1000 into an account that pays an annual interest rate r of 5% compounded quarterly. Use $A = P\left(1 + \frac{r}{n}\right)^{nt}$.
- How long will it take until you have \$1500 in your account?
 - How long it will take for your money to double?

Example 8

Solve $5^{3x} > 7^{x+1}$.

$$5^{3x} > 7^{x+1} \quad \text{Original inequality}$$

$$\log 5^{3x} > \log 7^{x+1} \quad \text{Property of Inequality}$$

$$3x \log 5 > (x+1) \log 7 \quad \text{Power Property}$$

$$3x \log 5 > x \log 7 + \log 7 \quad \text{Distributive Property}$$

$$3x \log 5 - x \log 7 > \log 7 \quad \text{Subtract } x \log 7.$$

$$x(3 \log 5 - \log 7) > \log 7 \quad \text{Distributive Property}$$

$$x > \frac{\log 7}{3 \log 5 - \log 7} \quad \text{Divide by } 3 \log 5 - \log 7.$$

$$x > 0.6751 \quad \text{Use a calculator.}$$

The solution set is $\{x \mid x > 0.6751\}$.

7-7 Base e and Natural Logarithms

Solve each equation or inequality. Round to the nearest ten-thousandth.

56. $4e^x - 11 = 17$ 57. $2e^{-x} + 1 = 15$
58. $\ln 2x = 6$ 59. $2 + e^x > 9$
60. $\ln(x+3)^5 < 5$ 61. $e^{-x} > 18$
62. **SAVINGS** If you deposit \$2000 in an account paying 6.4% interest compounded continuously, how long will it take for your money to triple? Use $A = Pe^{rt}$.

Example 9

Solve $3e^{5x} + 1 = 10$. Round to the nearest ten-thousandth.

$$3e^{5x} + 1 = 10 \quad \text{Original equation}$$

$$3e^{5x} = 9 \quad \text{Subtract 1 from each side.}$$

$$e^{5x} = 3 \quad \text{Divide each side by 3.}$$

$$\ln e^{5x} = \ln 3 \quad \text{Property of Equality}$$

$$5x = \ln 3 \quad \text{In } e^x = x$$

$$x = \frac{\ln 3}{5} \quad \text{Divide each side by 5.}$$

$$x \approx 0.2197 \quad \text{Use a calculator.}$$

7-8 Using Exponential and Logarithmic Functions

63. **CARS** Abe bought a used car for \$2500. It is expected to depreciate at a rate of 25% per year. What will be the value of the car in 3 years?
64. **BIOLOGY** For a certain strain of bacteria, k is 0.728 when t is measured in days. Using the formula $y = ae^{kt}$, how long will it take 10 bacteria to increase to 675 bacteria?
65. **POPULATION** The population of a city 20 years ago was 24,330. Since then, the population has increased at a steady rate each year. If the population is currently 55,250, find the annual rate of growth for this city.

Example 10

A certain culture of bacteria will grow from 250 to 2000 bacteria in 1.5 hours. Find the constant k for the growth formula. Use $y = ae^{kt}$.

$$y = ae^{kt} \quad \text{Exponential Growth Formula}$$

$$2000 = 250e^{k(1.5)} \quad \text{Replace } y \text{ with 2000, } a \text{ with 250, and } t \text{ with 1.5.}$$

$$8 = e^{1.5k} \quad \text{Divide each side by 250.}$$

$$\ln 8 = \ln e^{1.5k} \quad \text{Property of Equality}$$

$$\ln 8 = 1.5k \quad \text{Inverse Property}$$

$$\frac{\ln 8}{1.5} = k \quad \text{Divide each side by 1.5.}$$

$$1.3863 \approx k \quad \text{Use a calculator.}$$

7 Practice Test

Graph each function. State the domain and range.

- $f(x) = 3^{x-3} + 2$
- $f(x) = 2\left(\frac{3}{4}\right)^{x+1} - 3$

Solve each equation or inequality. Round to the nearest ten-thousandth if necessary.

- $8^c + 1 = 16^{2c+3}$
- $9^{x-2} > \left(\frac{1}{27}\right)^x$
- $2^a + 3 = 3^{2a-1}$
- $\log_2(x^2 - 7) = \log_2 6x$
- $\log_5 x > 2$
- $\log_3 x + \log_3(x - 3) = \log_3 4$
- $6^{n-1} \leq 11^n$
- $4e^{2x} - 1 = 5$
- $\ln(x + 2)^2 > 2$

Use $\log_5 11 \approx 1.4899$ and $\log_5 2 \approx 0.4307$ to approximate the value of each expression.

- $\log_5 44$
- $\log_5 \frac{11}{2}$
- 14. POPULATION** The population of a city 10 years ago was 150,000. Since then, the population has increased at a steady rate each year. The population is currently 185,000.
 - Write an exponential function that could be used to model the population after x years if the population changes at the same rate.
 - What will the population be in 25 years?
- Write $\log_9 27 = \frac{3}{2}$ in exponential form.
- 16. AGRICULTURE** An equation that models the decline in the number of U.S. farms is $y = 3,962,520(0.98)^x$, where x is the number of years since 1960 and y is the number of farms.
 - How can you tell that the number is declining?
 - By what annual rate is the number declining?
 - Predict when the number of farms will be less than 1 million.

17. MULTIPLE CHOICE What is the value of $\log_4 \frac{1}{64}$?

- 3
- $-\frac{1}{3}$
- $\frac{1}{3}$
- 3

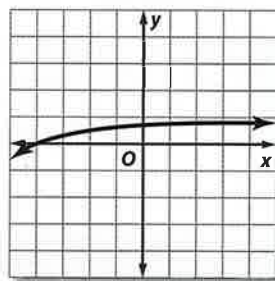
18. SAVINGS You put \$7500 in a savings account paying 3% interest compounded continuously.

- Assuming there are no deposits or withdrawals from the account, what is the balance after 5 years?
- How long will it take your savings to double?
- In how many years will you have \$10,000 in your account?

19. MULTIPLE CHOICE What is the solution of $\log_4 16 - \log_4 x = \log_4 8$?

- $\frac{1}{2}$
- 2
- 4
- 8

20. MULTIPLE CHOICE Which function is graphed below?



- $y = \log_{10}(x - 5)$
- $y = 5 \log_{10} x$
- $y = \log_{10}(x + 5)$
- $y = -5 \log_{10} x$

21. Write $2 \ln 6 + 3 \ln 4 - 5 \ln \left(\frac{1}{3}\right)$ as a single logarithm.



Using Technology

Your calculator can be a useful tool in taking standardized tests. Some problems that you encounter might have steps or computations that require the use of a calculator. A calculator may also help you solve a problem more quickly.

Strategies for Using Technology

Step 1

A calculator is a useful tool, but typically it should be used sparingly. Standardized tests are designed to measure your ability to reason and solve problems, not to measure your ability to punch keys on a calculator.

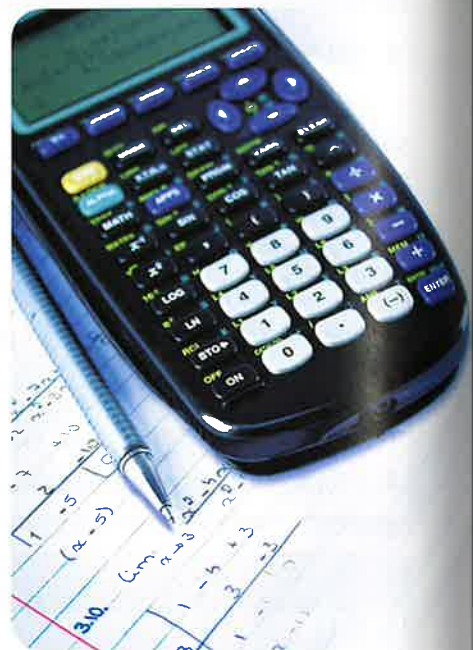
Before using a calculator, ask yourself:

- How would I normally solve this type of problem?
- Are there any steps that I cannot perform mentally or by using paper and pencil?
- Is a calculator absolutely necessary to solve this problem?
- Would a calculator help me solve this problem more quickly or efficiently?

Step 2

When might a calculator come in handy?

- solving problems that involve large, complex computations
- solving certain problems that involve graphing functions, evaluating functions, solving equations, and so on
- checking solutions of problems



Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

A certain can of soda contains 60 milligrams of caffeine. The caffeine is eliminated from the body at a rate of 15% per hour. What is the *half-life* of the caffeine? That is, how many hours does it take for half of the caffeine to be eliminated from the body?

- | | |
|--------------|--------------|
| A 4 hours | C 4.5 hours |
| B 4.25 hours | D 4.75 hours |

Read the problem carefully. The problem can be solved using an exponential function. Use the exponential decay formula to model the problem and solve for the half-life of caffeine.

$$y = a(1 - r)^t$$

$$y = 60(1 - 0.15)^t$$

Half of 60 milligrams is 30. So, let $y = 30$ and solve for t .

$$30 = 60(1 - 0.15)^t$$

$$0.5 = (0.85)^t$$

Take the log of each side and use the power property.

$$\log 0.5 = \log (0.85)^t$$

$$\log 0.5 = t \log 0.85$$

$$\frac{\log 0.5}{\log 0.85} = t$$

At this point, it is necessary to use a calculator to evaluate the logarithms and solve the problem. Doing so shows that $t \approx 4.265$. So, the half-life of caffeine is about 4.25 hours. The correct answer is B.

Exercises

Read each problem. Identify what you need to know. Then use the information in the problem to solve.

- Jason recently purchased a new truck for \$34,750. The value of the truck decreases by 12% each year. What will the approximate value of the truck be 7 years after Jason purchased it?
 - \$13,775
 - \$13,890
 - \$14,125
 - \$14,200
- A baseball is thrown upward at a velocity of 105 feet per second, and is released when it is 5 feet above the ground. The height of the baseball t seconds after being thrown is given by the formula $h(t) = -16t^2 + 105t + 5$. Find the time at which the baseball reaches its maximum height.

F 1.0 s	H 6.6 s
G 3.3 s	J 177.3 s

- Lucinda deposited \$2500 in a CD with the terms described below.

Super CD!

Earn 4.25% interest compounded daily!

(Minimum deposit of \$1,000 over a period of at least 12 months.)

Use the formula below to solve for t , the number of years needed to earn \$250 in interest with the CD.

$$2750 = 2500 \left(1 + \frac{0.0425}{365} \right)^{365t}$$

- about 2.15 years
- about 2.24 years
- about 2.35 years
- about 2.46 years

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. What is the y -intercept of the exponential function below?

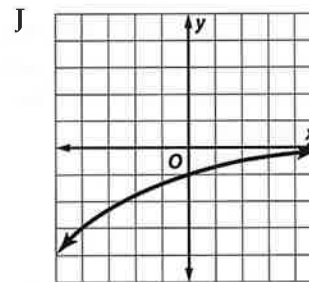
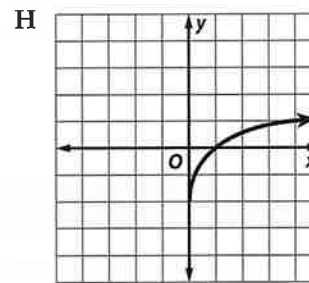
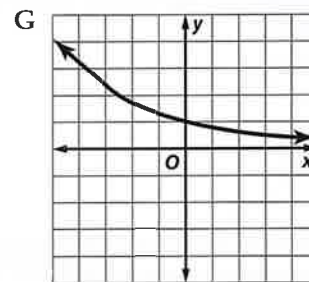
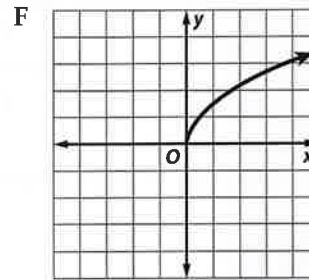
$$y = 4^x - 1$$

- A 0 B 1 C 2 D 3
2. Suppose there are only 3500 birds of a particular endangered species left on an island and the population decreases at a rate of about 5% each year. The logarithmic function $t = \log_{0.95} \frac{p}{3500}$ predicts how many years t it will be for the population to decrease to a number p . About how long will it take for the population to reach 3000 birds?
- F 2 years H 5 years
G 3 years J 8 years
3. Suppose a certain bacteria duplicates to reproduce itself every 20 minutes. If you begin with one cell of the bacteria, how many will there be after 2 hours?
- A 2 B 6 C 32 D 64
4. Lucas determined that the total cost C to rent a car could be represented by the equation $C = 0.35m + 125$, where m is the number of miles that he drives. If the total cost to rent the car was \$363, how many miles did he drive?
- F 125 H 520
G 238 J 680
5. Which of the following best describes the graph of $3y = 4x - 3$ and $8y = -6x - 5$?
- A The lines have the same y -intercept.
B The lines have the same x -intercept.
C The lines are perpendicular.
D The lines are parallel

Test-Taking Tip

Question 2 Use technology and the properties of logarithms to find t when $p = 3000$.

6. Graph $y = \log_5 x$.

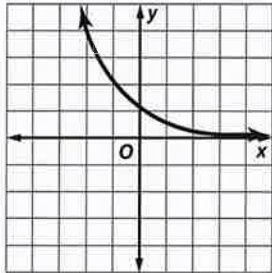


7. Ray's Book Store sells two used books for \$7.99. The total cost c of purchasing n books can be found by—
- A multiplying n by c .
B multiplying n by 5.
C multiplying n by the cost of 1 book.
D dividing n by c

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

8. The function $y = \left(\frac{1}{2}\right)^x$ is graphed below. What is the domain of the function?



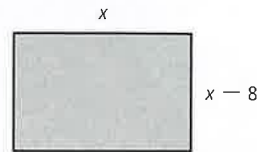
9. Suzanne bought a new car this year for \$33,750. The value of the car is expected to decrease by 10.5% per year. What will be the *approximate* value of the car 6 years after Suzanne purchased it? Show your work.
10. Julio is solving the system of equations $8x - 2y = 12$ and $-15x + 2y = -19$ by using elimination. His work is shown below.

$$\begin{array}{r} 8x - 2y = 12 \\ -15x + 2y = -19 \\ \hline -7x = 7 \\ x = -1 \end{array} \qquad \begin{array}{r} 8x - 2y = 12 \\ 8(-1) - 2y = 12 \\ -8 - 2y = 12 \\ -2y = 20 \\ y = -10 \end{array}$$

The solution is $(-1, -10)$.

- a. What error does Julio make?
- b. What is the correct solution of the system of equations? Show your work.
11. **GRIDDED RESPONSE** If $f(x) = 3x$ and $g(x) = x^2 - 1$, what is the value of $f(g(-3))$?
12. Simplify $(-2a^{-2}b^{-6})(-3a^{-1}b^8)$.

13. **GRIDDED RESPONSE** For what value of x would the rectangle below have an area of 48 square units?



Extended Response

Record your answers on a sheet of paper. Show your work.

14. Suppose the number of whitetail deer in a particular region has increased at an annual rate of about 10% since 1995. There were 135,000 deer in 1995.
- a. Write a function to model the number of whitetail deer after t years.
- b. About how many whitetail deer inhabited the region in 2000? Round your answer to the nearest hundred deer.
15. Sandy inherited \$250,000 from her aunt in 1998. She invested the money and increased it as shown in the table below.

Year	Amount
1998	\$250,000
2006	\$329,202
2011	\$390,989

- a. Write an exponential function that could be used to predict the amount of money A after investing for t years.
- b. If the money continues to grow at the same rate, in what year will it be worth \$500,000?

Need ExtraHelp?

If you missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Go to Lesson...	7-1	7-4	7-2	1-3	3-1	7-3	2-4	7-1	7-6	3-1	6-1	5-1	4-3	7-6	7-2

