

Rational Functions and Relations



Then

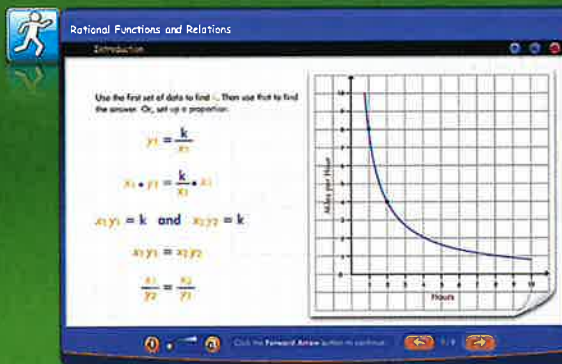
- You used factoring to solve quadratic equations and you graphed quadratic equations.

Now

- You will:
 - Simplify rational expressions.
 - Graph rational functions.
 - Solve direct, joint, and inverse variation problems.
 - Solve rational equations and inequalities.

Why? ▲

- TRAVEL** Whether you travel by boat, car, bicycle, or airplane, rational functions can be used to find distance traveled, time spent traveling, and speed. If you want to arrive at a destination on time, rational relations can tell you at what speed you need to travel to reach your goal. When graphing rational functions you see clearly how the speed at which you travel affects the time it takes to get there.



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Animation



Vocabulary



eGlossary



Personal Tutor



Virtual Manipulatives



Graphing Calculator



Audio



Foldables



Self-Check Practice



Worksheets



Get Ready for the Chapter

Diagnose Readiness | You have two options for checking prerequisite skills.

1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Solve each equation. Write in simplest form.

1. $\frac{5}{14} = \frac{1}{3}x$

2. $\frac{1}{8}m = \frac{7}{3}$

3. $\frac{8}{5} = \frac{1}{4}k$

4. $\frac{10}{9}p = 7$

5. **TRUCKS** Martin used $\frac{1}{3}$ of a tank of gas in his truck to get to work. He began with a full tank of gas. If he has 18 gallons of gas left, how many gallons does his tank hold?

Simplify each expression.

6. $\frac{3}{4} - \frac{7}{8}$

7. $\frac{8}{9} - \frac{7}{6} + \frac{1}{3}$

8. $\frac{9}{10} - \frac{4}{15} + \frac{1}{3}$

9. $\frac{10}{3} + \frac{5}{6} + 3$

10. **BAKING** Annie baked cookies for a bake sale. She used $\frac{2}{3}$ cup of flour for one recipe and $4\frac{1}{2}$ cups of flour for the other recipe. How many cups did she use in all?

Solve each proportion.

11. $\frac{9}{12} = \frac{p}{36}$

12. $\frac{9}{18} = \frac{6}{m}$

13. $\frac{2}{7} = \frac{5}{k}$

14. **SALES TAX** Kirsten pays \$4.40 tax on \$55 worth of clothes. What amount of tax will she pay on \$35 worth of clothes?

QuickReview

Example 1

Solve $\frac{9}{11} = \frac{7}{8}r$. Write in simplest form.

$$\frac{9}{11} = \frac{7}{8}r$$

$$\frac{72}{11} = 7r \quad \text{Multiply each side by 8.}$$

$$\frac{72}{77} = r \quad \text{Divide each side by 7.}$$

Since the GCF of 72 and 77 is 1, the solution is in simplest form.

Example 2

Simplify $\frac{1}{3} + \frac{3}{4} - \frac{5}{6}$.

$$\frac{1}{3} + \frac{3}{4} - \frac{5}{6}$$

$$= \frac{1}{3}\left(\frac{4}{4}\right) + \frac{3}{4}\left(\frac{3}{3}\right) - \frac{5}{6}\left(\frac{2}{2}\right)$$

The GCF of 3, 4, and 6 is 12.

$$= \frac{4}{12} + \frac{9}{12} - \frac{10}{12}$$

Simplify.

$$= \frac{3}{12}$$

Add and subtract.

$$= \frac{3 \div 3}{12 \div 3} \text{ or } \frac{1}{4}$$

Simplify.

Example 3

Solve $\frac{5}{8} = \frac{u}{11}$.

$$\frac{5}{8} = \frac{u}{11} \quad \text{Write the equation.}$$

$$5(11) = 8u \quad \text{Find the cross products.}$$

$$55 = 8u \quad \text{Simplify.}$$

$$\frac{55}{8} = u \quad \text{Divide each side by 8.}$$

Since the GCF of 55 and 8 is 1, the answer is in simplified form.

$$u = \frac{55}{8} \text{ or } 6\frac{7}{8}$$

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.



Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 8. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

FOLDABLES StudyOrganizer

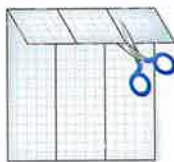


Rational Functions and Relations Make this Foldable to help you organize your Chapter 8 notes about rational functions and relations. Begin with an $8\frac{1}{2}'' \times 11''$ sheet of grid paper.

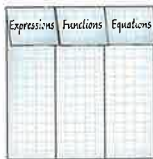
- 1** Fold in thirds along the height.



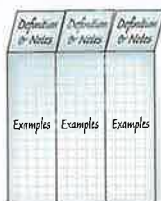
- 2** Fold the top edge down making a 2'' tab at the top. Cut along the folds.



- 3** Label the outside tabs *Expressions*, *Functions*, and *Equations*. Use the inside tabs for definitions and notes.



- 4** Write examples of each topic in the space below each tab.



New Vocabulary

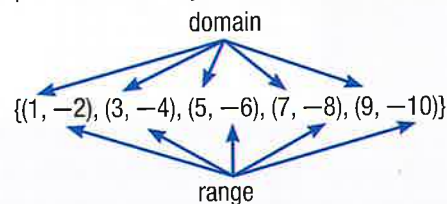


English		Español
rational expression	p. 529	expresión racional
complex fraction	p. 532	fracción compleja
reciprocal function	p. 545	función recíproca
hyperbola	p. 545	hipérbola
rational function	p. 553	función racional
vertical asymptote	p. 553	asíntota vertical
horizontal asymptote	p. 553	asíntota horizontal
oblique asymptote	p. 555	asíntota oblicua
point discontinuity	p. 556	discontinuidad evitable
direct variation	p. 562	variación directa
constant of variation	p. 562	constante de variación
joint variation	p. 563	variación conjunta
inverse variation	p. 564	variación inversa
combined variation	p. 565	variación combinada
rational equation	p. 570	ecuación racional
weighted average	p. 572	media ponderada
rational inequality	p. 575	desigualdad racional

Review Vocabulary



function **función** a relation in which each element of the domain is paired with exactly one element of the range



least common multiple **mínimo común múltiplo** the least number that is a common multiple of two or more numbers

rational number **número racional** a number expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$

LESSON 8-1 Multiplying and Dividing Rational Expressions

Then

- You factored polynomials.

Now

- Simplify rational expressions.
- Simplify complex fractions.

Why?

- If a scuba diver goes to depths greater than 33 feet, the rational function $T(d) = \frac{1700}{d-33}$ gives the maximum time a diver can remain at those depths and still surface at a steady rate with no stops. $T(d)$ represents the dive time in minutes, and d represents the depth in feet.



New Vocabulary
rational expression
complex fraction



Common Core State Standards

Content Standards
A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Mathematical Practices

- 8 Look for and express regularity in repeated reasoning.

1 Simplify Rational Expressions A ratio of two polynomial expressions such as $\frac{1700}{d-33}$ is called a **rational expression**.

Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar. Just as with reducing fractions, to simplify a rational expression, you divide the numerator and denominator by their greatest common factor (GCF).

$$\frac{8}{12} = \frac{2 \cdot \cancel{4}}{3 \cdot \cancel{4}} = \frac{2}{3}$$

GCF = 4

$$\frac{x^2 - 4x + 3}{x^2 - 6x + 5} = \frac{(x-3)\cancel{(x-1)}}{(x-5)\cancel{(x-1)}} = \frac{(x-3)}{(x-5)}$$

GCF = $(x-1)$

Example 1 Simplify a Rational Expression



a. Simplify $\frac{5x(x^2 + 4x + 3)}{(x-6)(x^2 - 9)}$.

$$\frac{5x(x^2 + 4x + 3)}{(x-6)(x^2 - 9)} = \frac{5x(x+3)(x+1)}{(x-6)(x+3)(x-3)}$$

Factor numerator and denominator.

$$= \frac{5x(x+1)}{(x-6)(x-3)} \cdot \frac{\cancel{(x+3)}}{\cancel{(x+3)}}$$

Eliminate common factors.

$$= \frac{5x(x+1)}{(x-6)(x-3)}$$

Simplify.

b. Under what conditions is this expression undefined?

The original factored denominator is $(x-6)(x+3)(x-3)$.

Determine the values that would make the denominator equal to 0.

These values are 6, -3, or 3, so the expression is undefined when $x = 6, -3$ or 3.

Guided Practice

Simplify each expression. Under what conditions is the expression undefined?

1A. $\frac{4y(y-3)(y+4)}{y(y^2-y-6)}$

1B. $\frac{2z(z+5)(z^2+2z-8)}{(z-1)(z+5)(z-2)}$



Standardized Test Example 2 Use Elimination

For what value(s) is $\frac{x^2(x^2 - 5x - 14)}{4x(x^2 + 6x + 8)}$ undefined?

A -2, -4

C 0, -2, -4

B -2, 7

D 0, -2, -4, 7

Read the Test Item

You want to determine which values of x make the denominator equal to 0.

Solve the Test Item

With $4x$ in the denominator, x cannot equal 0. So, choices A and B can be eliminated. Next, factor the denominator.

$x^2 + 6x + 8 = (x + 2)(x + 4)$, so the denominator is $4x(x + 2)(x + 4)$.

Because the denominator equals 0 when $x = 0, -2,$ and -4 , the answer is C.

Test-Taking Tip

Eliminating Choices
Sometimes you can save time by looking at the possible answers and eliminating choices.

Guided Practice

2. For what value(s) of x is $\frac{x(x^2 + 8x + 12)}{-6(x^2 - 3x - 10)}$ undefined?

F 0, 5, -2

G 5, -2

H 0, -2, -6

J 5, -2, -6

Sometimes you can factor out -1 in the numerator or denominator to help simplify a rational expression.

Example 3 Simplify Using -1

Simplify $\frac{(4w^2 - 3wy)(w + y)}{(3y - 4w)(5w + y)}$.

$$\frac{(4w^2 - 3wy)(w + y)}{(3y - 4w)(5w + y)} = \frac{w(4w - 3y)(w + y)}{(3y - 4w)(5w + y)}$$

Factor.

$$= \frac{w(-1)(3y - 4w)(w + y)}{(3y - 4w)(5w + y)}$$

$$4w - 3y = -1(3y - 4w)$$

$$= \frac{(-w)(w + y)}{5w + y}$$

Simplify.

Guided Practice

Simplify each expression.

3A. $\frac{(xz - 4z)}{z^2(4 - x)}$

3B. $\frac{ab^2 - 5ab}{(5 + b)(5 - b)}$

The method for multiplying and dividing fractions also works with rational expressions. Remember that to multiply two fractions, you multiply the numerators and multiply the denominators. To divide two fractions, you multiply by the multiplicative inverse, or the reciprocal, of the divisor.

Multiplication

$$\frac{2}{9} \cdot \frac{15}{4} = \frac{\cancel{2} \cdot \cancel{3} \cdot 5}{\cancel{3} \cdot \cancel{3} \cdot 2} = \frac{5}{3 \cdot 2} = \frac{5}{6}$$

Division

$$\frac{3}{5} \div \frac{6}{35} = \frac{3}{5} \cdot \frac{35}{6} = \frac{\cancel{3} \cdot \cancel{5} \cdot 7}{\cancel{3} \cdot 2 \cdot \cancel{5}} = \frac{7}{2}$$



The following table summarizes the rules for multiplying and dividing rational expressions.

KeyConcept	
Multiplying Rational Expressions	
Words	To multiply rational expressions, multiply the numerators and multiply the denominators.
Symbols	For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
Dividing Rational Expressions	
Words	To divide rational expressions, multiply by the reciprocal of the divisor.
Symbols	For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$, $c \neq 0$, and $d \neq 0$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

StudyTip

Eliminating Common Factors

Be sure to eliminate factors from both the numerator and denominator.

Example 4 Multiply and Divide Rational Expressions

Simplify each expression.

a. $\frac{6c}{5d} \cdot \frac{15cd^2}{8a}$

$$\frac{6c}{5d} \cdot \frac{15cd^2}{8a} = \frac{2 \cdot 3 \cdot c \cdot 5 \cdot 3 \cdot c \cdot d \cdot d}{5 \cdot d \cdot 2 \cdot 2 \cdot 2 \cdot a}$$

Factor.

$$= \frac{\overset{1}{\cancel{2}} \cdot 3 \cdot c \cdot \overset{1}{\cancel{5}} \cdot 3 \cdot c \cdot \overset{1}{\cancel{d}} \cdot d}{\overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{d}} \cdot 2 \cdot 2 \cdot 2 \cdot a}$$

Eliminate common factors.

$$= \frac{3 \cdot 3 \cdot c \cdot c \cdot d}{2 \cdot 2 \cdot a}$$

Simplify.

$$= \frac{9c^2d}{4a}$$

Simplify.

b. $\frac{18xy^3}{7a^2b^2} \div \frac{12x^2y}{35a^2b}$

$$\frac{18xy^3}{7a^2b^2} \div \frac{12x^2y}{35a^2b} = \frac{18xy^3}{7a^2b^2} \cdot \frac{35a^2b}{12x^2y}$$

Multiply by reciprocal of the divisor.

$$= \frac{2 \cdot 3 \cdot 3 \cdot x \cdot y \cdot y \cdot y \cdot 5 \cdot 7 \cdot a \cdot a \cdot b}{7 \cdot a \cdot a \cdot b \cdot b \cdot 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y}$$

Factor.

$$= \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot 3 \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{y}} \cdot y \cdot y \cdot 5 \cdot \overset{1}{\cancel{7}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{b}}}{\overset{1}{\cancel{7}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{b}} \cdot b \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{y}}}$$

Eliminate common factors.

$$= \frac{3 \cdot 5 \cdot y \cdot y}{2 \cdot b \cdot x}$$

Simplify.

$$= \frac{15y^2}{2bx}$$

Simplify.

GuidedPractice

4A. $\frac{12c^3d^2}{21ab} \cdot \frac{14a^2b}{8c^2d}$

4B. $\frac{6xy}{15ab^2} \cdot \frac{21a^3}{18x^4y}$

4C. $\frac{16m^2t^2}{21a^4b^3} \div \frac{24m^3}{7a^2b^2}$

4D. $\frac{12x^4y^2}{40a^4b^4} \div \frac{6x^2y^4}{16a^2x}$



Sometimes you must factor the numerator and/or the denominator first before you can simplify a product or a quotient of rational expressions.

StudyTip

CCSS Regularity When simplifying rational expressions, factors in one polynomial will often reappear in other polynomials. In Example 5a, $x - 8$ appears four times. Use this as a guide when factoring challenging polynomials.

Example 5 Polynomials in the Numerator and Denominator

Simplify each expression.

a. $\frac{x^2 - 6x - 16}{x^2 - 16x + 64} \cdot \frac{x - 8}{x^2 + 5x + 6}$

$$\frac{x^2 - 6x - 16}{x^2 - 16x + 64} \cdot \frac{x - 8}{x^2 + 5x + 6} = \frac{(x - 8)(x + 2)}{(x - 8)(x - 8)} \cdot \frac{x - 8}{(x + 3)(x + 2)}$$

Factor.

$$= \frac{\overset{1}{(x-8)}\overset{1}{(x+2)}}{\underset{1}{(x-8)}\underset{1}{(x-8)}} \cdot \frac{\overset{1}{x-8}}{\underset{1}{(x+3)}\underset{1}{(x+2)}}$$

Eliminate common factors.

$$= \frac{1}{x + 3}$$

Simplify.

b. $\frac{x^2 - 16}{12y + 36} \div \frac{x^2 - 12x + 32}{y^2 - 3y - 18}$

$$\frac{x^2 - 16}{12y + 36} \div \frac{x^2 - 12x + 32}{y^2 - 3y - 18} = \frac{x^2 - 16}{12y + 36} \cdot \frac{y^2 - 3y - 18}{x^2 - 12x + 32}$$

Multiply by reciprocal.

$$= \frac{(x + 4)(x - 4)}{12(y + 3)} \cdot \frac{(y - 6)(y + 3)}{(x - 4)(x - 8)}$$

Factor.

$$= \frac{(x + 4)\overset{1}{(x-4)}}{\underset{1}{12}(y + 3)} \cdot \frac{(y - 6)\overset{1}{(y+3)}}{\underset{1}{(x-4)}(x - 8)}$$

Eliminate common factors.

$$= \frac{(x + 4)(y - 6)}{12(x - 8)}$$

Simplify.

GuidedPractice

5A. $\frac{8x - 20}{x^2 + 2x - 35} \cdot \frac{x^2 - 7x + 10}{4x^2 - 16}$

5B. $\frac{x^2 - 9x + 20}{x^2 + 10x + 21} \div \frac{x^2 - x - 12}{6x + 42}$

2 Simplify Complex Fractions A **complex fraction** is a rational expression with a numerator and/or denominator that is also a rational expression. The following expressions are complex fractions.

$$\frac{\frac{c}{6}}{5d} \quad \frac{\frac{8}{x}}{x-2} \quad \frac{\frac{x-3}{8}}{\frac{x-2}{x+4}} \quad \frac{\frac{4}{a} + 6}{\frac{12}{a} - 3}$$

To simplify a complex fraction, first rewrite it as a division expression.

Example 6 Simplify Complex Fractions

Simplify each expression.

a. $\frac{\frac{a+b}{4}}{\frac{a^2+b^2}{4}}$

$$\frac{\frac{a+b}{4}}{\frac{a^2+b^2}{4}} = \frac{a+b}{4} \div \frac{a^2+b^2}{4}$$

Express as a division expression.

$$= \frac{a+b}{4} \cdot \frac{4}{a^2+b^2}$$

Multiply by the reciprocal.

$$= \frac{a+b}{\underset{1}{\cancel{4}}} \cdot \frac{\overset{1}{\cancel{4}}}{a^2+b^2} \text{ or } \frac{a+b}{a^2+b^2}$$

Simplify.



$$\begin{aligned}
 \text{b. } & \frac{\frac{x^2}{x^2 - y^2}}{\frac{4x}{y - x}} \\
 & \frac{x^2}{x^2 - y^2} \div \frac{4x}{y - x} \\
 & = \frac{x^2}{x^2 - y^2} \cdot \frac{y - x}{4x} \\
 & = \frac{x \cdot x}{(x + y)(x - y)} \cdot \frac{(-1)(x - y)}{4x} \\
 & = \frac{\overset{1}{x} \cdot \overset{1}{\cancel{x}}}{(x + y)\overset{1}{\cancel{(x - y)}}} \cdot \frac{(-1)\overset{1}{\cancel{(x - y)}}}{4\overset{1}{\cancel{x}}} \\
 & = \frac{-x}{4(x + y)}
 \end{aligned}$$

Express as a division expression.

Multiply by the reciprocal.

Factor.

Eliminate Factors.

Simplify.

Guided Practice

Simplify each expression.

$$\text{6A. } \frac{\frac{(x - 2)^2}{2(x^2 - 5x + 4)}}{\frac{x^2 - 4}{4x - 10}}$$

$$\text{6B. } \frac{\frac{x^2 - y^2}{y^2 - 49}}{\frac{y - x}{y + 7}}$$

Check Your Understanding

Step-by-Step Solutions begin on page R14.



Example 1 Simplify each expression.

$$1. \frac{x^2 - 5x - 24}{x^2 - 64}$$

$$2. \frac{c + d}{3c^2 - 3d^2}$$

Example 2

3. MULTIPLE CHOICE Identify all values of x for which $\frac{x + 7}{x^2 - 3x - 28}$ is undefined.

A $-7, 4$

B $7, 4$

C $4, -7, 7$

D $-4, 7$

Examples 3–6 Simplify each expression.

$$4. \frac{y^2 + 3y - 40}{25 - y^2}$$

$$5. \frac{a^2x - b^2x}{by - ay}$$

$$6. \frac{27x^2y^4}{16yz^3} \cdot \frac{8z}{9xy^3}$$

$$7. \frac{12x^3y}{13ab^2} \div \frac{36xy^3}{26b}$$

$$8. \frac{x^2 - 4x - 21}{x^2 - 6x + 8} \cdot \frac{x - 4}{x^2 - 2x - 35}$$

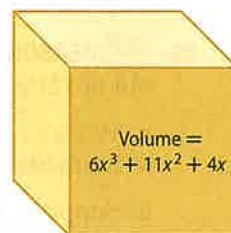
$$9. \frac{a^2 - b^2}{3a^2 - 6a + 3} \div \frac{4a + 4b}{a^2 - 1}$$

$$10. \frac{\frac{a^3b^3}{xy^4}}{\frac{a^2b}{x^2y}}$$

$$11. \frac{\frac{4x}{x + 6}}{\frac{x^2 - 3x}{x^2 + 3x - 18}}$$

12. CCSS SENSE-MAKING The volume of a shipping container in the shape of a rectangular prism can be represented by the polynomial $6x^3 + 11x^2 + 4x$, where the height is x .

- Find the length and width of the container.
- Find the ratio of the three dimensions of the container when $x = 2$.
- Will the ratio of the three dimensions be the same for all values of x ?



Example 1 Simplify each expression.

13. $\frac{x(x-3)(x+6)}{x^2+x-12}$

15. $\frac{(x^2-9)(x^2-z^2)}{4(x+z)(x-3)}$

17. $\frac{x^2(x+2)(x-4)}{6x(x^2+x-20)}$

14. $\frac{y^2(y^2+3y+2)}{2y(y-4)(y+2)}$

16. $\frac{(x^2-16x+64)(x+2)}{(x^2-64)(x^2-6x-16)}$

18. $\frac{3y(y-8)(y^2+2y-24)}{15y^2(y^2-12y+32)}$

Example 2 19. **MULTIPLE CHOICE** Identify all values of x for which $\frac{(x-3)(x+6)}{(x^2-7x+12)(x^2-36)}$ is undefined.

F 3, -6

G 4, 6

H -6, 6

J -6, 3, 4, 6

Example 3 Simplify each expression.

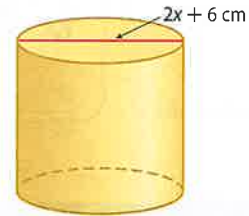
20. $\frac{x^2-5x-14}{28+3x-x^2}$

21. $\frac{x^3-9x^2}{x^2-3x-54}$

22. $\frac{(x-4)(x^2+2x-48)}{(36-x^2)(x^2+4x-32)}$

23. $\frac{16-c^2}{c^2+c-20}$

24. **GEOMETRY** The cylinder at the right has a volume of $(x+3)(x^2-3x-18)\pi$ cubic centimeters. Find the height of the cylinder.



Examples 4-6 Simplify each expression.

25. $\frac{3ac^3f^3}{8a^2bcf^4} \cdot \frac{12ab^2c}{18ab^3c^2f}$

26. $\frac{14xy^2z^3}{21w^4x^2yz} \cdot \frac{7wxyz}{12w^2y^3z}$

27. $\frac{64a^2b^5}{35b^2c^3f^4} \div \frac{12a^4b^3c}{70abcf^2}$

28. $\frac{9x^2yz}{5z^4} \div \frac{12x^4y^2}{50xy^4z^2}$

29. $\frac{15a^2b^2}{21ac} \cdot \frac{14a^4c^2}{6ab^3}$

30. $\frac{14c^2f^5}{9a^2} \div \frac{35cf^4}{18ab^3}$

31. $\frac{y^2+8y+15}{y-6} \cdot \frac{y^2-9y+18}{y^2-9}$

32. $\frac{c^2-6c-16}{c^2-d^2} \div \frac{c^2-8c}{c+d}$

33. $\frac{x^2+9x+20}{8x+16} \cdot \frac{4x^2+16x+16}{x^2-25}$

34. $\frac{3a^2+6a+3}{a^2-3a-10} \div \frac{12a^2-12}{a^2-4}$

35. $\frac{\frac{x^2-9}{6x-12}}{\frac{x^2+10x+21}{x^2-x-2}}$

36. $\frac{\frac{y-x}{z^3}}{\frac{x-y}{6z^2}}$

37. $\frac{\frac{a^2-b^2}{b^3}}{\frac{b^2-ab}{a^2}}$

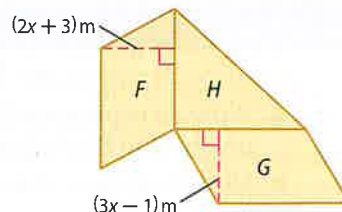
38. $\frac{\frac{x-y}{a+b}}{\frac{x^2-y^2}{b^2-a^2}}$

39. **CCSS REASONING** At the end of her high school soccer career, Ashley had made 33 goals out of 121 attempts.

- a. Write a ratio to represent the ratio of the number of goals made to goals attempted by Ashley at the end of her high school career.
- b. Suppose Ashley attempted a goals and made m goals during her first year at college. Write a rational expression to represent the ratio of the number of career goals made to the number of career goals attempted at the end of her first year in college.



40. **GEOMETRY** Parallelogram F has an area of $8x^2 + 10x - 3$ square meters and a height of $2x + 3$ meters. Parallelogram G has an area of $6x^2 + 13x - 5$ square meters and a height of $3x - 1$ meters. Find the area of right triangle H .

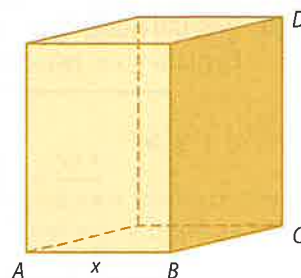


41. **POLLUTION** The thickness of an oil spill from a ruptured pipe on a rig is modeled by the function $T(x) = \frac{0.4(x^2 - 2x)}{x^3 + x^2 - 6x}$, where T is the thickness of the oil slick in meters and x is the distance from the rupture in meters.
- Simplify the function.
 - How thick is the slick 100 meters from the rupture?

Simplify each expression.

42. $\frac{x^2 - 16}{3x^3 + 18x^2 + 24x} \cdot \frac{x^3 - 4x}{2x^2 - 7x - 4}$
43. $\frac{3x^2 - 17x - 6}{4x^2 - 20x - 24} \div \frac{6x^2 - 7x - 3}{2x^2 - x - 3}$
44. $\frac{9 - x^2}{x^2 - 4x - 21} \cdot \left(\frac{2x^2 + 7x + 3}{2x^2 - 15x + 7}\right)^{-1}$
45. $\left(\frac{2x^2 + 2x - 12}{x^2 + 4x - 5}\right)^{-1} \cdot \frac{2x^3 - 8x}{x^2 - 2x - 35}$
46. $\left(\frac{3xy^3z}{2a^2bc^2}\right)^3 \cdot \frac{16a^4b^3c^5}{15x^7yz^3}$
47. $\frac{20x^2y^6z^{-2}}{3a^3c^2} \cdot \left(\frac{16x^3y^3}{9acz}\right)^{-1}$
48. $\left(\frac{2xy^3}{3abc}\right)^{-2} \div \frac{6a^2b}{x^2y^4}$
49. $\frac{\frac{8x^2 - 10x - 3}{10x^2 + 35x - 20}}{\frac{2x^2 + x - 6}{4x^2 + 18x + 8}}$
50. $\frac{\frac{2x^2 + 7x - 30}{-6x^2 + 13x + 5}}{\frac{4x^2 + 12x - 72}{3x^2 - 11x - 4}}$
51. $\frac{\frac{4x^2 - 1}{3x^3 - 6x^2 - 24x}}{\frac{12x^2 + 12x - 9}{-2x^2 + 5x + 12}}$

52. **GEOMETRY** The area of the base of the rectangular prism at the right is 20 square centimeters.
- Find the length of \overline{BC} in terms of x .
 - If $DC = 3BC$, determine the area of the shaded region in terms of x .
 - Determine the volume of the prism in terms of x .



Simplify each expression.

53. $\frac{x^2 + 4x - 32}{2x^2 + 9x - 5} \cdot \frac{3x^2 - 75}{3x^2 - 11x - 4} \div \frac{6x^2 - 18x - 60}{x^3 - 4x}$
54. $\frac{8x^2 + 10x - 3}{3x^2 - 12x - 36} \div \frac{2x^2 - 5x - 12}{3x^2 - 17x - 6} \cdot \frac{4x^2 + 3x - 1}{4x^2 - 40x + 24}$
55. $\frac{4x^2 - 9x - 9}{3x^2 + 6x - 18} \div \frac{-2x^2 + 5x + 3}{x^2 - 4x - 32} \div \frac{8x^2 + 10x + 3}{6x^2 - 6x - 12}$
56. **CCSS PERSEVERANCE** Use the formula $d = rt$ and the following information. An airplane is traveling at a rate r of 500 miles per hour for a time t of $(6 + x)$ hours. A second airplane travels at the rate r of $(540 + 90x)$ miles per hour for a time t of 6 hours.
- Write a rational expression to represent the ratio of the distance d traveled by the first airplane to the distance d traveled by the second airplane.
 - Simplify the rational expression. What does this expression tell you about the distances traveled by the two airplanes?
 - Under what condition is the rational expression undefined? Describe what this condition would tell you about the two airplanes.



- 57 TRAINS** Trying to get into a train yard one evening, all of the trains are backed up for 2 miles along a system of tracks. Assume that each car occupies an average of 75 feet of space on a track and that the train yard has 5 tracks.
- Write an expression that could be used to determine the number of train cars involved in the backup.
 - How many train cars are involved in the backup?
 - Suppose that there are 8 attendants doing safety checks on each car, and it takes each vehicle an average of 45 seconds for each check. Approximately how many hours will it take for all the vehicles in the backup to exit?
- 58. MULTIPLE REPRESENTATIONS** In this problem, you will investigate the graph of a rational function.
- Algebraic** Simplify $\frac{x^2 - 5x + 4}{x - 4}$.
 - Tabular** Let $f(x) = \frac{x^2 - 5x + 4}{x - 4}$. Use the expression you wrote in part a to write the related function $g(x)$. Use a graphing calculator to make a table for both functions for $0 \leq x \leq 10$.
 - Analytical** What are $f(4)$ and $g(4)$? Explain the significance of these values.
 - Graphical** Graph the functions on the graphing calculator. Use the TRACE function to investigate each graph, using the \blacktriangle and \blacktriangledown keys to switch from one graph to the other. Compare and contrast the graphs.
 - Verbal** What conclusions can you draw about the expressions and the functions?

H.O.T. Problems Use Higher-Order Thinking Skills

- 59. REASONING** Compare and contrast $\frac{(x-6)(x+2)(x+3)}{x+3}$ and $(x-6)(x+2)$.
- 60. CCSS CRITIQUE** Troy and Beverly are simplifying $\frac{x+y}{x-y} \div \frac{4}{y-x}$. Is either of them correct? Explain your reasoning.

Troy

$$\frac{x+y}{x-y} \div \frac{4}{y-x} = \frac{x-y}{x+y} \cdot \frac{4}{y-x}$$

$$= \frac{-4}{x+y}$$

Beverly

$$\frac{x+y}{x-y} \div \frac{4}{y-x} = \frac{x+y}{x-y} \cdot \frac{y-x}{4}$$

$$= -\frac{x+y}{4}$$

- 61. CHALLENGE** Find the expression that makes the following statement true.
- $$\frac{x-6}{x+3} \cdot \frac{?}{x-6} = x-2$$
- 62. WHICH ONE DOESN'T BELONG?** Identify the expression that does not belong with the other three. Explain your reasoning.

$$\frac{1}{x-1}$$

$$\frac{x^2+3x+2}{x-5}$$

$$\frac{x+1}{\sqrt{x+3}}$$

$$\frac{x^2+1}{3}$$

- 63. REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

A rational function that has a variable in the denominator is defined for all real values of x .

- 64. OPEN ENDED** Write a rational expression that simplifies to $\frac{x-1}{x+4}$.
- 65. WRITING IN MATH** The rational expression $\frac{x^2+3x}{4x}$ is simplified to $\frac{x+3}{4}$. Explain why this new expression is not defined for all values of x .



Standardized Test Practice

66. SAT/ACT The Mason family wants to drive an average of 250 miles per day on their vacation. On the first five days, they travel 220 miles, 300 miles, 210 miles, 275 miles, and 240 miles. How many miles must they travel on the sixth day to meet their goal?

- A 235 miles D 275 miles
 B 251 miles E 315 miles
 C 255 miles

67. Which of the following equations gives the relationship between N and T in the table?

N	1	2	3	4	5	6
T	1	4	7	10	13	16

- F $T = 2 - N$ H $T = 3N + 1$
 G $T = 4 - 3N$ J $T = 3N - 2$

68. A monthly cell phone plan costs \$39.99 for up to 300 minutes and 20 cents per minute thereafter. Which of the following represents the total monthly bill (in dollars) to talk for x minutes if x is greater than 300?

- A $39.99 + 0.20(300 - x)$
 B $39.99 + 0.20(x - 300)$
 C $39.99 + 0.20x$
 D $39.99 + 20x$

69. SHORT RESPONSE The area of a circle 6 meters in diameter exceeds the combined areas of a circle 4 meters in diameter and a circle 2 meters in diameter by how many square meters?

Spiral Review

70. ANTHROPOLOGY An anthropologist studying the bones of a prehistoric person finds there is so little remaining Carbon-14 in the bones that instruments cannot measure it. This means that there is less than 0.5% of the amount of Carbon-14 the bones would have contained when the person was alive. The half-life of Carbon-14 is 5760 years. How long ago did the person die? (Lesson 7-8)

Solve each equation. Round to the nearest ten-thousandth. (Lesson 7-7)

71. $3e^x + 1 = 5$ 72. $2e^x - 1 = 0$ 73. $-3e^{4x} + 11 = 2$ 74. $8 + 3e^{3x} = 26$

75. NOISE ORDINANCE A proposed city ordinance will make it illegal in a residential area to create sound that exceeds 72 decibels during the day and 55 decibels during the night. How many times as intense is the noise level allowed during the day than at night? (Lesson 7-3)

Simplify. (Lesson 6-5)

76. $\sqrt{50x^4}$ 77. $\sqrt[3]{16y^3}$ 78. $\sqrt{18x^2y^3}$ 79. $\sqrt{40a^3b^4}$

80. AUTOMOBILES The length of the cargo space in a sport-utility vehicle is 4 inches greater than the height of the space. The width is 16 inches less than twice the height. The cargo space has a total volume of 55,296 cubic inches. (Lesson 5-8)

- Write a polynomial function that represents the volume of the cargo space.
- Will a package 34 inches long, 44 inches wide, and 34 inches tall fit in the cargo space? Explain.



Skills Review

Simplify.

81. $(2a + 3b) + (8a - 5b)$ 82. $(x^2 - 4x + 3) - (4x^2 + 3x - 5)$ 83. $(5y + 3y^2) + (-8y - 6y^2)$
 84. $2x(3y + 9)$ 85. $(x + 6)(x + 3)$ 86. $(x + 1)(x^2 - 2x + 3)$



Adding and Subtracting Rational Expressions

Then

- You added and subtracted polynomial expressions.

Now

- Determine the LCM of polynomials.
- Add and subtract rational expressions.

Why?

- As a fire engine moves toward a person, the pitch of the siren sounds higher to that person than it would if the fire engine were at rest. This is because the sound waves are compressed closer together, referred to as the *Doppler effect*. The Doppler effect can be represented by the rational expression $P_0 \left(\frac{s_0}{s_0 - v} \right)$, where P_0 is the actual pitch of the siren, v is the speed of the fire truck, and s_0 is the speed of sound in air.



Common Core State Standards

Content Standards

A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Mathematical Practices

3 Construct viable arguments and critique the reasoning of others.

1 LCM of Polynomials Just as with rational numbers in fractional form, to add or subtract two rational expressions that have unlike denominators, you must first find the least common denominator (LCD). The LCD is the least common multiple (LCM) of the denominators.

To find the LCM of two or more numbers or polynomials, factor them. The LCM contains each factor the greatest number of times it appears as a factor.

Numbers

$$\frac{5}{6} + \frac{4}{9}$$

LCM of 6 and 9

$$6 = 2 \cdot 3$$

$$9 = 3 \cdot 3$$

$$\text{LCM} = 2 \cdot 3 \cdot 3 \text{ or } 18$$

Polynomials

$$\frac{3}{x^2 - 3x + 2} + \frac{5}{2x^2 - 2}$$

LCM of $x^2 - 3x + 2$ and $2x^2 - 2$

$$x^2 - 3x + 2 = (x - 1)(x - 2)$$

$$2x^2 - 2 = 2 \cdot (x - 1)(x + 1)$$

$$\text{LCM} = 2(x - 1)(x - 2)(x + 1)$$

Example 1 LCM of Monomials and Polynomials

Find the LCM of each set of polynomials.

a. $6xy$, $15x^2$, and $9xy^4$

$$6xy = 2 \cdot 3 \cdot x \cdot y$$

$$15x^2 = 3 \cdot 5 \cdot x^2$$

$$9xy^4 = 3 \cdot 3 \cdot x \cdot y^4$$

$$\begin{aligned} \text{LCM} &= 2 \cdot 3 \cdot 3 \cdot 5 \cdot x^2 \cdot y^4 \\ &= 90x^2y^4 \end{aligned}$$

Factor the first monomial.

Factor the second monomial.

Factor the third monomial.

Use each factor the greatest number of times it appears.

Then simplify.

b. $y^4 + 8y^3 + 15y^2$ and $y^2 - 3y - 40$

$$y^4 + 8y^3 + 15y^2 = y^2(y + 5)(y + 3)$$

$$y^2 - 3y - 40 = (y + 5)(y - 8)$$

$$\text{LCM} = y^2(y + 5)(y + 3)(y - 8)$$

Factor the first polynomial.

Factor the second polynomial.

Use each factor the greatest number of times it appears as a factor.

Guided Practice

1A. $12a^2b$, $15abc$, $8b^3c^4$

1B. $4a^2 - 12a - 16$ and $a^3 - 9a^2 + 20a$



2 Add and Subtract Rational Expressions

As with fractions, rational expressions must have common denominators in order to be added or subtracted.

Key Concept

Adding Rational Expressions

Words To add rational expressions, find the least common denominator (LCD). Rewrite each expression with the LCD. Then add.

Symbols For all $\frac{a}{b}$ and $\frac{c}{d}$, with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$.

Subtracting Rational Expressions

Words To subtract rational expressions, find the least common denominator (LCD). Rewrite each expression with the LCD. Then subtract.

Symbols For all $\frac{a}{b}$ and $\frac{c}{d}$, with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$.



Example 2 Monomial Denominators

Simplify $\frac{3y}{2x^3} + \frac{5z}{8xy^2}$.

$$\begin{aligned} \frac{3y}{2x^3} + \frac{5z}{8xy^2} &= \frac{3y}{2x^3} \cdot \frac{4y^2}{4y^2} + \frac{5z}{8xy^2} \cdot \frac{x^2}{x^2} \\ &= \frac{12y^3}{8x^3y^2} + \frac{5x^2z}{8x^3y^2} \\ &= \frac{12y^3 + 5x^2z}{8x^3y^2} \end{aligned}$$

The LCD is $8x^3y^2$.

Multiply fractions.

Add the numerators.

Guided Practice

Simplify each expression.

2A. $\frac{4}{5a^3b^2} + \frac{9c}{10ab}$

2B. $\frac{3a^2}{16b^2} - \frac{8x}{5a^3b}$

The LCD is also used to combine rational expressions with polynomial denominators.

Example 3 Polynomial Denominators

Simplify $\frac{5}{6x-18} - \frac{x-1}{4x^2-14x+6}$.

$$\begin{aligned} \frac{5}{6x-18} - \frac{x-1}{4x^2-14x+6} &= \frac{5}{6(x-3)} - \frac{x-1}{2(2x-1)(x-3)} \\ &= \frac{5(2x-1)}{6(x-3)(2x-1)} - \frac{(x-1)(3)}{2(2x-1)(x-3)(3)} \\ &= \frac{10x-5-3x+3}{6(x-3)(2x-1)} \\ &= \frac{7x-2}{6(x-3)(2x-1)} \end{aligned}$$

Factor denominators.

Multiply by missing factors.

Subtract numerators.

Simplify.

Study Tip

Simplifying Rational Expressions After you add or subtract rational expressions, it is possible that the resulting expression can be further simplified.

Guided Practice

Simplify each expression.

3A. $\frac{x-1}{x^2-x-6} - \frac{4}{5x+10}$

3B. $\frac{x-8}{4x^2+21x+5} + \frac{6}{12x+3}$



One way to simplify a complex fraction is to simplify the numerator and the denominator separately, and then simplify the resulting expressions.

StudyTip

Undefined Terms Remember that there are restrictions on variables in the denominator.

Example 4 Complex Fractions with Different LCDs

Simplify $\frac{1 + \frac{1}{x}}{1 - \frac{x}{y}}$.

$$\frac{1 + \frac{1}{x}}{1 - \frac{x}{y}} = \frac{\frac{x}{x} + \frac{1}{x}}{\frac{y}{y} - \frac{x}{y}}$$

$$= \frac{\frac{x+1}{x}}{\frac{y-x}{y}}$$

$$= \frac{x+1}{x} \div \frac{y-x}{y}$$

$$= \frac{x+1}{x} \cdot \frac{y}{y-x}$$

$$= \frac{xy+y}{xy-x^2}$$

The LCD of the numerator is x .

The LCD of the denominator is y .

Simplify the numerator and denominator.

Write as a division expression.

Multiply by the reciprocal of the divisor.

Simplify.

GuidedPractice

Simplify each expression.

4A. $\frac{1 - \frac{y}{x}}{\frac{1}{y} + \frac{1}{x}}$

4B. $\frac{\frac{c}{d} - \frac{d}{c}}{\frac{d}{c} + 2}$

Another method of simplifying complex fractions is to find the LCD of all of the denominators. Then, the denominators are all eliminated by multiplying by the LCD.

Example 5 Complex Fractions with Same LCDs

Simplify $\frac{1 + \frac{1}{x}}{1 - \frac{x}{y}}$.

$$\frac{1 + \frac{1}{x}}{1 - \frac{x}{y}} = \frac{\left(1 + \frac{1}{x}\right) \cdot xy}{\left(1 - \frac{x}{y}\right) \cdot xy}$$

$$= \frac{xy+y}{xy-x^2}$$

The LCD of all of the denominators is xy .

Multiply by $\frac{xy}{xy}$.

Distribute xy .

Notice that the same problem is solved in Examples 4 and 5 using different methods, but both produce the same answer. So, how you solve problems similar to these is left up to your own discretion.

GuidedPractice

Simplify each expression.

5A. $\frac{1 + \frac{2}{x}}{\frac{3}{y} - \frac{4}{x}}$

5B. $\frac{\frac{1}{d} - \frac{d}{c}}{\frac{1}{c} + 6}$

5C. $\frac{\frac{1}{y} + \frac{1}{x}}{\frac{1}{y} - \frac{1}{x}}$

5D. $\frac{\frac{a}{b} + 1}{1 - \frac{b}{a}}$





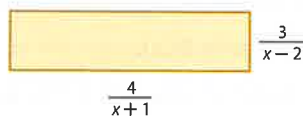
Example 1 Find the LCM of each set of polynomials.

1. $16x, 8x^2y^3, 5x^3y$
2. $7a^2, 9ab^3, 21abc^4$
3. $3y^2 - 9y, y^2 - 8y + 15$
4. $x^3 - 6x^2 - 16x, x^2 - 4$

Examples 2–3 Simplify each expression.

5. $\frac{12y}{5x} + \frac{5x}{4y^3}$
6. $\frac{5}{6ab} + \frac{3b^2}{14a^3}$
7. $\frac{7b}{12a} - \frac{1}{18ab^3}$
8. $\frac{y^2}{8c^2d^2} - \frac{3x}{14c^4d}$
9. $\frac{4x}{x^2 + 9x + 18} + \frac{5}{x + 6}$
10. $\frac{8}{y - 3} + \frac{2y - 5}{y^2 - 12y + 27}$
11. $\frac{4}{3x + 6} - \frac{x + 1}{x^2 - 4}$
12. $\frac{3a + 2}{a^2 - 16} - \frac{7}{6a + 24}$

13. **GEOMETRY** Find the perimeter of the rectangle.



Examples 4–5 Simplify each expression.

14. $\frac{4 + \frac{2}{x}}{3 - \frac{2}{x}}$
15. $\frac{6 + \frac{4}{y}}{2 + \frac{6}{y}}$
16. $\frac{\frac{3}{x} + \frac{2}{y}}{1 + \frac{4}{y}}$
17. $\frac{\frac{2}{b} + \frac{5}{a}}{\frac{3}{a} - \frac{8}{b}}$

Practice and Problem Solving

Extra Practice is on page R8.

Example 1 Find the LCM of each set of polynomials.

18. $24cd, 40a^2c^3d^4, 15abd^3$
19. $4x^2y^3, 18xy^4, 10xz^2$
20. $x^2 - 9x + 20, x^2 + x - 30$
21. $6x^2 + 21x - 12, 4x^2 + 22x + 24$

Examples 2–3 **CCSS PERSEVERANCE** Simplify each expression.

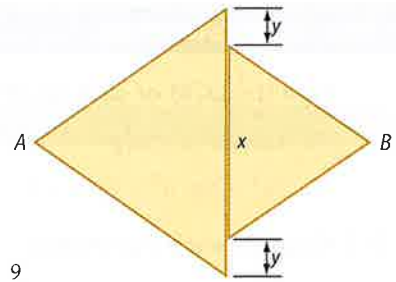
22. $\frac{5a}{24cf^4} + \frac{a}{36bc^4f^3}$
23. $\frac{4b}{15x^3y^2} - \frac{3b}{35x^2y^4z}$
24. $\frac{5b}{6a} + \frac{3b}{10a^2} + \frac{2}{ab^2}$
25. $\frac{4}{3x} + \frac{8}{x^3} + \frac{2}{5xy}$
26. $\frac{8}{3y} + \frac{2}{9} - \frac{3}{10y^2}$
27. $\frac{1}{16a} + \frac{5}{12b} - \frac{9}{10b^3}$
28. $\frac{8}{x^2 - 6x - 16} + \frac{9}{x^2 - 3x - 40}$
29. $\frac{6}{y^2 - 2y - 35} + \frac{4}{y^2 + 9y + 20}$
30. $\frac{12}{3y^2 - 10y - 8} - \frac{3}{y^2 - 6y + 8}$
31. $\frac{6}{2x^2 + 11x - 6} - \frac{8}{x^2 + 3x - 18}$
32. $\frac{2x}{4x^2 + 9x + 2} + \frac{3}{2x^2 - 8x - 24}$
33. $\frac{4x}{3x^2 + 3x - 18} - \frac{2x}{2x^2 + 11x + 15}$

34. **BIOLOGY** After a person eats something, the pH or acid level A of his or her mouth can be determined by the formula $A = \frac{20.4t}{t^2 + 36} + 6.5$, where t is the number of minutes that have elapsed since the food was eaten.

- a. Simplify the equation.
- b. What would the acid level be after 30 minutes?



35. **GEOMETRY** Both triangles in the figure at the right are equilateral. If the area of the smaller triangle is 200 square centimeters and the area of the larger triangle is 300 square centimeters, find the minimum distance from A to B in terms of x and y and simplify.



Examples 4–5 Simplify each expression.

36.
$$\frac{\frac{2}{x-3} + \frac{3x}{x^2-9}}{\frac{3}{x+3} - \frac{4x}{x^2-9}}$$

38.
$$\frac{\frac{5}{x+6} - \frac{2x}{2x-1}}{\frac{x}{2x-1} + \frac{4}{x+6}}$$

37.
$$\frac{\frac{4}{x+5} + \frac{9}{x-6}}{\frac{5}{x-6} - \frac{8}{x+5}}$$

39.
$$\frac{\frac{8}{x-9} - \frac{x}{3x+2}}{\frac{3}{3x+2} + \frac{4x}{x-9}}$$

40. **OIL PRODUCTION** Managers of an oil company have estimated that oil will be pumped from a certain well at a rate based on the function $R(x) = \frac{20}{x} + \frac{200x}{3x^2 + 20}$, where $R(x)$ is the rate of production in thousands of barrels per year x years after pumping begins.
- Simplify $R(x)$.
 - At what rate will oil be pumping from the well in 50 years?

Find the LCM of each set of polynomials.

41. $12xy^4, 14x^4y^2, 5xyz^3, 15x^5y^3$

42. $-6abc^2, 18a^2b^2, 15a^4c, 8b^3$

43. $x^2 - 3x - 28, 2x^2 + 9x + 4, x^2 - 16$

44. $x^2 - 5x - 24, x^2 - 9, 3x^2 + 8x - 3$

Simplify each expression.

45.
$$\frac{1}{12a} + 6 - \frac{3}{5a^2}$$

46.
$$\frac{5}{16y^2} - 4 - \frac{8}{3x^2y}$$

47.
$$\frac{5}{6x^2 + 46x - 16} + \frac{2}{6x^2 + 57x + 72}$$

48.
$$\frac{1}{8x^2 - 20x - 12} + \frac{4}{6x^2 + 27x + 12}$$

49.
$$\frac{x^2 + y^2}{x^2 - y^2} + \frac{y}{x+y} - \frac{x}{x-y}$$

50.
$$\frac{x^2 + x}{x^2 - 9x + 8} + \frac{4}{x-1} - \frac{3}{x-8}$$

51.
$$\frac{\frac{2}{a-1} + \frac{3}{a-4}}{\frac{6}{a^2 - 5a + 4}}$$

52.
$$\frac{\frac{1}{x} + \frac{1}{y}}{\left(\frac{1}{x} - \frac{1}{y}\right)(x+y)}$$

53. **GEOMETRY** An expression for the length of one rectangle is $\frac{x^2 - 9}{x - 2}$. The length of a similar rectangle is expressed as $\frac{x + 3}{x^2 - 4}$. What is the scale factor of the lengths of the two rectangles? Write in simplest form.
54. **CCSS MODELING** Cameron is taking a 20-mile kayaking trip. He travels half the distance at one rate. The rest of the distance he travels 2 miles per hour slower.
- If x represents the faster pace in miles per hour, write an expression that represents the time spent at that pace.
 - Write an expression for the amount of time spent at the slower pace.
 - Write an expression for the amount of time Cameron needed to complete the trip.

Find the slope of the line that passes through each pair of points.

55. $A\left(\frac{2}{p}, \frac{1}{2}\right)$ and $B\left(\frac{1}{3}, \frac{3}{p}\right)$

56. $C\left(\frac{1}{4}, \frac{4}{q}\right)$ and $D\left(\frac{5}{q}, \frac{1}{5}\right)$

57. $E\left(\frac{7}{w}, \frac{1}{7}\right)$ and $F\left(\frac{1}{7}, \frac{7}{w}\right)$

58. $G\left(\frac{6}{n}, \frac{1}{6}\right)$ and $H\left(\frac{1}{6}, \frac{6}{n}\right)$



59. **PHOTOGRAPHY** The focal length of a lens establishes the field of view of the camera. The shorter the focal length is, the larger the field of view. For a camera with a fixed focal length of 70 mm to focus on an object x mm from the lens, the film must be placed a distance y from the lens. This is represented by $\frac{1}{x} + \frac{1}{y} = \frac{1}{70}$.
- Express y as a function of x .
 - What happens to the focusing distance when the object is 70 mm away?
60. **PHARMACOLOGY** Two drugs are administered to a patient. The concentrations in the bloodstream of each are given by $f(t) = \frac{2t}{3t^2 + 9t + 6}$ and $g(t) = \frac{3t}{2t^2 + 6t + 4}$ where t is the time, in hours, after the drugs are administered.
- Add the two functions together to determine a function for the total concentration of drugs in the patient's bloodstream.
 - What is the concentration of drugs after 8 hours?
61. **DOPPLER EFFECT** Refer to the application at the beginning of the lesson. George is equidistant from two fire engines traveling toward him from opposite directions.
- Let x be the speed of the faster fire engine and y be the speed of the slower fire engine. Write and simplify a rational expression representing the difference in pitch between the two sirens according to George.
 - If one is traveling at 45 meters per second and the other is traveling at 70 meters per second, what is the difference in their pitches according to George? The speed of sound in air is 332 meters per second, and both engines have a siren with a pitch of 500 Hz.
62. **RESEARCH** A student studying learning behavior performed an experiment in which a rat was repeatedly sent through a maze. It was determined that the time it took the rat to complete the maze followed the rational function $T(x) = 4 + \frac{10}{x}$, where x represented the number of trials.
- What is the domain of the function?
 - Graph the function for $0 \leq x \leq 10$.
 - Make a table of the function for $x = 20, 50, 100, 200,$ and 400 .
 - If it were possible to have an infinite number of trials, what do you think would be the rat's best time? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

63. **CHALLENGE** Simplify $\frac{5x^{-2} - \frac{x+1}{x}}{\frac{4}{3-x^{-1}} + 6x^{-1}}$.
64. **CCSS ARGUMENTS** The sum of any two rational numbers is always a rational number. So, the set of rational numbers is said to be closed under addition. Determine whether the set of rational expressions is closed under addition, subtraction, multiplication, and division by a nonzero rational expression. Justify your reasoning.
65. **OPEN ENDED** Write three monomials with an LCM of $180a^4b^6c$.
66. **WRITING IN MATH** Write a how-to manual for adding rational expressions that have unlike denominators. How does this compare to adding rational numbers?

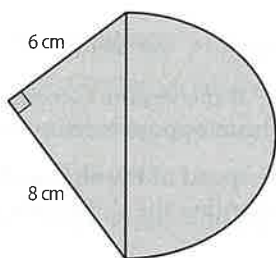


Standardized Test Practice

67. PROBABILITY A drawing is to be held to select the winner of a new bike. There are 100 seniors, 150 juniors, and 200 sophomores who had correct entries. The drawing will contain 3 tickets for each senior name, 2 for each junior, and 1 for each sophomore. What is the probability that a senior's ticket will be chosen?

- A $\frac{1}{8}$ C $\frac{2}{7}$
 B $\frac{2}{9}$ D $\frac{3}{8}$

68. SHORT RESPONSE Find the area of the figure.



69. SAT/ACT If Mauricio receives b books in addition to the number of books he had, he will have t times as many as he had originally. In terms of b and t , how many books did Mauricio have at the beginning?

- F $\frac{b}{t-1}$ J $\frac{b}{t}$
 G $\frac{b}{t+1}$ K $\frac{t}{b}$
 H $\frac{t+1}{b}$

70. If $\frac{2a}{a} + \frac{1}{a} = 4$, then $a =$ ____.

- A 2 C $\frac{1}{8}$
 B $\frac{1}{2}$ D $-\frac{1}{8}$

Spiral Review

Simplify each expression. (Lesson 8-1)

71. $\frac{-4ab}{21c} \cdot \frac{14c^2}{22a^2}$

72. $\frac{x^2 - y^2}{6y} \div \frac{x + y}{36y^2}$

73. $\frac{n^2 - n - 12}{n + 2} \div \frac{n - 4}{n^2 - 4n - 12}$

74. BIOLOGY Bacteria usually reproduce by a process known as *binary fission*. In this type of reproduction, one bacterium divides, forming two bacteria. Under ideal conditions, some bacteria reproduce every 20 minutes. (Lesson 7-8)

- Find the constant k for this type of bacteria under ideal conditions.
- Write the equation for modeling the exponential growth of this bacterium.

Graph each function. State the domain and range of each function. (Lesson 6-3)

75. $y = -\sqrt{2x + 1}$

76. $y = \sqrt{5x - 3}$

77. $y = \sqrt{x + 6} - 3$

78. $y = 5 - \sqrt{x + 4}$

79. $y = \sqrt{3x - 6} + 4$

80. $y = 2\sqrt{3 - 4x} + 3$

Solve each equation. State the number and type of roots. (Lesson 5-7)

81. $3x + 8 = 0$

82. $2x^2 - 5x + 12 = 0$

83. $x^3 + 9x = 0$

84. $x^4 - 81 = 0$

Skills Review

Graph each function.

85. $y = 4(x + 3)^2 + 1$

86. $y = -(x - 5)^2 - 3$

87. $y = \frac{1}{4}(x - 2)^2 + 4$

88. $y = \frac{1}{2}(x - 3)^2 - 5$

89. $y = x^2 + 6x + 2$

90. $y = x^2 - 8x + 18$



8-3

Graphing Reciprocal Functions

Then

- You graphed polynomial functions.

Now

- Determine properties of reciprocal functions.
- Graph transformations of reciprocal functions.

Why?

- The East High School Chorale wants to raise \$5000 to fund a trip to a national competition in Tampa. They have decided to sell candy bars. They will make a \$1 profit on each candy bar they sell, so they need to sell 5000 candy bars.

If c represents the number of candy bars each student has to sell and n represents the number of students, then $c = \frac{5000}{n}$.



New Vocabulary
reciprocal function
hyperbola

Common Core State Standards

Content Standards

A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

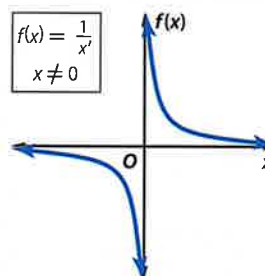
Mathematical Practices

2 Reason abstractly and quantitatively.

1 Vertical and Horizontal Asymptotes The function $c = \frac{5000}{n}$ is a reciprocal function. A **reciprocal function** has an equation of the form $f(x) = \frac{1}{a(x)}$, where $a(x)$ is a linear function and $a(x) \neq 0$.

Key Concept Parent Function of Reciprocal Functions

Parent function:	$f(x) = \frac{1}{x}$
Type of graph:	hyperbola
Domain and range:	all nonzero real numbers
Asymptotes:	$x = 0$ and $f(x) = 0$
Intercepts:	none
Not defined:	$x = 0$



The domain of a reciprocal function is limited to values for which the function is defined.

Functions:	$f(x) = \frac{-3}{x+2}$	$g(x) = \frac{4}{x-5}$	$h(x) = \frac{3}{x}$
Not defined at:	$x = -2$	$x = 5$	$x = 0$

Example 1 Limitations on Domain

Determine the value of x for which $f(x) = \frac{3}{2x+5}$ is not defined.

Find the value for which the denominator of the expression equals 0.

$$\frac{3}{2x+5} \rightarrow 2x+5=0$$

$$x = -\frac{5}{2} \quad \text{The function is undefined for } x = -\frac{5}{2}.$$

Guided Practice

Determine the value of x for which each function is not defined.

1A. $f(x) = \frac{2}{x-1}$

1B. $f(x) = \frac{7}{3x+2}$



The graphs of reciprocal functions may have breaks in continuity for excluded values. Some may have an asymptote, which is a line that the graph of the function approaches.

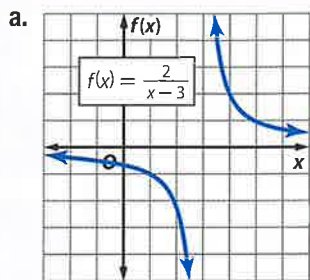


StudyTip

CCSS Structure Vertical asymptotes show where a function is undefined, while horizontal asymptotes show the end behavior of a graph.

Example 2 Determine Properties of Reciprocal Functions

Identify the asymptotes, domain, and range of each function.



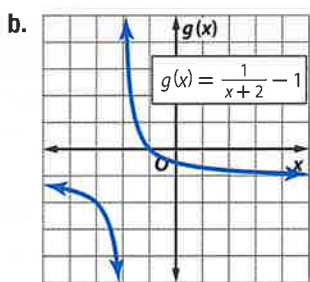
Identify x -values for which $f(x)$ is undefined.

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

$f(x)$ is not defined when $x = 3$. So there is an asymptote at $x = 3$.

From $x = 3$, as x -values decrease, $f(x)$ -values approach 0, and as x -values increase, $f(x)$ -values approach 0. So there is an asymptote at $f(x) = 0$.

The domain is all real numbers not equal to 3 and the range is all real numbers not equal to 0.



Identify x -values for which $g(x)$ is undefined.

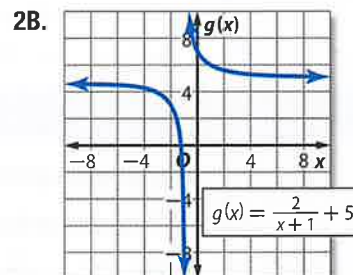
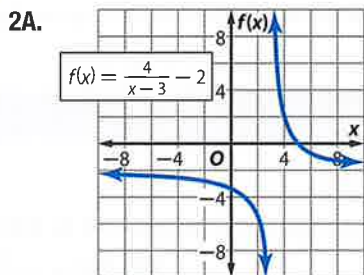
$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

$g(x)$ is not defined when $x = -2$. So there is an asymptote at $x = -2$.

From $x = -2$, as x -values decrease, $g(x)$ -values approach -1 , and as x -values increase, $g(x)$ -values approach -1 . So there is an asymptote at $g(x) = -1$.

The domain is all real numbers not equal to -2 and the range is all real numbers not equal to -1 .

Guided Practice



2 Transformations of Reciprocal Functions The same techniques used to transform the graphs of other functions you have studied can be applied to the graphs of reciprocal functions. In Example 2, note that the asymptotes have been moved along with the graphs of the functions.

StudyTip

Asymptotes The asymptotes of a reciprocal function move with the graph of the function and intersect at (h, k) .

KeyConcept Transformations of Reciprocal Functions

$$f(x) = \frac{a}{x-h} + k$$

h – Horizontal Translation

h units right if h is positive
 $|h|$ units left if h is negative
 The vertical asymptote is at $x = h$.

k – Vertical Translation

k units up if k is positive
 $|k|$ units down if k is negative
 The horizontal asymptote is at $f(x) = k$.

a – Orientation and Shape

If $a < 0$, the graph is reflected across the x -axis.
 If $|a| > 1$, the graph is stretched vertically.
 If $0 < |a| < 1$, the graph is compressed vertically.

Example 3 Graph Transformations



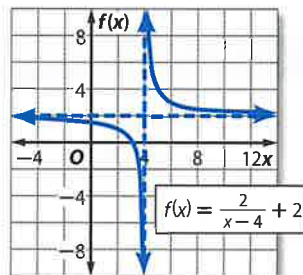
Graph each function. State the domain and range.

a. $f(x) = \frac{2}{x-4} + 2$

This represents a transformation of the graph of $f(x) = \frac{1}{x}$.

- $a = 2$: The graph is stretched vertically.
- $h = 4$: The graph is translated 4 units right. There is an asymptote at $x = 4$.
- $k = 2$: The graph is translated 2 units up. There is an asymptote at $f(x) = 2$.

Domain: $\{x \mid x \neq 4\}$ Range: $\{f(x) \mid f(x) \neq 2\}$

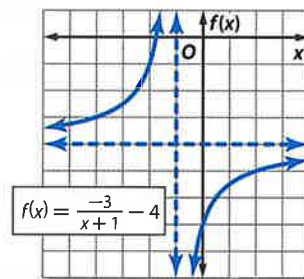


b. $f(x) = \frac{-3}{x+1} - 4$

This represents a transformation of the graph of $f(x) = \frac{1}{x}$.

- $a = -3$: The graph is stretched vertically and reflected across the x -axis.
- $h = -1$: The graph is translated 1 unit left. There is an asymptote at $x = -1$.
- $k = -4$: The graph is translated 4 units down. There is an asymptote at $f(x) = -4$.

Domain: $\{x \mid x \neq -1\}$ Range: $\{f(x) \mid f(x) \neq -4\}$

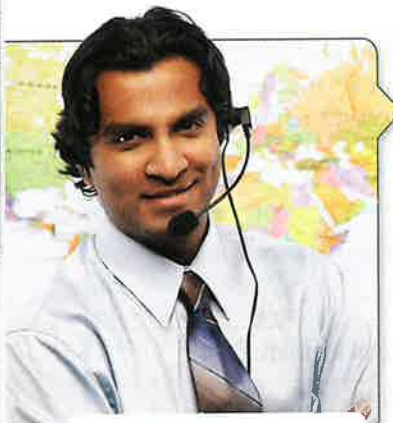


GuidedPractice

3A. $f(x) = \frac{-2}{x+4} + 1$

3B. $g(x) = \frac{1}{3(x-1)} - 2$





Real-World Career

Travel Agent Travel agents assess individual and business needs to help make the best possible travel arrangements. They may specialize by type of travel, such as leisure or business, or by destination, such as Europe or Africa. A high school diploma is required, and vocational training is preferred.

Real-World Example 4 Write Equations

TRAVEL An airline has a daily nonstop flight between Los Angeles, California, and Sydney, Australia. A one-way trip is about 7500 miles.

- a. Write an equation to represent the travel time from Los Angeles to Sydney as a function of flight speed. Then graph the equation.

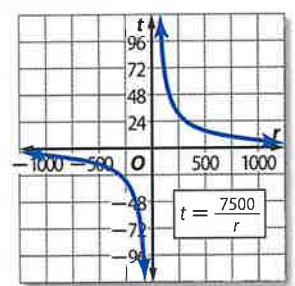
Solve $rt = d$ for t .

$rt = d$ Original formula

$t = \frac{d}{r}$ Divide each side by r .

$t = \frac{7500}{r}$ $d = 7500$

Graph $t = \frac{7500}{r}$.



- b. Explain any limitations to the range or domain in this situation.

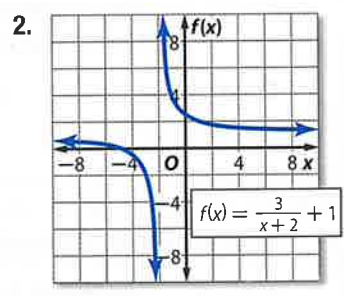
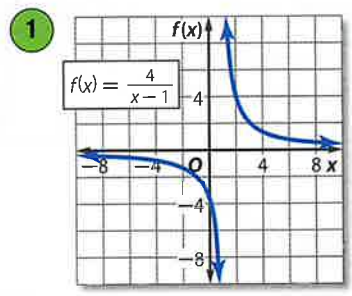
In this situation, the range and domain are limited to all real numbers greater than zero because negative values do not make sense. There will be further restrictions to the domain because the aircraft has minimum and maximum speeds at which it can travel.

Guided Practice

4. **HOMECOMING DANCE** The junior and senior class officers are sponsoring a homecoming dance. The total cost for the facilities and catering is \$45 per person plus a \$2500 deposit. Write and graph an equation to represent the average cost per person. Then explain any limitations to the domain and range.

Check Your Understanding = Step-by-Step Solutions begin on page R14.

Examples 1–2 Identify the asymptotes, domain, and range of each function.



Example 3 Graph each function. State the domain and range.

3. $f(x) = \frac{5}{x}$

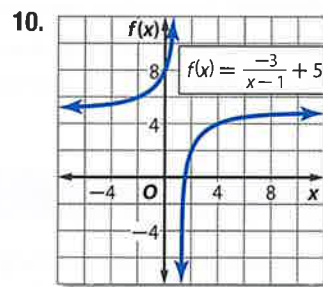
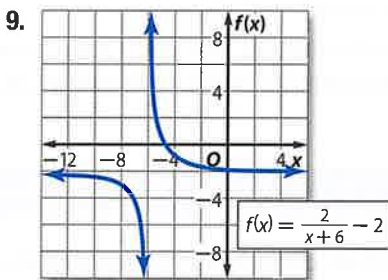
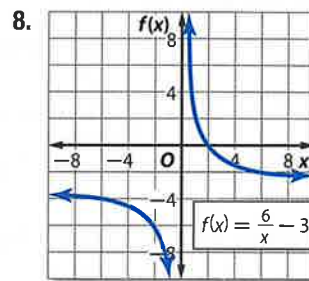
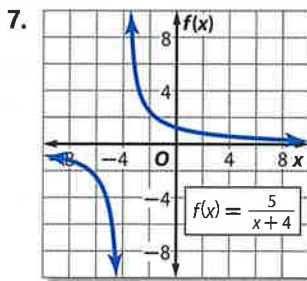
4. $f(x) = \frac{2}{x+3}$

5. $f(x) = \frac{-1}{x-2} + 4$

Example 4 6. **CCSS SENSE-MAKING** A group of friends plans to get their youth group leader a gift certificate for a day at a spa. The certificate costs \$150.

- a. If c represents the cost for each friend and f represents the number of friends, write an equation to represent the cost to each friend as a function of how many friends give.
- b. Graph the function.
- c. Explain any limitations to the range or domain in this situation.

Examples 1–2 Identify the asymptotes, domain, and range of each function.



Example 3 Graph each function. State the domain and range.

11. $f(x) = \frac{3}{x}$

12. $f(x) = \frac{-4}{x+2}$

13. $f(x) = \frac{2}{x-6}$

14. $f(x) = \frac{6}{x} - 5$

15. $f(x) = \frac{2}{x} + 3$

16. $f(x) = \frac{8}{x}$

17. $f(x) = \frac{-2}{x-5}$

18. $f(x) = \frac{3}{x-7} - 8$

19. $f(x) = \frac{9}{x+3} + 6$

20. $f(x) = \frac{8}{x+3}$

21. $f(x) = \frac{-6}{x+4} - 2$

22. $f(x) = \frac{-5}{x-2} + 2$

Example 4 23. **CYCLING** Marina's New Year's resolution is to ride her bike 5000 miles.

- If m represents the mileage Marina rides each day and d represents the number of days, write an equation to represent the mileage each day as a function of the number of days that she rides.
- Graph the function.
- If she rides her bike every day of the year, how many miles should she ride each day to meet her goal?

24. **CCSS MODELING** Parker has 200 grams of an unknown liquid. Knowing the density will help him discover what type of liquid this is.

- Density of a liquid is found by dividing the mass by the volume. Write an equation to represent the density of this unknown as a function of volume.
- Graph the function.
- From the graph, identify the asymptotes, domain, and range of the function.

Graph each function. State the domain and range.

25. $f(x) = \frac{3}{2x-4}$

26. $f(x) = \frac{5}{3x}$

27. $f(x) = \frac{2}{4x+1}$

28. $f(x) = \frac{1}{2x+3}$



29 BASEBALL The distance from the pitcher's mound to home plate is 60.5 feet.

- If r represents the speed of the pitch and t represents the time it takes the ball to get to the plate, write an equation to represent the speed as a function of time.
- Graph the function.
- If a two-seam fastball reaches the plate in 0.48 second, what was its speed?

Graph each function. State the domain and range, and identify the asymptotes.

30. $f(x) = \frac{-3}{x+7} - 1$

31. $f(x) = \frac{-4}{x+2} - 5$

32. $f(x) = \frac{6}{x-1} + 2$

33. $f(x) = \frac{2}{x-4} + 3$

34. $f(x) = \frac{-7}{x-8} - 9$

35. $f(x) = \frac{-6}{x-7} - 8$

36. FINANCIAL LITERACY Lawanda's car went 440 miles on one tank of gas.

- If g represents the number of miles to the gallon that the car gets and t represents the size of the gas tank, write an equation to represent the miles to the gallon as a function of tank size.
- Graph the function.
- How many miles does the car get per gallon if it has a 15-gallon tank?

37. MULTIPLE REPRESENTATIONS In this problem you will investigate the similarities and differences between power functions with positive and negative exponents.

- TABULAR** Make a table of values for $a(x) = x^2$, $b(x) = x^{-2}$, $c(x) = x^3$, and $d(x) = x^{-3}$.
- GRAPHICAL** Graph $a(x)$ and $b(x)$ on the same coordinate plane.
- VERBAL** Compare the domain, range, end behavior, and behavior at $x = 0$ for $a(x)$ and $b(x)$.
- GRAPHICAL** Graph $c(x)$ and $d(x)$ on the same coordinate plane.
- VERBAL** Compare the domain, range, end behavior, and behavior at $x = 0$ for $c(x)$ and $d(x)$.
- ANALYTICAL** What conclusions can you make about the similarities and differences between power functions with positive and negative exponents?

H.O.T. Problems Use Higher-Order Thinking Skills

38. OPEN ENDED Write a reciprocal function for which the graph has a vertical asymptote at $x = -4$ and a horizontal asymptote at $f(x) = 6$.

39. REASONING Compare and contrast the graphs of each pair of equations.

a. $y = \frac{1}{x}$ and $y - 7 = \frac{1}{x}$ b. $y = \frac{1}{x}$ and $y = 4\left(\frac{1}{x}\right)$ c. $y = \frac{1}{x}$ and $y = \frac{1}{x+5}$

d. Without making a table of values, use what you observed in parts a–c to sketch a graph of $y - 7 = 4\left(\frac{1}{x+5}\right)$.

40. CCSS ARGUMENTS Find the function that does not belong. Explain.

$$f(x) = \frac{3}{x+1}$$

$$g(x) = \frac{x+2}{x^2+1}$$

$$h(x) = \frac{5}{x^2+2x+1}$$

$$j(x) = \frac{20}{x-7}$$

41. CHALLENGE Write two different reciprocal functions with graphs having the same vertical and horizontal asymptotes. Then graph the functions.

42. WRITING IN MATH Refer to the beginning of the lesson. Explain how rational functions can be used in fundraising. Explain why only part of the graph is meaningful in the context of the problem.



Standardized Test Practice

43. **SHORT RESPONSE** What is the value of $(x + y)(x + y)$ if $xy = -3$ and $x^2 + y^2 = 10$?

44. **GRIDDED RESPONSE** If $x = 2y$, $y = 4z$, $2z = w$, and $w \neq 0$, then $\frac{x}{w} = \underline{\hspace{2cm}}$.

45. If $c = 1 + \frac{1}{d}$ and $d > 1$, then c could equal $\underline{\hspace{2cm}}$.

A $\frac{5}{7}$

C $\frac{15}{7}$

B $\frac{9}{7}$

D $\frac{19}{7}$

46. **SAT/ACT** A car travels m miles at the rate of t miles per hour. How many hours does the trip take?

F $\frac{m}{t}$

J $\frac{t}{m}$

G $m - t$

K $t - m$

H mt

47. If $-1 < a < b < 0$, then which of the following has the greatest value?

A $a - b$

C $a + b$

B $b - a$

D $2b - a$

Spiral Review

48. **BUSINESS** A small corporation decides that 8% of its profits will be divided among its six managers. There are two sales managers and four nonsales managers. Fifty percent will be split equally among all six managers. The other 50% will be split among the four nonsales managers. Let p represent the profits. (Lesson 8-2)

- Write an expression to represent the share of the profits each nonsales manager will receive.
- Simplify this expression.
- Write an expression in simplest form to represent the share of the profits each sales manager will receive.

Simplify each expression. (Lesson 8-1)

49. $\frac{\frac{p^3}{2n}}{\frac{p^2}{4n}}$

50. $\frac{\frac{m+q}{5}}{\frac{m^2+q^2}{5}}$

51. $\frac{\frac{x+y}{2x-y}}{\frac{x+y}{2x+y}}$

Graph each function. State the domain and range. (Lesson 7-1)

52. $y = 2(3)^x$

53. $y = 5(2)^x$

54. $y = 0.5(4)^x$

55. $y = 4\left(\frac{1}{3}\right)^x$

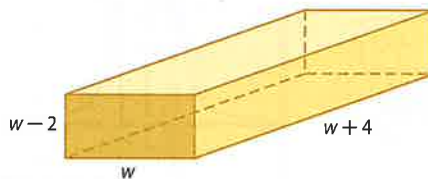
Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. (Lesson 6-1)

56. $f(x) = x + 9$
 $g(x) = x - 9$

57. $f(x) = 2x - 3$
 $g(x) = 4x + 9$

58. $f(x) = 2x^2$
 $g(x) = 8 - x$

59. **GEOMETRY** The width of a rectangular prism is w centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism. (Lesson 5-5)



Skills Review

Graph each polynomial function. Estimate the x -coordinates at which the relative maxima and relative minima occur. State the domain and range for each function.

60. $f(x) = x^3 + 2x^2 - 3x - 5$

61. $f(x) = x^4 - 8x^2 + 10$



WatchOut!

Zeros vs. Vertical Asymptotes

Zeros of rational functions occur at the values that make the numerator equal to zero. Vertical asymptotes occur at the values that make the denominator equal to zero.

StudyTip

Graphing Calculator The TABLE feature of a graphing calculator can be used to calculate decimal values for x and y .

The asymptotes of a rational function can be used to draw the graph of the function. Additionally, the asymptotes can be used to divide a graph into regions to find ordered pairs on the graph.



Example 1 Graph with no Horizontal Asymptote

$$\text{Graph } f(x) = \frac{x^3}{x-1}.$$

Step 1 Find the zeros.

$$x^3 = 0 \quad \text{Set } a(x) = 0.$$

$$x = 0 \quad \text{Take the cube root of each side.}$$

There is a zero at $x = 0$.

Step 2 Draw the asymptotes.

Find the vertical asymptote.

$$x - 1 = 0 \quad \text{Set } b(x) = 0.$$

$$x = 1 \quad \text{Add 1 to each side.}$$

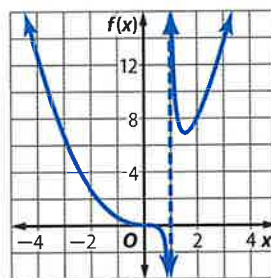
There is a vertical asymptote at $x = 1$.

The degree of the numerator is greater than the degree of the denominator. So, there is no horizontal asymptote.

Step 3 Draw the graph.

Use a table to find ordered pairs on the graph. Then connect the points.

x	$f(x)$
-3	6.75
-2	2.67
-1	0.5
0	0
0.5	-0.25
1.5	6.75
2	8
3	13.5



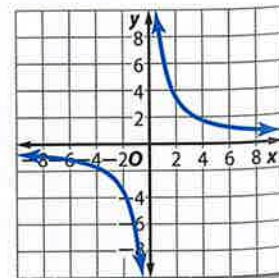
GuidedPractice

Graph each function.

1A. $f(x) = \frac{x^2 - x - 6}{x + 1}$

1B. $f(x) = \frac{(x + 1)^3}{(x + 2)^2}$

In the real world, sometimes values on the graph of a rational function are not meaningful. In the graph at the right, x -values such as time, distance, and number of people cannot be negative in the context of the problem. So, you do not even need to consider that portion of the graph.



Real-World Example 2 Use Graphs of Rational Functions

AVERAGE SPEED A boat traveled upstream at r_1 miles per hour. During the return trip to its original starting point, the boat traveled at r_2 miles per hour. The average speed for the entire trip R is given by the formula $R = \frac{2r_1r_2}{r_1 + r_2}$.

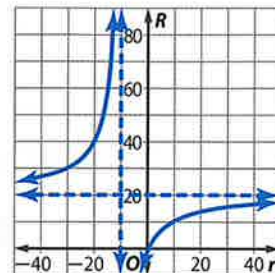
- a. Let r_1 be the independent variable, and let R be the dependent variable. Draw the graph if $r_2 = 10$ miles per hour.

The function is $R = \frac{2r_1(10)}{r_1 + (10)}$ or $R = \frac{20r_1}{r_1 + 10}$.

The vertical asymptote is $r_1 = -10$.

Graph the vertical asymptote and the function.

Notice that the horizontal asymptote is $R = 20$.



- b. What is the R -intercept of the graph?

The R -intercept is 0.

- c. What domain and range values are meaningful in the context of the problem?

In the problem context, speeds are nonnegative values. Therefore, only values of r_1 greater than or equal to 0 and values of R between 0 and 20 are meaningful.

Guided Practice

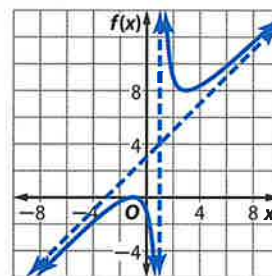
2. **SALARIES** A company uses the formula $S(x) = \frac{45x + 25}{x + 1}$ to determine the salary in thousands of dollars of an employee during his x th year. Graph $S(x)$. What domain and range values are meaningful in the context of the problem? What is the meaning of the horizontal asymptote for the graph?

2 Oblique Asymptotes and Point Discontinuity An **oblique asymptote**, sometimes called a *slant asymptote*, is an asymptote that is neither horizontal nor vertical.

Key Concept Oblique Asymptotes

Words If $f(x) = \frac{a(x)}{b(x)}$, $a(x)$ and $b(x)$ are polynomial functions with no common factors other than 1 and $b(x) \neq 0$, then $f(x)$ has an oblique asymptote if the degree of $a(x)$ minus the degree of $b(x)$ equals 1. The equation of the asymptote is $f(x) = \frac{a(x)}{b(x)}$ with no remainder.

Example $f(x) = \frac{x^4 + 3x^3}{x^3 - 1}$
 Vertical asymptote: $x = 1$
 Oblique asymptote: $f(x) = x + 3$



Real-World Career

U.S. Coast Guard Boatswain's Mate

The most versatile member of the U.S. Coast Guard's operational team is the boatswain's mate. BMs are capable of performing almost any task. Training for BMs is accomplished through 12 weeks of intensive training.



StudyTip

Oblique Asymptotes

Oblique asymptotes occur for rational functions that have a numerator polynomial that is one degree higher than the denominator polynomial.

Example 3 Determine Oblique Asymptotes

Graph $f(x) = \frac{x^2 + 4x + 4}{2x - 1}$.

Step 1 Find the zeros.

$$\begin{aligned}
 x^2 + 4x + 4 &= 0 && \text{Set } a(x) = 0. \\
 (x + 2)^2 &= 0 && \text{Factor.} \\
 x + 2 &= 0 && \text{Take the square root of each side.} \\
 x &= -2 && \text{Subtract 2 from each side.}
 \end{aligned}$$

There is a zero at $x = -2$.

Step 2 Find the asymptotes.

$$\begin{aligned}
 2x - 1 &= 0 && \text{Set } b(x) = 0. \\
 2x &= 1 && \text{Add 1 to each side.} \\
 x &= \frac{1}{2} && \text{Divide each side by 2.}
 \end{aligned}$$

There is a vertical asymptote at $x = \frac{1}{2}$.

The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote.

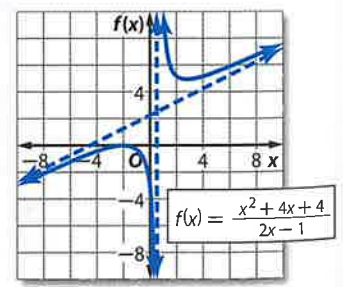
Divide the numerator by the denominator to determine the equation of the oblique asymptote.

The equation of the asymptote is the quotient excluding any remainder.

Thus, the oblique asymptote is the line $f(x) = \frac{1}{2}x + \frac{9}{4}$.

$$\begin{array}{r}
 \frac{1}{2}x + \frac{9}{4} \\
 2x - 1 \overline{) x^2 + 4x + 4} \\
 \underline{(-)x^2 - \frac{1}{2}x} \\
 \frac{9}{2}x + 4 \\
 \underline{(-)\frac{9}{2}x - \frac{9}{4}} \\
 \phantom{\frac{9}{2}x} \frac{25}{4}
 \end{array}$$

Step 3 Draw the asymptotes, and then use a table of values to graph the function.



GuidedPractice

Graph each function.

3A. $f(x) = \frac{x^2}{x - 2}$

3B. $f(x) = \frac{x^3 - 1}{x^2 - 4}$

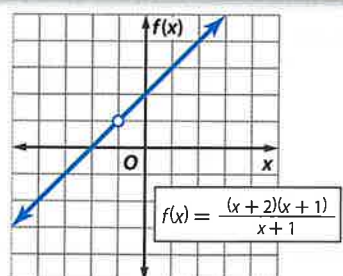
In some cases, graphs of rational functions may have **point discontinuity**, which looks like a hole in the graph. This is because the function is undefined at that point.

KeyConcept Point Discontinuity



Words If $f(x) = \frac{a(x)}{b(x)}$, $b(x) \neq 0$, and $x - c$ is a factor of both $a(x)$ and $b(x)$, then there is a point discontinuity at $x = c$.

Example $f(x) = \frac{(x+2)(x+1)}{x+1}$
 $= x + 2; x \neq -1$



WatchOut!

Holes Remember that a common factor in the numerator and denominator can signal a hole.

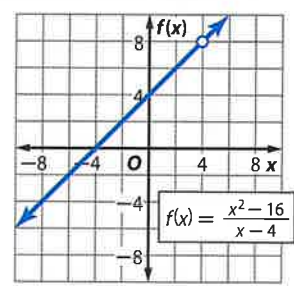
Example 4 Graph with Point Discontinuity



Graph $f(x) = \frac{x^2 - 16}{x - 4}$.

Notice that $\frac{x^2 - 16}{x - 4} = \frac{(x+4)\cancel{(x-4)}}{\cancel{x-4}}$ or $x + 4$.

Therefore, the graph of $f(x) = \frac{x^2 - 16}{x - 4}$ is the graph of $f(x) = x + 4$ with a hole at $x = 4$.



GuidedPractice

Graph each function.

4A. $f(x) = \frac{x^2 + 4x - 5}{x + 5}$

4B. $f(x) = \frac{x^3 + 2x^2 - 9x - 18}{x^2 - 9}$

Check Your Understanding

= Step-by-Step Solutions begin on page R14.



Example 1 Graph each function.

1. $f(x) = \frac{x^4 - 2}{x^2 - 1}$

2. $f(x) = \frac{x^3}{x + 2}$

Example 2

3. **CCSS REASONING** Eduardo is a kicker for his high school football team. So far this season, he has made 7 out of 11 field goals. He would like to improve his field goal percentage. If he can make x consecutive field goals, his field goal percentage can be determined using the function $P(x) = \frac{7+x}{11+x}$.
- Graph the function.
 - What part of the graph is meaningful in the context of this problem?
 - Describe the meaning of the intercept of the vertical axis.
 - What is the equation of the horizontal asymptote? Explain its meaning with respect to Eduardo's field goal percentage.

Examples 3-4 Graph each function.

4. $f(x) = \frac{6x^2 - 3x + 2}{x}$

5. $f(x) = \frac{x^2 + 8x + 20}{x + 2}$

6. $f(x) = \frac{x^2 - 4x - 5}{x + 1}$

7. $f(x) = \frac{x^2 + x - 12}{x + 4}$



Example 1 Graph each function.

$$8. f(x) = \frac{x^4}{6x + 12}$$

$$10. f(x) = \frac{x^4 - 16}{x^2 - 1}$$

$$9. f(x) = \frac{x^3}{8x - 4}$$

$$11. f(x) = \frac{x^3 + 64}{16x - 24}$$

- Example 2** 12. **SCHOOL SPIRIT** As president of Student Council, Brandy is getting T-shirts made for a pep rally. Each T-shirt costs \$9.50, and there is a set-up fee of \$75. The student council plans to sell the shirts, but each of the 15 council members will get one for free.
- Write a function for the average cost of a T-shirt to be sold. Graph the function.
 - What is the average cost if 200 shirts are ordered? if 500 shirts are ordered?
 - How many T-shirts must be ordered to bring the average cost under \$9.75?

Examples 2–3 Graph each function.

$$13. f(x) = \frac{x}{x + 2}$$

$$15. f(x) = \frac{4}{(x - 2)^2}$$

$$17. f(x) = \frac{1}{(x + 4)^2}$$

$$19. f(x) = \frac{(x - 4)^2}{x + 2}$$

$$21. f(x) = \frac{x^3 + 1}{x^2 - 4}$$

$$23. f(x) = \frac{3x^2 + 8}{2x - 1}$$

$$25. f(x) = \frac{x^4 - 2x^2 + 1}{x^3 + 2}$$

$$14. f(x) = \frac{5}{(x - 1)(x + 4)}$$

$$16. f(x) = \frac{x - 3}{x + 1}$$

$$18. f(x) = \frac{2x}{(x + 2)(x - 5)}$$

$$20. f(x) = \frac{(x + 3)^2}{x - 5}$$

$$22. f(x) = \frac{4x^3}{2x^2 + x - 1}$$

$$24. f(x) = \frac{2x^2 + 5}{3x + 4}$$

$$26. f(x) = \frac{x^4 - x^2 - 12}{x^3 - 6}$$

27. **CCSS PERSEVERANCE** The current in amperes in an electrical circuit with three resistors in a series is given by the equation $I = \frac{V}{R_1 + R_2 + R_3}$, where V is the voltage in volts in the circuit and R_1 , R_2 , and R_3 are the resistances in ohms of the three resistors.
- Let R_1 be the independent variable, and let I be the dependent variable. Graph the function if $V = 120$ volts, $R_2 = 25$ ohms, and $R_3 = 75$ ohms.
 - Give the equation of the vertical asymptote and the R_1 - and I -intercepts of the graph.
 - Find the value of I when the value of R_1 is 140 ohms.
 - What domain and range values are meaningful in the context of the problem?

Example 4 Graph each function.

$$28. f(x) = \frac{x^2 - 2x - 8}{x - 4}$$

$$30. f(x) = \frac{x^2 - 25}{x + 5}$$

$$32. f(x) = \frac{(x - 4)(x^2 - 4)}{x^2 - 6x + 8}$$

$$34. f(x) = \frac{3x^4 + 6x^3 + 3x^2}{x^2 + 2x + 1}$$

$$29. f(x) = \frac{x^2 + 4x - 12}{x - 2}$$

$$31. f(x) = \frac{x^2 - 64}{x - 8}$$

$$33. f(x) = \frac{(x + 5)(x^2 + 2x - 3)}{x^2 + 8x + 15}$$

$$35. f(x) = \frac{2x^4 + 10x^3 + 12x^2}{x^2 + 5x + 6}$$



36. **BUSINESS** Liam purchased a snow plow for \$4500 and plows the parking lots of local businesses. Each time he plows a parking lot, he incurs a cost of \$50 for gas and maintenance.
- Write and graph the rational function representing his average cost per customer as a function of the number of parking lots.
 - What are the asymptotes of the graph?
 - Why is the first quadrant in the graph the only relevant quadrant?
 - How many total parking lots does Liam need to plow for his average cost per parking lot to be less than \$80?
37. **FINANCIAL LITERACY** Kristina bought a new cell phone with Internet access. The phone cost \$150, and her monthly usage charge is \$30 plus \$10 for the Internet access.
- Write and graph the rational function representing her average monthly cost as a function of the number of months Kristina uses the phone.
 - What are the asymptotes of the graph?
 - Why is the first quadrant in the graph the only relevant quadrant?
 - After how many months will the average monthly charge be \$45?
38. **CCSS SENSE-MAKING** Alana plays softball for Centerville High School. So far this season she has gotten a hit 4 out of 12 times at bat. She is determined to improve her batting average. If she can get x consecutive hits, her batting average can be determined using $B(x) = \frac{4+x}{12+x}$.
- Graph the function.
 - What part of the graph is meaningful in the context of the problem?
 - Describe the meaning of the intercept of the vertical axis.
 - What is the equation of the horizontal asymptote? Explain its meaning with respect to Alana's batting average.

Graph each function.

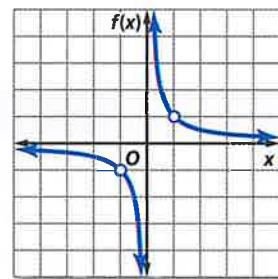
39. $f(x) = \frac{x+1}{x^2+6x+5}$

40. $f(x) = \frac{x^2-10x-24}{x+2}$

41. $f(x) = \frac{6x^2+4x+2}{x+2}$

H.O.T. Problems Use Higher-Order Thinking Skills

42. **OPEN ENDED** Sketch the graph of a rational function with a horizontal asymptote $y = 1$ and a vertical asymptote $x = -2$.
43. **CHALLENGE** Compare and contrast $g(x) = \frac{x^2-1}{x(x^2-2)}$ and $f(x)$ shown at the right.
44. **REASONING** What is the difference between the graphs of $f(x) = x - 2$ and $g(x) = \frac{(x+3)(x-2)}{x+3}$?
45. **PROOF** A rational function has an equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$. Show that $f(x) = \frac{x}{a-b} + c$ is a rational function.



46. **WRITING IN MATH** How can factoring be used to determine the vertical asymptotes or point discontinuity of a rational function?



Standardized Test Practice

47. PROBABILITY Of the 6 courses offered by the music department at her school, Kaila must choose exactly 2 of them. How many different combinations of 2 courses are possible for Kaila if there are no restrictions on which 2 courses she can choose?

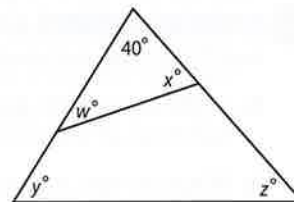
- A 48 C 15
B 18 D 12

48. The projected sales of a game cartridge is given by the function $S(p) = \frac{3000}{2p + a}$, where $S(p)$ is the number of cartridges sold, in thousands, p is the price per cartridge, in dollars, and a is a constant. If 100,000 cartridges are sold at \$10 per cartridge, how many cartridges will be sold at \$20 per cartridge?

- F 20,000 H 60,000
G 50,000 J 150,000

49. GRIDDED RESPONSE Five distinct points lie in a plane such that 3 of the points are on line ℓ and 3 of the points are on a different line m . What is the total number of lines that can be drawn so that each line passes through exactly 2 of these 5 points?

50. GEOMETRY In the figure below, what is the value of $w + x + y + z$?



- A 140 C 320
B 280 D 360

Spiral Review

Graph each function. State the domain and range. (Lesson 8-3)

51. $f(x) = \frac{-5}{x+2}$

52. $f(x) = \frac{4}{x-1} - 3$

53. $f(x) = \frac{1}{x+6} + 1$

Simplify each expression. (Lesson 8-2)

54. $\frac{m}{m^2-4} + \frac{2}{3m+6}$

55. $\frac{y}{y+3} - \frac{6y}{y^2-9}$

56. $\frac{5}{x^2-3x-28} + \frac{7}{2x-14}$

57. $\frac{d-4}{d^2+2d-8} - \frac{d+2}{d^2-16}$

Simplify each expression. (Lesson 6-6)

58. $y^{\frac{5}{3}} \cdot y^{\frac{7}{3}}$

59. $x^{\frac{3}{4}} \cdot x^{\frac{9}{4}}$

60. $\left(\frac{1}{b^3}\right)^{\frac{3}{5}}$

61. $\left(a^{-\frac{2}{3}}\right)^{-\frac{1}{6}}$

Skills Review

62. TRAVEL Mr. and Mrs. Wells are taking their daughter to college. The table shows their distances from home after various amounts of time.

- a. Find the average rate of change in their distances from home between 1 and 3 hours after leaving home.
b. Find the average rate of change in their distances from home between 0 and 5 hours after leaving home.

Time (h)	Distance (mi)
0	0
1	55
2	110
3	165
4	165
5	225



8-4 Graphing Technology Lab

Graphing Rational Functions



A TI-83/84 Plus graphing calculator can be used to explore graphs of rational functions. These graphs have some features that never appear in the graphs of polynomial functions.

Activity 1 Graph with Asymptotes



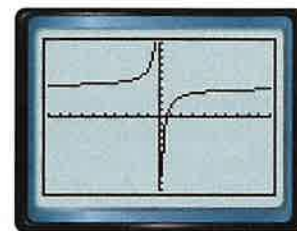
Graph $y = \frac{8x - 5}{2x}$ in the standard viewing window. Find the equations of any asymptotes. State the domain and range of the function.

Step 1 Enter the equation in the Y= list, and then graph.

KEYSTROKES: $\boxed{Y=}$ $\boxed{(}$ $\boxed{8}$ $\boxed{X,T,\theta,n}$ $\boxed{-}$ $\boxed{5}$ $\boxed{)}$ $\boxed{\div}$
 $\boxed{(}$ $\boxed{2}$ $\boxed{X,T,\theta,n}$ $\boxed{)}$ \boxed{ZOOM} $\boxed{6}$

Step 2 Examine the graph.

By looking at the equation, we can determine that if $x = 0$, the function is undefined. The equation of the vertical asymptote is $x = 0$. Notice what happens to the y -values as x grows larger and as x gets smaller. The y -values approach 4. So, the equation for the horizontal asymptote is $y = 4$. The domain is $\{x \mid x \neq 0\}$, and the range is all real numbers.



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

Activity 2 Graph with Point Discontinuity

Graph $y = \frac{x^2 - 16}{x + 4}$ in the window $[-5, 4.4]$ by $[-10, 2]$ with scale factors of 1.

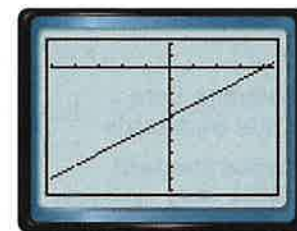
Step 1 Because the function is not continuous, put the calculator in dot mode.

KEYSTROKES: $\boxed{\text{MODE}}$ $\boxed{\blacktriangledown}$ $\boxed{\blacktriangledown}$ $\boxed{\blacktriangledown}$ $\boxed{\blacktriangledown}$ $\boxed{\blacktriangleright}$ $\boxed{\text{ENTER}}$

Step 2 Examine the graph.

This graph looks like a line with a break in continuity at $x = -4$. This happens because the denominator is 0 when $x = -4$. Therefore, the function is undefined when $x = -4$.

If you TRACE along the graph, when you come to $x = -4$, you will see that there is no corresponding y -value.



$[-5, 4.4]$ scl: 1 by $[-10, 2]$ scl: 1

Exercises

Use a graphing calculator to graph each function. Write the x -coordinates of any points of discontinuity and/or the equations of any asymptotes. State the domain and range.

1. $f(x) = \frac{1}{x}$

2. $f(x) = \frac{x}{x+2}$

3. $f(x) = \frac{2}{x-4}$

4. $f(x) = \frac{2x}{3x-6}$

5. $f(x) = \frac{4x+2}{x-1}$

6. $f(x) = \frac{x^2-9}{x+3}$

8-5 Variation Functions

Then

- You wrote and graphed linear equations.

Now

- 1 Recognize and solve direct and joint variation problems.
- 2 Recognize and solve inverse and combined variation problems.

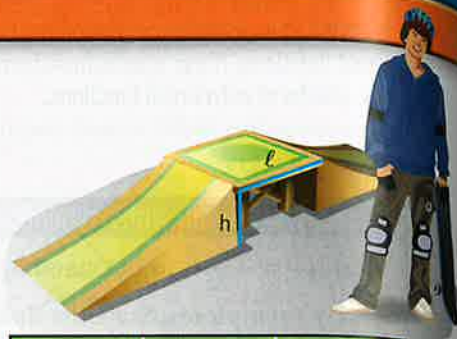
Why?

- While building skateboard ramps, Yu determined that the best ramps were the ones in which the length of the top of the ramp was 1.5 times as long as the height of the ramp.

As shown in the table, the length of the top of the ramp depends on the height of a ramp. The length increases as the height increases, but the ratio remains the same, or is *constant*.

The equation $\frac{\ell}{h} = 1.5$ can be written as $\ell = 1.5h$.

The length *varies directly* with the height of the ramp.



Length (ℓ)	Height (h)	Ratio $\frac{\ell}{h}$
3	2	1.5
6	4	1.5
9	6	1.5
12	8	1.5



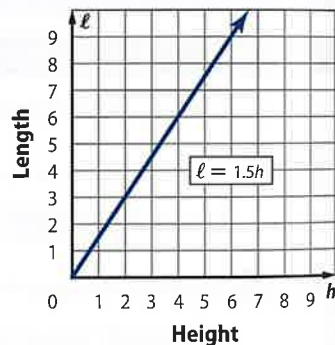
New Vocabulary

- direct variation
- constant of variation
- joint variation
- inverse variation
- combined variation

1 Direct Variation and Joint Variation The relationship given by $\ell = 1.5h$ is an example of direct variation. A **direct variation** can be expressed in the form $y = kx$. In this equation, k is called the **constant of variation**.

Notice that the graph of $\ell = 1.5h$ is a straight line through the origin. A direct variation is a special case of an equation written in slope-intercept form, $y = mx + b$. When $m = k$ and $b = 0$, $y = mx + b$ becomes $y = kx$. So the slope of a direct variation equation is its constant of variation.

To express a direct variation, we say that y varies directly as x . In other words, as x increases, y increases or decreases at a constant rate.



Common Core State Standards

Content Standards
A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Mathematical Practices

- 1 Make sense of problems and persevere in solving them.
- 4 Model with mathematics.

Key Concept Direct Variation

Words y varies directly as x if there is some nonzero constant k such that $y = kx$. k is called the *constant of variation*.

Example If $y = 3x$ and $x = 7$, then $y = 3(7)$ or 21.

If you know that y varies directly as x and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1 \quad \text{and} \quad y_2 = kx_2$$

$$\frac{y_1}{x_1} = k \quad \frac{y_2}{x_2} = k \quad \text{Therefore, } \frac{y_1}{x_1} = \frac{y_2}{x_2}$$

Using the properties of equality, you can find many other proportions that relate these same x - and y -values.





Example 1 Direct Variation

If y varies directly as x and $y = 15$ when $x = -5$, find y when $x = 7$.

Use a proportion that relates the values.

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} \quad \text{Direct variation}$$

$$\frac{15}{-5} = \frac{y_2}{7} \quad y_1 = 15, x_1 = -5, \text{ and } x_2 = 7$$

$$15(7) = -5(y_2) \quad \text{Cross multiply.}$$

$$105 = -5y_2 \quad \text{Simplify.}$$

$$-21 = y_2 \quad \text{Divide each side by } -5.$$

Guided Practice

1. If r varies directly as t and $r = -20$ when $t = 4$, find r when $t = -6$.

Another type of variation is joint variation. **Joint variation** occurs when one quantity varies directly as the product of two or more other quantities.

StudyTip

Joint Variation Some mathematicians consider joint variation a special type of combined variation.

KeyConcept Joint Variation

Words y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.

Example If $y = 5xz$, $x = 6$, and $z = -2$, then $y = 5(6)(-2)$ or -60 .

If you know that y varies jointly as x and z and one set of values, you can use a proportion to find the other set of corresponding values.

$$\begin{array}{l} y_1 = kx_1z_1 \quad \text{and} \quad y_2 = kx_2z_2 \\ \frac{y_1}{x_1z_1} = k \quad \quad \quad \frac{y_2}{x_2z_2} = k \quad \quad \quad \text{Therefore, } \frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}. \end{array}$$



Example 2 Joint Variation

Suppose y varies jointly as x and z . Find y when $x = 9$ and $z = 2$, if $y = 20$ when $z = 3$ and $x = 5$.

Use a proportion that relates the values.

$$\frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2} \quad \text{Joint variation}$$

$$\frac{20}{5(3)} = \frac{y_2}{9(2)} \quad y_1 = 20, x_1 = 5, z_1 = 3, x_2 = 9, \text{ and } z_2 = 2$$

$$20(9)(2) = 5(3)(y_2) \quad \text{Cross multiply.}$$

$$360 = 15y_2 \quad \text{Simplify.}$$

$$24 = y_2 \quad \text{Divide each side by } 15.$$

Guided Practice

2. Suppose r varies jointly as v and t . Find r when $v = 2$ and $t = 8$, if $r = 70$ when $v = 10$ and $t = 4$.



2 Inverse Variation and Combined Variation Another type of variation is inverse variation. If two quantities x and y show **inverse variation**, their product is equal to a constant k .

Inverse variation is often described as one quantity increasing while the other quantity is decreasing. For example, speed and time for a fixed distance vary inversely with each other; the faster you go, the less time it takes you to get there.

Key Concept Inverse Variation

Words y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Example If $xy = 2$, and $x = 6$, then $y = \frac{2}{6}$ or $\frac{1}{3}$.

StudyTip

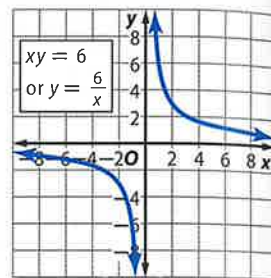
Direct and Inverse Variation

You can identify the type of variation by looking at a table of values for x and y . If the quotient $\frac{y}{x}$ has a constant value, y varies directly as x . If the product xy has a constant value, y varies inversely as x .

Suppose y varies inversely as x such that $xy = 6$ or $y = \frac{6}{x}$.

The graph of this equation is shown at the right. Since k is a positive value, as the values of x increase, the values of y decrease.

Notice that the graph of an inverse variation is a reciprocal function.



A proportion can be used with inverse variation to solve problems in which some quantities are known. The following proportion is only one of several that can be formed.

$$x_1y_1 = k \text{ and } x_2y_2 = k$$

$$x_1y_1 = x_2y_2 \quad \text{Substitution Property of Equality}$$

$$\frac{x_1}{y_2} = \frac{x_2}{y_1} \quad \text{Divide each side by } y_1y_2.$$

Example 3 Inverse Variation

If a varies inversely as b and $a = 28$ when $b = -2$, find a when $b = -10$.

Use a proportion that relates the values.

$$\frac{a_1}{b_2} = \frac{a_2}{b_1} \quad \text{Inverse Variation}$$

$$\frac{28}{-10} = \frac{a_2}{-2} \quad a_1 = 28, b_1 = -2, \text{ and } b_2 = -10$$

$$28(-2) = -10(a_2) \quad \text{Cross multiply.}$$

$$-56 = -10(a_2) \quad \text{Simplify.}$$

$$5\frac{3}{5} = a_2 \quad \text{Divide each side by } -10.$$

Guided Practice

3. If x varies inversely as y and $x = 24$ when $y = 4$, find x when $y = 12$.

Inverse variation is often used in real-world situations.





Real-WorldLink

When you pluck a string, it vibrates back and forth. This causes mechanical energy to travel through the air in waves. The number of times per second these waves hit our ear is called the *frequency*. The more waves per second, the higher the pitch.

StudyTip

Combined Variation

Quantities that vary directly appear in the numerator. Quantities that vary inversely appear in the denominator.

Real-World Example 4 Write and Solve an Inverse Variation



MUSIC The length of a violin string varies inversely as the frequency of its vibrations. A violin string 10 inches long vibrates at a frequency of 512 cycles per second. Find the frequency of an 8-inch violin string.

Let $v_1 = 10$, $f_1 = 512$, and $v_2 = 8$. Solve for f_2 .

$$v_1 f_1 = v_2 f_2 \quad \text{Original equation}$$

$$10 \cdot 512 = 8 \cdot f_2 \quad v_1 = 10, f_1 = 512, \text{ and } v_2 = 8$$

$$\frac{5120}{8} = f_2 \quad \text{Divide each side by 8.}$$

$$640 = f_2 \quad \text{Simplify.}$$

The 8-inch violin string vibrates at a frequency of 640 cycles per second.

GuidedPractice

- The apparent length of an object is inversely proportional to one's distance from the object. Earth is about 93 million miles from the Sun. Jupiter is about 483.6 million miles from the Sun. Find how many times as large the diameter of the Sun would appear on Earth as on Jupiter.

Another type of variation is combined variation. **Combined variation** occurs when one quantity varies directly and/or inversely as two or more other quantities.

If you know that y varies directly as x , y varies inversely as z , and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = \frac{kx_1}{z_1} \quad \text{and} \quad y_2 = \frac{kx_2}{z_2}$$

$$\frac{y_1 z_1}{x_1} = k \quad \frac{y_2 z_2}{x_2} = k \quad \text{Therefore, } \frac{y_1 z_1}{x_1} = \frac{y_2 z_2}{x_2}$$



Example 5 Combined Variation

Suppose f varies directly as g , and f varies inversely as h . Find g when $f = 18$ and $h = -3$, if $g = 24$ when $h = 2$ and $f = 6$.

First set up a correct proportion for the information given.

$$f_1 = \frac{kg_1}{h_1} \text{ and } f_2 = \frac{kg_2}{h_2} \quad \text{g varies directly as f, so g goes in the numerator. h varies inversely as f, so h goes in the denominator.}$$

$$k = \frac{f_1 h_1}{g_1} \text{ and } k = \frac{f_2 h_2}{g_2} \quad \text{Solve for k.}$$

$$\frac{f_1 h_1}{g_1} = \frac{f_2 h_2}{g_2} \quad \text{Set the two proportions equal to each other.}$$

$$\frac{6(2)}{24} = \frac{18(-3)}{g_2} \quad f_1 = 6, g_1 = 24, h_1 = 2, f_2 = 18, \text{ and } h_2 = -3$$

$$24(18)(-3) = 6(2)(g_2) \quad \text{Cross multiply.}$$

$$-1296 = 12g_2 \quad \text{Simplify.}$$

$$-108 = g_2 \quad \text{Divide each side by 12.}$$

When $f = 18$ and $h = -3$, the value of g is -108 .

GuidedPractice

- Suppose p varies directly as r , and p varies inversely as t . Find t when $r = 10$ and $p = -5$, if $t = 20$ when $p = 4$ and $r = 2$.



Check Your Understanding



= Step-by-Step Solutions begin on page R14.



- Examples 1–3**
1. If y varies directly as x and $y = 12$ when $x = 8$, find y when $x = 14$.
 2. Suppose y varies jointly as x and z . Find y when $x = 9$ and $z = -3$, if $y = -50$ when z is 5 and x is -10 .
 3. If y varies inversely as x and $y = -18$ when $x = 16$, find x when $y = 9$.

Example 4

4. **TRAVEL** A map of Illinois is scaled so that 2 inches represents 15 miles. How far apart are Chicago and Rockford if they are 12 inches apart on the map?

- Example 5**
5. Suppose a varies directly as b , and a varies inversely as c . Find b when $a = 8$ and $c = -3$, if $b = 16$ when $c = 2$ and $a = 4$.
 6. Suppose d varies directly as f , and d varies inversely as g . Find g when $d = 6$ and $f = -7$, if $g = 12$ when $d = 9$ and $f = 3$.

Practice and Problem Solving

Extra Practice is on page R8.

Example 1 If x varies directly as y , find x when $y = 8$.

7. $x = 6$ when $y = 32$
8. $x = 11$ when $y = -3$
9. $x = 14$ when $y = -2$
10. $x = -4$ when $y = 10$

11. **MOON** Astronaut Neil Armstrong, the first man on the Moon, weighed 360 pounds on Earth with all his equipment on, but weighed only 60 pounds on the Moon. Write an equation that relates weight on the Moon m with weight on Earth w .

Example 2 If a varies jointly as b and c , find a when $b = 4$ and $c = -3$.

12. $a = -96$ when $b = 3$ and $c = -8$
13. $a = -60$ when $b = -5$ and $c = 4$
14. $a = -108$ when $b = 2$ and $c = 9$
15. $a = 24$ when $b = 8$ and $c = 12$

16. **CCSS MODELING** According to the A.C. Nielsen Company, the average American watches 4 hours of television a day.

- a. Write an equation to represent the average number of hours spent watching television by m household members during a period of d days.
- b. Assume that members of your household watch the same amount of television each day as the average American. How many hours of television would the members of your household watch in a week?

Example 3 If f varies inversely as g , find f when $g = -6$.

17. $f = 15$ when $g = 9$
18. $f = 4$ when $g = 28$
19. $f = -12$ when $g = 19$
20. $f = 0.6$ when $g = -21$

21. **COMMUNITY SERVICE** Every year students at West High School collect canned goods for a local food pantry. They plan to distribute flyers to homes in the community asking for donations. Last year, 12 students were able to distribute 1000 flyers in four hours.

- a. Write an equation that relates the number of students s to the amount of time t it takes to distribute 1000 flyers.
- b. How long would it take 15 students to hand out the same number of flyers this year?



Example 4

- 22. BIRDS** When a group of snow geese migrate, the distance that they fly varies directly with the amount of time they are in the air.
- A group of snow geese migrated 375 miles in 7.5 hours. Write a direct variation equation that represents this situation.
 - Every year, geese migrate 3000 miles from their winter home in the southwest United States to their summer home in the Canadian Arctic. Estimate the number of hours of flying time that it takes for the geese to migrate.

Example 5

- 23.** Suppose a varies directly as b , and a varies inversely as c . Find b when $a = 5$ and $c = -4$, if $b = 12$ when $c = 3$ and $a = 8$.
- 24.** Suppose x varies directly as y , and x varies inversely as z . Find z when $x = 10$ and $y = -7$, if $z = 20$ when $x = 6$ and $y = 14$.

Determine whether each relation shows *direct* or *inverse* variation, or *neither*.

25.

x	y
4	12
8	24
16	48
32	96

26.

x	y
8	2
4	4
-2	-8
-8	-2

27.

x	y
2	4
3	9
4	16
5	25

28. If y varies inversely as x and $y = 6$ when $x = 19$, find y when $x = 2$.

29. If x varies inversely as y and $x = 16$ when $y = 5$, find x when $y = 20$.

30. Suppose a varies directly as b , and a varies inversely as c . Find b when $a = 7$ and $c = -8$, if $b = 15$ when $c = 2$ and $a = 4$.

31. Suppose x varies directly as y , and x varies inversely as z . Find z when $x = 8$ and $y = -6$, if $z = 26$ when $x = 8$ and $y = 13$.

State whether each equation represents a *direct*, *joint*, *inverse*, or *combined* variation. Then name the constant of variation.

32. $\frac{x}{y} = 2.75$

33. $fg = -2$

34. $a = 3bc$

35. $10 = \frac{xy^2}{z}$

36. $y = -11x$

37. $\frac{n}{p} = 4$

38. $9n = pr$

39. $-2y = z$

40. $a = 27b$

41. $c = \frac{7}{d}$

42. $-10 = gh$

43. $m = 20cd$

44. CCSS PRECISION The volume of a gas v varies inversely as the pressure p and directly as the temperature t .

- Write an equation to represent the volume of a gas in terms of pressure and temperature. Is your equation a *direct*, *joint*, *inverse*, or *combined* variation?
- A certain gas has a volume of 8 liters, a temperature of 275 Kelvin, and a pressure of 1.25 atmospheres. If the gas is compressed to a volume of 6 liters and is heated to 300 Kelvin, what will the new pressure be?
- If the volume stays the same, but the pressure drops by half, then what must have happened to the temperature?

45. VACATION The time it takes the Levensteins to reach Lake Tahoe varies inversely with their average rate of speed.

- If they are 800 miles away, write and graph an equation relating their travel time to their average rate of speed.
- What minimum average speed will allow them to arrive within 18 hours?



46. **MUSIC** The maximum number of songs that a digital audio player can hold depends on the lengths and the quality of the songs that are recorded. A song will take up more space on the player if it is recorded at a higher quality, like from a CD, than at a lower quality, like from the Internet.
- If a certain player has 5400 megabytes of storage space, write a function that represents the number of songs the player can hold as a function of the average size of the songs.
 - Is your function a *direct*, *joint*, *inverse*, or *combined* variation?
 - Suppose the average file size for a high-quality song is 8 megabytes and the average size for a low-quality song is 5 megabytes. Determine how many more songs the player can hold if they are low quality than if they are high quality.
47. **GRAVITY** According to the Law of Universal Gravitation, the attractive force F in newtons between any two bodies in the universe is directly proportional to the product of the masses m_1 and m_2 in kilograms of the two bodies and inversely proportional to the square of the distance d in meters between the bodies. That is, $F = \frac{Gm_1m_2}{d^2}$. G is the universal gravitational constant. Its value is $6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$.
- The distance between Earth and the Moon is about 3.84×10^8 meters. The mass of the Moon is 7.36×10^{22} kilograms. The mass of Earth is 5.97×10^{24} kilograms. What is the gravitational force that the Moon and Earth exert upon each other?
 - The distance between Earth and the Sun is about 1.5×10^{11} meters. The mass of the Sun is about 1.99×10^{30} kilograms. What is the gravitational force that the Sun and Earth exert upon each other?
 - Find the gravitational force exerted on each other by two 1000-kilogram iron balls at a distance of 0.1 meter apart.

H.O.T. Problems Use Higher-Order Thinking Skills

48. **CCSS CRITIQUE** Jamil and Savannah are setting up a proportion to begin solving the combined variation in which z varies directly as x and z varies inversely as y . Who has set up the correct proportion? Explain your reasoning.

Jamil

$$z_1 = \frac{kx_1}{y_1} \text{ and } z_2 = \frac{kx_2}{y_2}$$

$$k = \frac{z_1y_1}{x_1} \text{ and } k = \frac{z_2y_2}{x_2}$$

$$\frac{z_1y_1}{x_1} = \frac{z_2y_2}{x_2}$$

Savannah

$$z_1 = \frac{kx_1}{y_1} \text{ and } z_2 = \frac{kx_2}{y_2}$$

$$k = \frac{z_1x_1}{y_1} \text{ and } k = \frac{z_2x_2}{y_2}$$

$$\frac{z_1x_1}{y_1} = \frac{z_2x_2}{y_2}$$

49. **CHALLENGE** If a varies inversely as b , c varies jointly as b and f , and f varies directly as g , how are a and g related?
50. **REASONING** Explain why some mathematicians consider every joint variation a combined variation, but not every combined variation a joint variation.
51. **OPEN ENDED** Describe three real-life quantities that vary jointly with each other.
52. **WRITING IN MATH** Determine the type(s) of variation(s) for which 0 cannot be one of the values. Explain your reasoning.



Standardized Test Practice

- 53. SAT/ACT** Rafael left the dorm and drove toward the cabin at an average speed of 40 km/h. Monica left some time later driving in the same direction at an average speed of 48 km/h. After driving for five hours, Monica caught up with Rafael. How long did Rafael drive before Monica caught up?
- A 1 hour D 6 hours
B 2 hours E 8 hours
C 4 hours
- 54.** 75% of 88 is the same as 60% of what number?
- F 100
G 105
H 108
J 110
- 55. EXTENDED RESPONSE** Audrey's hair is 7 inches long and is expected to grow at an average rate of 3 inches per year.
- a. Make a table that shows the expected length of Audrey's hair after each of the first 4 years.
b. Write a function that can be used to determine the length of her hair after each year.
c. If she does not get a haircut, determine the length of her hair after 9 years.
- 56.** Which of the following is equal to the sum of two consecutive even integers?
- A 144 C 147
B 146 D 148

Spiral Review

Determine any vertical asymptotes and holes in the graph of each rational function.

(Lesson 8-4)

57. $f(x) = \frac{1}{x^2 + 5x + 6}$

58. $f(x) = \frac{x + 2}{x^2 + 3x - 4}$

59. $f(x) = \frac{x^2 + 4x + 3}{x + 3}$

- 60. PHOTOGRAPHY** The formula $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$ can be used to determine how far the film should be placed from the lens of a camera to create a perfect photograph. The variable q represents the distance from the lens to the film, f represents the focal length of the lens, and p represents the distance from the object to the lens. (Lesson 8-3)
- a. Solve the formula for $\frac{1}{p}$.
b. Write the expression containing f and q as a single rational expression.
c. If a camera has a focal length of 8 centimeters and the lens is 10 centimeters from the film, how far should an object be from the lens so that the picture will be in focus?

Solve each equation. Check your solutions. (Lesson 7-5)

61. $\log_3 42 - \log_3 n = \log_3 7$

62. $\log_2(3x) + \log_2 5 = \log_2 30$

63. $2 \log_5 x = \log_5 9$

64. $\log_{10} a + \log_{10} (a + 21) = 2$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. (Lesson 5-6)

65. $2x^3 - 5x^2 - 28x + 15; x - 5$

66. $3x^3 + 10x^2 - x - 12; x + 3$

Skills Review

Find the LCM of each set of polynomials.

67. $a, 2a, a + 1$

68. $x, 4y, x - y$

69. $8, 24x, 12$

70. $x^4, 3x^2, 2xy$

71. $12a, 15, 4b^2$

72. $x + 2, x - 3, x^2 - x - 6$



Solving Rational Equations and Inequalities

Then

- You simplified rational expressions.

Now

- Solve rational equations.
- Solve rational inequalities.

Why?

- A gaming club charges \$20 per month for membership. Members also have to pay \$5 each time they visit the club. If a member visits the club x times in one month, then the charge for that month will be $20 + 5x$. The actual cost per visit will be $\frac{20 + 5x}{x}$. To determine how many visits are needed for the cost per visit to be \$6, you would need to solve the equation $\frac{20 + 5x}{x} = 6$.



New Vocabulary

rational equation
weighted average
rational inequality



Common Core State Standards

Content Standards

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Mathematical Practices

6 Attend to precision.

1 Solve Rational Equations Equations that contain one or more rational expressions are called **rational equations**. These equations are often easier to solve once the fractions are eliminated. You can eliminate the fractions by multiplying each side by the least common denominator (LCD).



Example 1 Solve a Rational Equation

Solve $\frac{4}{x+3} + \frac{5}{6} = \frac{23}{18}$. Check your solution.

The LCD for the terms is $18(x+3)$.

$$\frac{4}{x+3} + \frac{5}{6} = \frac{23}{18}$$

Original equation

$$18(x+3)\left(\frac{4}{x+3}\right) + 18(x+3)\left(\frac{5}{6}\right) = 18(x+3)\left(\frac{23}{18}\right)$$

Multiply by LCD.

$$18\cancel{(x+3)}\left(\frac{4}{\cancel{x+3}}\right) + 18\cancel{(x+3)}\left(\frac{5}{\cancel{6}}\right) = 18\cancel{(x+3)}\left(\frac{23}{\cancel{18}}\right)$$

Divide common factors.

$$72 + 15x + 45 = 23x + 69$$

Multiply.

$$15x + 117 = 23x + 69$$

Simplify.

$$48 = 8x$$

Subtract $15x$ and 69 .

$$6 = x$$

Divide.

CHECK $\frac{4}{x+3} + \frac{5}{6} = \frac{23}{18}$

Original equation

$$\frac{4}{6+3} + \frac{5}{6} = \frac{23}{18}$$

$$x = 6$$

$$\frac{4}{9} + \frac{5}{6} = \frac{23}{18}$$

Simplify.

$$\frac{8}{18} + \frac{15}{18} = \frac{23}{18}$$

Simplify.

$$\frac{23}{18} = \frac{23}{18} \checkmark$$

Add.

Guided Practice

Solve each equation. Check your solution.

1A. $\frac{2}{x+3} + \frac{3}{2} = \frac{19}{10}$

1B. $\frac{7}{12} + \frac{9}{x-4} = \frac{55}{48}$





Math HistoryLink

Brook Taylor (1685–1731)
English mathematician Taylor developed a theorem used in calculus known as Taylor's Theorem that relies on the remainders after computations with rational expressions.

Multiplying each side of an equation by the LCD of rational expressions can yield results that are not solutions of the original equation. These are extraneous solutions.



Example 2 Solve a Rational Equation

Solve $\frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} = \frac{3}{x+3}$. Check your solution.

The LCD for the terms is $(x+3)(x+5)$.

$$\frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} = \frac{3}{x+3}$$

Original equation

$$\frac{(x+3)(x+5)(2x)}{(x+3)(x+5)(x+5)} - \frac{(x+3)(x+5)(x^2 - x - 10)}{(x+3)(x+5)(x^2 + 8x + 15)} = \frac{(x+3)(x+5)3}{(x+3)(x+5)(x+3)}$$

Multiply by LCD.

Divide common factors.

$$\frac{(x+3)\cancel{(x+5)}(2x)}{\cancel{(x+3)}\cancel{(x+5)}(x+5)} - \frac{\cancel{(x+3)}\cancel{(x+5)}(x^2 - x - 10)}{\cancel{(x+3)}\cancel{(x+5)}(x^2 + 8x + 15)} = \frac{(x+5)\cancel{(x+3)}3}{\cancel{(x+3)}\cancel{(x+5)}(x+3)}$$

$$(x+3)(2x) - (x^2 - x - 10) = 3(x+5)$$

Simplify.

$$2x^2 + 6x - x^2 + x + 10 = 3x + 15$$

Distribute.

$$x^2 + 7x + 10 = 3x + 15$$

Simplify.

$$x^2 + 4x - 5 = 0$$

Subtract $3x + 15$.

$$(x+5)(x-1) = 0$$

Factor.

$$x+5 = 0 \quad \text{or} \quad x-1 = 0$$

Zero Product Property

$$x = -5 \qquad \qquad x = 1$$

CHECK Try $x = -5$.

$$\begin{aligned} \frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} &= \frac{3}{x+3} \\ \frac{2(-5)}{-5+5} - \frac{(-5)^2 - (-5) - 10}{(-5)^2 + 8(-5) + 15} &\stackrel{?}{=} \frac{3}{-5+3} \\ \frac{-10}{0} - \frac{25 + 5 - 10}{25 - 40 + 15} &\neq \frac{3}{-2} \quad \times \end{aligned}$$

Try $x = 1$.

$$\begin{aligned} \frac{2x}{x+5} - \frac{x^2 - x - 10}{x^2 + 8x + 15} &= \frac{3}{x+3} \\ \frac{2(1)}{1+5} - \frac{1^2 - 1 - 10}{1^2 + 8(1) + 15} &\stackrel{?}{=} \frac{3}{1+3} \\ \frac{2}{6} - \frac{-10}{24} &\stackrel{?}{=} \frac{3}{4} \\ \frac{8}{24} + \frac{10}{24} &\stackrel{?}{=} \frac{3}{4} \\ \frac{3}{4} &= \frac{3}{4} \quad \checkmark \end{aligned}$$

When solving a rational equation, any possible solution that results in a zero in the denominator must be excluded from your list of solutions.

Since $x = -5$ results in a zero in the denominator, it is extraneous. Eliminate -5 from the list of solutions. The solution is 1.

Guided Practice

2A. $\frac{5}{y-2} + 2 = \frac{17}{6}$

2B. $\frac{2}{z+1} - \frac{1}{z-1} = \frac{-2}{z^2-1}$

2C. $\frac{7n}{3n+3} - \frac{5}{4n-4} = \frac{3n}{2n+2}$

2D. $\frac{1}{p-2} = \frac{2p+1}{p^2+2p-8} + \frac{2}{p+4}$

Review Vocabulary

extraneous solutions
solutions that do not satisfy the original equation



The **weighted average** is a method for finding the mean of a set of numbers in which some elements of the set carry more importance, or weight, than others. Many real-world problems involving mixtures, work, distance, and interest can be solved by using rational equations.



Real-World Example 3 Mixture Problem

CHEMISTRY Mia adds a 70% acid solution to 12 milliliters of a solution that is 15% acid. How much of the 70% acid solution should be added to create a solution that is 60% acid?

Understand Mia needs to know how much of a solution needs to be added to an original solution to create a new solution.

Plan Each solution has a certain percentage that is acid. The percentage of acid in the final solution must equal the amount of acid divided by the total solution.

StudyTip

CCSS Modeling Tables like the one in Example 3 are useful in organizing and solving mixture, work, weighted average, and distance problems.

	Original	Added	New
Amount of Acid	0.15(12)	0.7(x)	0.15(12) + 0.7x
Total Solution	12	x	12 + x

$$\text{Percentage of acid in solution} = \frac{\text{amount of acid}}{\text{total solution}}$$

Solve

$$\frac{\text{percent}}{100} = \frac{\text{amount of acid}}{\text{total solution}} \quad \text{Write a proportion.}$$

$$\frac{60}{100} = \frac{0.15(12) + 0.7x}{12 + x} \quad \text{Substitute.}$$

$$\frac{60}{100} = \frac{1.8 + 0.7x}{12 + x} \quad \text{Simplify numerator.}$$

$$100(12 + x) \frac{60}{100} = 100(12 + x) \frac{1.8 + 0.7x}{12 + x} \quad \begin{array}{l} \text{LCD is } 100(12 + x). \\ \text{Multiply by LCD.} \end{array}$$

$$\frac{100(12 + x)}{1} \frac{60}{100} = 100 \frac{(12 + x)}{1} \frac{1.8 + 0.7x}{12 + x} \quad \text{Divide common factors.}$$

$$(12 + x)60 = 100(1.8 + 0.7x) \quad \text{Simplify.}$$

$$720 + 60x = 180 + 70x \quad \text{Distribute.}$$

$$540 = 10x \quad \text{Subtract } 60x \text{ and } 180.$$

$$54 = x \quad \text{Divide by } 10.$$

Check

$$\frac{60}{100} = \frac{0.15(12) + 0.7x}{12 + x} \quad \text{Original equation}$$

$$\frac{60}{100} \stackrel{?}{=} \frac{0.15(12) + 0.7(54)}{12 + 54} \quad x = 54$$

$$\frac{60}{100} \stackrel{?}{=} \frac{37.8}{66} \quad \text{Simplify.}$$

$$0.6 = 0.6 \quad \checkmark \quad \text{Simplify.}$$

Mia needs to add 54 milliliters of the 70% acid solution.

Guided Practice

3. Jimmy adds a 65% fruit juice solution to 15 milliliters of a drink that is 10% fruit juice. How much of the 65% fruit juice solution must be added to create a fruit punch that is 35% fruit juice?



The formula relating distance, rate, and time can also be used to solve rational equations. The most common use is $d = rt$. However, it can also be represented by $r = \frac{d}{t}$ and $t = \frac{d}{r}$.



Real-World Example 4 Distance Problem

ROWING Sandra is rowing a canoe on Stanhope Lake. Her rate in still water is 6 miles per hour. It takes Sandra 3 hours to travel 10 miles round trip. Assuming that Sandra rowed at a constant rate of speed, determine the rate of the current.

Understand We are given her speed in still water and the time it takes her to travel with the current and against it. We need to determine the speed of the current.

Plan She traveled 5 miles with the current and 5 miles against it. The formula that relates distance, rate, and time is $d = rt$, or $t = \frac{d}{r}$.

Time with the Current	Time Against the Current	Total Time
$\frac{5}{6+r}$	$\frac{5}{6-r}$	3 hours

Solve

$$\frac{5}{6+r} + \frac{5}{6-r} = 3$$

Write the equation.

$$(6+r)(6-r) \frac{5}{6+r} + (6+r)(6-r) \frac{5}{6-r} = (6+r)(6-r)3$$

LCD = $(6+r)(6-r)$
Multiply by LCD.

$$\cancel{(6+r)}(6-r) \frac{5}{\cancel{6+r}} + (6+r)\cancel{(6-r)} \frac{5}{\cancel{6-r}} = (6+r)(6-r)3$$

Divide common factors.

$$(6-r)5 + (6+r)5 = (36-r^2)3$$

Simplify.

$$30 - 5r + 30 + 5r = 108 - 3r^2$$

Distribute.

$$60 = 108 - 3r^2$$

Simplify.

$$0 = -3r^2 + 48$$

Subtract $10r$.

$$0 = -3(r+4)(r-4)$$

Factor.

$$0 = (r+4)(r-4)$$

Divide each side by -3 .

$$r = 4 \text{ or } -4$$

Zero Product Property

Check $\frac{5}{6+r} + \frac{5}{6-r} = 3$

Original equation

$$\frac{5}{6+4} + \frac{5}{6-4} \stackrel{?}{=} 3$$

$r = 4$

$$\frac{5}{10} + \frac{5}{2} \stackrel{?}{=} 3$$

Simplify.

$$\frac{1}{2} + \frac{5}{2} = \frac{6}{2} \checkmark$$

Simplify.

Since speed cannot be negative, the speed of the current is 4 miles per hour.

Guided Practice

4. **FLYING** The speed of the wind is 20 miles per hour. If it takes a plane 7 hours to fly 2368 miles round trip, determine the plane's speed in still air.

StudyTip

Distance Problems When distances involve round trips, the distance in one direction usually equals the distance in the other direction.



Real-world problems that involve work can often be solved using rational equations.



Real-WorldLink

Since 1997, students from Rock Point School in Burlington, Vermont, spend a week servicing communities throughout the world. While working with Habitat for Humanity, the students spent the time in rural Tennessee, starting and completing the roof of a Habitat home in one week.

Source: Vermont Community Work

Real-World Example 5 Work Problems

COMMUNITY SERVICE Every year, the junior and senior classes at Hillcrest High School build a house for the community. If it takes the senior class 24 days to complete a house and 18 days if they work with the junior class, how long would it take the junior class to complete a house if they worked alone?

Understand We are given how long it takes the senior class working alone and when the classes work together. We need to determine how long it would take the junior class by themselves.

Plan The senior class can complete 1 house in 24 days, so their rate is $\frac{1}{24}$ of a house per day.

The rate for the junior class is $\frac{1}{j}$.

The combined rate for both classes is $\frac{1}{18}$.

Senior Rate	Junior Rate	Combined Rate
$\frac{1}{24}$	$\frac{1}{j}$	$\frac{1}{18}$

Solve $\frac{1}{24} + \frac{1}{j} = \frac{1}{18}$

Write the equation.

$$72j \cdot \frac{1}{24} + 72j \cdot \frac{1}{j} = 72j \cdot \frac{1}{18}$$

LCD = 72j

Multiply by LCD.

$$\overset{3}{72j} \cdot \frac{1}{24} + 72j \cdot \frac{1}{\underset{1}{j}} = \overset{4}{72j} \cdot \frac{1}{\underset{1}{18}}$$

Divide common factors.

$$3j + 72 = 4j$$

Distribute.

$$72 = j$$

Subtract 3j.

Check Two methods are possible.

Method 1 Substitute values.

$$\frac{1}{24} + \frac{1}{j} = \frac{1}{18}$$

Original equation

$$\frac{1}{24} + \frac{1}{72} \stackrel{?}{=} \frac{1}{18}$$

$$j = 72$$

$$\frac{3}{72} + \frac{1}{72} \stackrel{?}{=} \frac{4}{72}$$

$$\text{LCD} = 72$$

$$\frac{4}{72} = \frac{4}{72} \checkmark$$

Simplify.

Method 2 Use a calculator.



It would take the junior class 72 days to complete the house by themselves.

Guided Practice

- 5A.** It took Anthony and Travis 6 hours to rake the leaves together last year. The previous year it took Travis 10 hours to do it alone. How long will it take Anthony if he rakes them by himself this year?
- 5B.** Noah and Owen paint houses together. If Noah can paint a particular house in 6 days and Owen can paint the same house in 5 days, how long would it take the two of them if they work together?



2 Solve Rational Inequalities

To solve **rational inequalities**, which are inequalities that contain one or more rational expressions, follow these steps.

KeyConcept Solving Rational Inequalities

- Step 1** State the excluded values. These are the values for which the denominator is 0.
- Step 2** Solve the related equation.
- Step 3** Use the values determined from the previous steps to divide a number line into intervals.
- Step 4** Test a value in each interval to determine which intervals contain values that satisfy the inequality.



Example 6 Solve a Rational Inequality

Solve $\frac{x}{3} - \frac{1}{x-2} < \frac{x+1}{4}$.

Step 1 The excluded value for this inequality is 2.

Step 2 Solve the related equation.

$$\begin{aligned} \frac{x}{3} - \frac{1}{x-2} &= \frac{x+1}{4} \\ \frac{4}{1} \cdot \frac{x}{3} - \frac{4}{1} \cdot \frac{1}{x-2} &= \frac{4}{1} \cdot \frac{x+1}{4} \\ 4x^2 - 8x - 12 &= 3x^2 - 3x - 6 \\ x^2 - 5x - 6 &= 0 \\ (x-6)(x+1) &= 0 \\ x &= 6 \text{ or } -1 \end{aligned}$$

Related equation

LCD is $12(x-2)$.

Multiply by LCD.

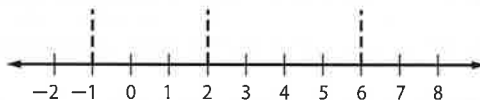
Distribute.

Subtract $3x^2 - 3x - 6$.

Factor.

Zero Product Property

Step 3 Draw vertical lines at the excluded value and at the solutions to separate the number line into intervals.



Step 4 Now test a sample value in each interval to determine whether the values in the interval satisfy the inequality.

Test $x = -3$.	Test $x = 0$.	Test $x = 4$.	Test $x = 8$.
$\frac{-3}{3} - \frac{1}{-3-2} \stackrel{?}{<} \frac{-3+1}{4}$	$\frac{0}{3} - \frac{1}{0-2} \stackrel{?}{<} \frac{0+1}{4}$	$\frac{4}{3} - \frac{1}{4-2} \stackrel{?}{<} \frac{4+1}{4}$	$\frac{8}{3} - \frac{1}{8-2} \stackrel{?}{<} \frac{8+1}{4}$
$-1 + \frac{1}{5} \stackrel{?}{<} -\frac{2}{4}$	$0 + \frac{1}{2} \stackrel{?}{<} \frac{1}{4}$	$\frac{4}{3} - \frac{1}{2} \stackrel{?}{<} \frac{5}{4}$	$\frac{32}{12} - \frac{2}{12} \stackrel{?}{<} \frac{27}{12}$
$-\frac{4}{5} < -\frac{1}{2}$ ✓	$\frac{1}{2} \not< \frac{1}{4}$	$\frac{5}{6} < \frac{5}{4}$ ✓	$\frac{30}{12} \not< \frac{27}{12}$

The statement is true for $x = -3$ and $x = 4$. Therefore, the solution is $x < -1$ or $2 < x < 6$.

StudyTip

Rational Inequalities It is possible that none or all of the intervals will produce a true statement.

GuidedPractice Solve each inequality.

6A. $\frac{5}{x} + \frac{6}{5x} > \frac{2}{3}$

6B. $\frac{4}{3x} + \frac{7}{x} < \frac{5}{9}$





Examples 1-2 Solve each equation. Check your solution.

1. $\frac{4}{7} + \frac{3}{x-3} = \frac{53}{56}$

2. $\frac{7}{3} - \frac{3}{x-5} = \frac{19}{12}$

3. $\frac{10}{2x+1} + \frac{4}{3} = 2$

4. $\frac{11}{4} - \frac{5}{y+3} = \frac{23}{12}$

5. $\frac{8}{x-5} - \frac{9}{x-4} = \frac{5}{x^2-9x+20}$

6. $\frac{14}{x+3} + \frac{10}{x-2} = \frac{122}{x^2+x-6}$

7. $\frac{14}{x-8} - \frac{5}{x-6} = \frac{82}{x^2-14x+48}$

8. $\frac{5}{x+2} - \frac{3}{x-2} = \frac{12}{x^2-4}$

Example 3

9. **CCSS STRUCTURE** Sara has 10 pounds of dried fruit selling for \$6.25 per pound. She wants to know how many pounds of mixed nuts selling for \$4.50 per pound she needs to make a trail mix selling for \$5 per pound.

a. Let m = the number of pounds of mixed nuts. Complete the following table.

	Pounds	Price per Pound	Total Price
Dried Fruit	10	-\$6.25	6.25(10)
Mixed Nuts			
Trail Mix			

b. Write a rational equation using the last column of the table.

c. Solve the equation to determine how many pounds of mixed nuts are needed.

Example 4

10. **DISTANCE** Alicia's average speed riding her bike is 11.5 miles per hour. She takes a round trip of 40 miles. It takes her 1 hour and 20 minutes with the wind and 2 hours and 30 minutes against the wind.

a. Write an expression for Alicia's time with the wind.

b. Write an expression for Alicia's time against the wind.

c. How long does it take to complete the trip?

d. Write and solve the rational equation to determine the speed of the wind.

Example 5

11. **WORK** Kendal and Chandi wax cars. Kendal can wax a particular car in 60 minutes and Chandi can wax the same car in 80 minutes. They plan on waxing the same car together and want to know how long it will take.

a. How much will Kendal complete in 1 minute?

b. How much will Kendal complete in x minutes?

c. How much will Chandi complete in 1 minute?

d. How much will Chandi complete in x minutes?

e. Write a rational equation representing Kendal and Chandi working together on the car.

f. Solve the equation to determine how long it will take them to finish the car.

Example 6

Solve each inequality. Check your solutions.

12. $\frac{3}{5x} + \frac{1}{6x} > \frac{2}{3}$

13. $\frac{1}{4c} + \frac{1}{9c} < \frac{1}{2}$

14. $\frac{4}{3y} + \frac{2}{5y} < \frac{3}{2}$

15. $\frac{1}{3b} + \frac{1}{4b} < \frac{1}{5}$



Examples 1–2 Solve each equation. Check your solutions.

$$16. \frac{9}{x-7} - \frac{7}{x-6} = \frac{13}{x^2 - 13x + 42}$$

$$18. \frac{14}{x-2} - \frac{18}{x+1} = \frac{22}{x^2 - x - 2}$$

$$20. \frac{x}{2x-1} + \frac{3}{x+4} = \frac{21}{2x^2 + 7x - 4}$$

$$17. \frac{13}{y+3} - \frac{12}{y+4} = \frac{18}{y^2 + 7y + 12}$$

$$19. \frac{11}{a+2} - \frac{10}{a+5} = \frac{36}{a^2 + 7a + 10}$$

$$21. \frac{2}{y-5} + \frac{y-1}{2y+1} = \frac{2}{2y^2 - 9y - 5}$$

Examples 3–5 **22. CHEMISTRY** How many milliliters of a 20% acid solution must be added to 40 milliliters of a 75% acid solution to create a 30% acid solution?

23. GROCERIES Ellen bought 3 pounds of bananas for \$0.90 per pound. How many pounds of apples costing \$1.25 per pound must she purchase so that the total cost for fruit is \$1 per pound?

24. BUILDING Bryan's volunteer group can build a garage in 12 hours. Sequoia's group can build it in 16 hours. How long would it take them if they worked together?

Example 6 Solve each inequality. Check your solutions.

$$25. 3 - \frac{4}{x} > \frac{5}{4x}$$

$$26. \frac{5}{3a} - \frac{3}{4a} > \frac{5}{6}$$

$$27. \frac{x-2}{x+2} + \frac{1}{x-2} > \frac{x-4}{x-2}$$

$$28. \frac{3}{4} - \frac{1}{x-3} > \frac{x}{x+4}$$

$$29. \frac{x}{5} + \frac{2}{3} < \frac{3}{x-4}$$

$$30. \frac{x}{x+2} + \frac{1}{x-1} < \frac{3}{2}$$

31. AIR TRAVEL It takes a plane 20 hours to fly to its destination against the wind. The return trip takes 16 hours. If the plane's average speed in still air is 500 miles per hour, what is the average speed of the wind during the flight?

32. FINANCIAL LITERACY Judie wants to invest \$10,000 in two different accounts. The risky account could earn 9% interest, while the other account earns 5% interest. She wants to earn \$750 interest for the year. Of tables, graphs, or equations, choose the best representation needed and determine how much should be invested in each account.

33. MULTIPLE REPRESENTATIONS Consider $\frac{2}{x-3} + \frac{1}{x} = \frac{x-1}{x-3}$.

a. Algebraic Solve the equation for x . Were any values of x extraneous?

b. Graphical Graph $y_1 = \frac{2}{x-3} + \frac{1}{x}$ and $y_2 = \frac{x-1}{x-3}$ on the same graph for $0 < x < 5$.

c. Analytical For what value(s) of x do they intersect? Do they intersect where x is extraneous for the original equation?

d. Verbal Use this knowledge to describe how you can use a graph to determine whether an apparent solution of a rational equation is extraneous.

Solve each equation. Check your solutions.

$$34. \frac{2}{y+3} - \frac{3}{4-y} = \frac{2y-2}{y^2 - y - 12}$$

$$35. \frac{2}{y+2} - \frac{y}{2-y} = \frac{y^2+4}{y^2-4}$$

H.O.T. Problems Use Higher-Order Thinking Skills

36. OPEN ENDED Give an example of a rational equation that can be solved by multiplying each side of the equation by $4(x+3)(x-4)$.

37. CHALLENGE Solve $\frac{1 + \frac{9}{x} + \frac{20}{x^2}}{1 - \frac{25}{x^2}} = \frac{x+4}{x-5}$.

38. CCSS TOOLS While using the table feature on the graphing calculator to explore $f(x) = \frac{1}{x^2 - x - 6}$, the values -2 and 3 say "ERROR." Explain its meaning.

39. WRITING IN MATH Why should you check solutions of rational equations and inequalities?



Standardized Test Practice

40. Nine pounds of mixed nuts containing 55% peanuts were mixed with 6 pounds of another kind of mixed nuts that contain 40% peanuts. What percent of the new mixture is peanuts?
 A 58% B 51% C 49% D 47%
41. Working alone, Dato can dig a 10-foot by 10-foot hole in five hours. Pedro can dig the same hole in six hours. How long would it take them if they worked together?
 F 1.5 hours H 2.52 hours
 G 2.34 hours J 2.73 hours
42. An aircraft carrier made a trip to Guam and back. The trip there took three hours and the trip back took four hours. It averaged 6 kilometers per hour on the return trip. Find the average speed of the trip to Guam.
 A 6 km/h C 10 km/h
 B 8 km/h D 12 km/h
43. **SHORT RESPONSE** If a line ℓ is perpendicular to a segment CD at point F and $CF = FD$, how many points on line ℓ are the same distance from point C as from point D ?

Spiral Review

Determine whether each relation shows *direct* or *inverse* variation, or *neither*. (Lesson 8-5)

44.

x	y
14	3
28	1.5
56	0.75
112	0.375

45.

x	y
0.2	24
0.6	72
1.8	216
5.4	648

46.

x	y
12	18
24	36
36	18
72	9

Graph each function. (Lesson 8-4)

47. $f(x) = \frac{x+4}{x^2+7x+12}$

48. $f(x) = \frac{x^2-5x-14}{x-7}$

49. $f(x) = \frac{x^2+3x-6}{x-2}$

50. **WEATHER** The atmospheric pressure P , in bars, of a given height on Earth is given by the formula $P = a \cdot e^{-\frac{k}{H}}$. In the formula, a is the surface pressure on Earth, which is approximately 1 bar, k is the altitude for which you want to find the pressure in kilometers, and H is always 7 kilometers. (Lesson 7-7)
- Find the pressure for 2, 4, and 7 kilometers.
 - What do you notice about the pressure as altitude increases?
51. **COMPUTERS** Since computers have been invented, computational speed has multiplied by a factor of 4 about every three years. (Lesson 7-1)
- If a typical computer operates with a computational speed s today, write an expression for the speed at which you can expect an equivalent computer to operate after x three-year periods.
 - Suppose your computer operates with a processor speed of 2.8 gigahertz and you want a computer that can operate at 5.6 gigahertz. If a computer with that speed is currently unavailable for home use, how long can you expect to wait until you can buy such a computer?

Skills Review

Determine whether the following are possible lengths of the sides of a right triangle.

52. 5, 12, 13

53. 60, 80, 100

54. 7, 24, 25



8-6 Graphing Technology Lab Solving Rational Equations and Inequalities



You can use a TI-83/84 Plus graphing calculator to solve rational equations by graphing or by using the table feature. Graph both sides of the equation, and locate the point(s) of intersection.

CCSS Common Core State Standards Content Standards

A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

A.REI.11 Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Mathematical Practices

5 Use appropriate tools strategically.



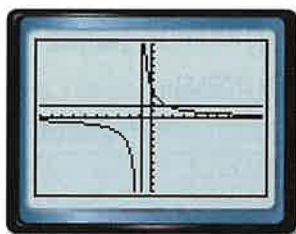
Activity 1 Rational Equation

Solve $\frac{4}{x+1} = \frac{3}{2}$.

Step 1 Graph each side of the equation.

Graph each side of the equation as a separate function. Enter $\frac{4}{x+1}$ as Y_1 and $\frac{3}{2}$ as Y_2 . Then graph the two equations in the standard viewing window.

KEYSTROKES: $Y=$ 4 \div (X,T,θ,n + 1)
 ENTER 3 \div 2 ZOOM 6



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

Because the calculator is in connected mode, a vertical line may appear connecting the two branches of the hyperbola. This line is not part of the graph.

Step 3 Use the TABLE feature.

Verify the solution using the TABLE feature. Set up the table to show x -values in increments of $\frac{1}{3}$.

KEYSTROKES: 2nd [TBLSET] 0 ENTER 1 \div 3 ENTER 2nd [TABLE]

X	Y ₁	Y ₂
0	4	1.5
.33333	4	1.5
.66667	4	1.5
1	1.3333	1.5
1.3333	1.7143	1.5
1.6667	1.5	1.5
2	1.3333	1.5

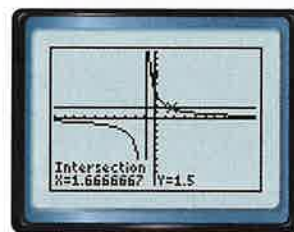
The table displays x -values and corresponding y -values for each graph. At $x = 1\frac{2}{3}$, both functions have a y -value of 1.5. Thus, the solution of the equation is $1\frac{2}{3}$.

Step 2 Use the intersect feature.

The **intersect** feature on the **CALC** menu allows you to approximate the ordered pair of the point at which the graphs cross.

KEYSTROKES: 2nd [CALC] 5

Select one graph and press ENTER . Select the other graph, press ENTER , and press ENTER again.



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

The solution is $1\frac{2}{3}$.

(continued on the next page)

Graphing Technology Lab

Solving Rational Equations and Inequalities *Continued*


You can use a similar procedure to solve rational inequalities using a graphing calculator.

Activity 2 Rational Inequality

Solve $\frac{3}{x} + \frac{7}{x} > 9$.

Step 1 Enter the inequalities.

Rewrite the problem as a system of inequalities.

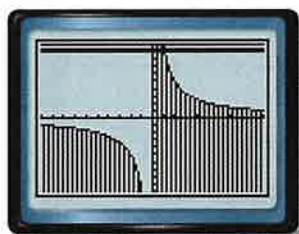
The first inequality is $\frac{3}{x} + \frac{7}{x} > y$ or $y < \frac{3}{x} + \frac{7}{x}$. Since this inequality includes the *less than* symbol, shade below the curve. First enter the boundary and then use the arrow and **ENTER** keys to choose the shade below icon, .

The second inequality is $y > 9$. Shade above the curve since this inequality contains *greater than*.

KEYSTROKES: **Y=** **◀** **◀** **ENTER** **ENTER** **ENTER** **▶** **▶** 3 **÷** **X,T,θ,n** **+** 7 **÷** **X,T,θ,n**
ENTER **◀** **◀** **ENTER** **ENTER** **▶** **▶** 9

Step 2 Graph the system.

KEYSTROKES: **GRAPH**



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

The solution set of the original inequality is the set of x -values of the points in the region where the shadings overlap. Using the calculator's **intersect** feature, you can conclude that the solution set is $\{x \mid 0 < x < 1\frac{1}{9}\}$.

Step 3 Use the TABLE feature.

Verify using the **TABLE** feature. Set up the table to show x -values in increments of $\frac{1}{9}$.

KEYSTROKES: **2nd** **[TBLSET]** 0 **ENTER** 1 **÷** 9 **ENTER**
2nd **[TABLE]**

X	Y1	Y2
.44444	22.5	9
.55556	18	9
.66667	15	9
.77778	12.857	9
.88889	11.25	9
1	10	9
1.11111	9	9

$X = 1.1111111111111111$

Scroll through the table. Notice that for x -values greater than 0 and less than $1\frac{1}{9}$, $Y1 > Y2$. This confirms that the solution of the inequality is $\{x \mid 0 < x < 1\frac{1}{9}\}$.

Exercises

Solve each equation or inequality.

1. $\frac{1}{x} + \frac{1}{2} = \frac{2}{x}$

4. $\frac{1}{1-x} = 1 - \frac{x}{x-1}$

7. $\frac{1}{x-1} + \frac{2}{x} < 0$

2. $\frac{1}{x-4} = \frac{2}{x-2}$

5. $\frac{1}{x+4} = \frac{2}{x^2+3x-4} - \frac{1}{1-x}$

8. $1 + \frac{5}{x-1} \leq 0$

3. $\frac{4}{x} = \frac{6}{x^2}$

6. $\frac{1}{x} + \frac{1}{2x} > 5$

9. $2 + \frac{1}{x-1} \geq 0$

8 Study Guide and Review

Study Guide

Key Concepts

Rational Expressions (Lessons 8-1 and 8-2)

- Multiplying and dividing rational expressions is similar to multiplying and dividing fractions.
- To simplify complex fractions, simplify the numerator and the denominator separately, and then simplify the resulting expression.

Reciprocal and Rational Functions (Lessons 8-3 and 8-4)

- A reciprocal function is of the form $f(x) = \frac{1}{a(x)}$, where $a(x)$ is a linear function and $a(x) \neq 0$.
- A rational function is of the form $\frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$.

Direct, Joint, and Inverse Variation (Lesson 8-5)

- Direct Variation: There is a nonzero number k such that $y = kx$.
- Joint Variation: There is a nonzero number k such that $y = kxz$.
- Inverse Variation: There is a nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Rational Equations and Inequalities (Lesson 8-6)

- Eliminate fractions in rational equations by multiplying each side of the equation by the LCD.
- Possible solutions of a rational equation must exclude values that result in zero in the denominator.

FOLDABLES Study Organizer

Be sure the Key Concepts are noted in your Foldable.

Definition of Notes	Definition of Notes	Definition of Notes
Examples	Examples	Examples

Key Vocabulary

- | | |
|--------------------------------|------------------------------|
| combined variation (p. 565) | point discontinuity (p. 556) |
| complex fraction (p. 532) | rational equation (p. 570) |
| constant of variation (p. 562) | rational expression (p. 529) |
| direct variation (p. 562) | rational function (p. 553) |
| horizontal asymptote (p. 553) | rational inequality (p. 575) |
| hyperbola (p. 545) | reciprocal function (p. 545) |
| inverse variation (p. 564) | vertical asymptote (p. 553) |
| joint variation (p. 563) | weighted average (p. 572) |
| oblique asymptote (p. 555) | |

Vocabulary Check

Choose a term from the list above that best completes each statement or phrase.

1. A(n) _____ is a rational expression whose numerator and/or denominator contains a rational expression.
2. If two quantities show _____, their product is equal to a constant k .
3. A(n) _____ asymptote is a linear asymptote that is neither horizontal nor vertical.
4. A(n) _____ can be expressed in the form $y = kx$.
5. Equations that contain one or more rational expressions are called _____.
6. The graph of $y = \frac{x}{x+2}$ has a(n) _____ at $x = -2$.
7. _____ occurs when one quantity varies directly as the product of two or more other quantities.
8. A ratio of two polynomial expressions is called a(n) _____.
9. _____ looks like a hole in a graph because the graph is undefined at that point.
10. _____ occurs when one quantity varies directly and/or inversely as two or more other quantities.

Lesson-by-Lesson Review



8-1 Multiplying and Dividing Rational Expressions

Simplify each expression.

11. $\frac{-16xy}{27z} \cdot \frac{15z^3}{8x^2}$

12. $\frac{x^2 - 2x - 8}{x^2 + x - 12} \cdot \frac{x^2 + 2x - 15}{x^2 + 7x + 10}$

13. $\frac{x^2 - 1}{x^2 - 4} \cdot \frac{x^2 - 5x - 14}{x^2 - 6x - 7}$

14. $\frac{x + y}{15x} \div \frac{x^2 - y^2}{3x^2}$

15. $\frac{\frac{x^2 + 3x - 18}{x + 4}}{\frac{x^2 + 7x + 6}{x + 4}}$

16. **GEOMETRY** A triangle has an area of $3x^2 + 9x - 54$ square centimeters. If the height of the triangle is $x + 6$ centimeters, find the length of the base.

Example 1

Simplify $\frac{4a}{3b} \cdot \frac{9b^4}{2a^2}$.

$$\begin{aligned} \frac{4a}{3b} \cdot \frac{9b^4}{2a^2} &= \frac{2 \cdot 2 \cdot a \cdot 3 \cdot 3 \cdot b \cdot b \cdot b \cdot b}{3 \cdot b \cdot 2 \cdot a \cdot a} \\ &= \frac{6b^3}{a} \end{aligned}$$

Example 2

Simplify $\frac{r^2 + 5r}{2r} \div \frac{r^2 - 25}{6r - 12}$.

$$\begin{aligned} \frac{r^2 + 5r}{2r} \div \frac{r^2 - 25}{6r - 12} &= \frac{r^2 + 5r}{2r} \cdot \frac{6r - 12}{r^2 - 25} \\ &= \frac{r(r + 5)}{2r} \cdot \frac{6(r - 2)}{(r + 5)(r - 5)} \\ &= \frac{3(r - 2)}{r - 5} \end{aligned}$$

8-2 Adding and Subtracting Rational Expressions

Simplify each expression.

17. $\frac{9}{4ab} + \frac{5a}{6b^2}$

18. $\frac{3}{4x - 8} - \frac{x - 1}{x^2 - 4}$

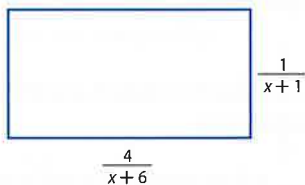
19. $\frac{y}{2x} + \frac{4y}{3x^2} - \frac{5}{6xy^2}$

20. $\frac{2}{x^2 - 3x - 10} - \frac{6}{x^2 - 8x + 15}$

21. $\frac{3}{3x^2 + 2x - 8} + \frac{4x}{2x^2 + 6x + 4}$

22. $\frac{\frac{3}{2x + 3} - \frac{x}{x + 1}}{\frac{2x}{x + 1} + \frac{5}{2x + 3}}$

23. **GEOMETRY** What is the perimeter of the rectangle?



Example 3

Simplify $\frac{3a}{a^2 - 4} - \frac{2}{a - 2}$.

$$\begin{aligned} \frac{3a}{a^2 - 4} - \frac{2}{a - 2} &= \frac{3a}{(a - 2)(a + 2)} - \frac{2}{a - 2} \\ &= \frac{3a}{(a - 2)(a + 2)} - \frac{2(a + 2)}{(a - 2)(a + 2)} \\ &= \frac{3a - 2(a + 2)}{(a - 2)(a + 2)} && \text{Subtract numerators.} \\ &= \frac{3a - 2a - 4}{(a - 2)(a + 2)} && \text{Distributive Property} \\ &= \frac{a - 4}{(a - 2)(a + 2)} && \text{Simplify.} \end{aligned}$$

8-3 Graphing Reciprocal Functions

Graph each function. State the domain and range.

24. $f(x) = \frac{10}{x}$

25. $f(x) = -\frac{12}{x} + 2$

26. $f(x) = \frac{3}{x+5}$

27. $f(x) = \frac{6}{x-9}$

28. $f(x) = \frac{7}{x-2} + 3$

29. $f(x) = -\frac{4}{x+4} - 8$

30. **CONSERVATION** The student council is planting 28 trees for a service project. The number of trees each person plants depends on the number of student council members.

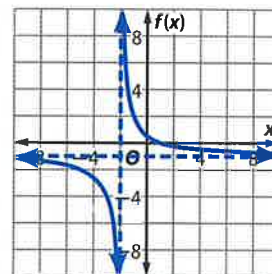
- Write a function to represent this situation.
- Graph the function.

Example 4

Graph $f(x) = \frac{3}{x+2} - 1$. State the domain and range.

- $a = 3$: The graph is stretched vertically.
 $h = -2$: The graph is translated 2 units left. There is an asymptote at $x = -2$.
 $k = -1$: The graph is translated 1 unit down. There is an asymptote at $f(x) = -1$.

Domain: $\{x \mid x \neq -2\}$,
 Range: $\{f(x) \mid f(x) \neq -1\}$



8-4 Graphing Rational Functions

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

31. $f(x) = \frac{3}{x^2 + 4x}$

32. $f(x) = \frac{x+2}{x^2 + 6x + 8}$

33. $f(x) = \frac{x^2 - 9}{x^2 - 5x - 24}$

Graph each rational function.

34. $f(x) = \frac{x+2}{(x+5)^2}$

35. $f(x) = \frac{x}{x+1}$

36. $f(x) = \frac{x^2 + 4x + 4}{x+2}$

37. $f(x) = \frac{x-1}{x^2 + 5x + 6}$

38. **SALES** Aliyah is selling magazine subscriptions. Out of the first 15 houses, she sold subscriptions to 10 of them. Suppose Aliyah goes to x more houses and sells subscriptions to all of them. The percentage of houses that she sold to out of the total houses can be determined using $P(x) = \frac{10+x}{15+x}$.

- Graph the function.
- What domain and range values are meaningful in the context of the problem?

Example 5

Determine the equation of any vertical asymptotes and the values of x for any holes in the graph of

$$f(x) = \frac{x^2 - 1}{x^2 + 2x - 3}$$

$$\frac{x^2 - 1}{x^2 + 2x - 3} = \frac{(x-1)(x+1)}{(x-1)(x+3)}$$

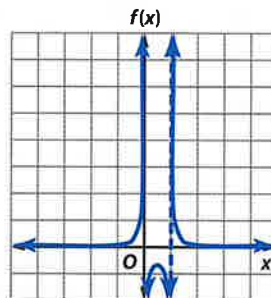
The function is undefined for $x = 1$ and $x = -3$.

Since $\frac{(x-1)(x+1)}{(x-1)(x+3)} = \frac{x+1}{x+3}$, $x = -3$ is a vertical asymptote, and $x = 1$ represents a hole in the graph.

Example 6

Graph $f(x) = \frac{1}{6x(x-1)}$.

The function is undefined for $x = 0$ and $x = 1$. Because $\frac{1}{6x(x-1)}$ is in simplest form, $x = 0$ and $x = 1$ are vertical asymptotes. Draw the two asymptotes and sketch the graph.



8-5 Variation Functions

39. If a varies directly as b and $b = 18$ when $a = 27$, find a when $b = 10$.
40. If y varies inversely as x and $y = 15$ when $x = 3.5$, find y when $x = -5$.
41. If y varies inversely as x and $y = -3$ when $x = 9$, find y when $x = 81$.
42. If y varies jointly as x and z , and $x = 8$ and $z = 3$ when $y = 72$, find y when $x = -2$ and $z = -5$.
43. If y varies jointly as x and z , and $y = 18$ when $x = 6$ and $z = 15$, find y when $x = 12$ and $z = 4$.
44. **JOBS** Lisa's earnings vary directly with how many hours she babysits. If she earns \$68 for 8 hours of babysitting, find her earnings after 5 hours of babysitting.

Example 7

If y varies inversely as x and $x = 24$ when $y = -8$, find x when $y = 15$.

$$\frac{x_1}{y_2} = \frac{x_2}{y_1}$$

Inverse variation

$$\frac{24}{15} = \frac{x_2}{-8}$$

$$x_1 = 24, y_1 = -8, y_2 = 15$$

$$24(-8) = 15(x_2)$$

Cross multiply.

$$-192 = 15x_2$$

Simplify.

$$-12\frac{4}{5} = x_2$$

Divide each side by 15.

When $y = 15$, the value of x is $-12\frac{4}{5}$.

8-6 Solving Rational Equations and Inequalities

Solve each equation or inequality. Check your solutions.

45. $\frac{1}{3} + \frac{4}{x-2} = 6$

46. $\frac{6}{x+5} - \frac{3}{x-3} = \frac{6}{x^2 + 2x - 15}$

47. $\frac{2}{x^2 - 9} = \frac{3}{x^2 - 2x - 3}$

48. $\frac{4}{2x-3} + \frac{x}{x+1} = \frac{-8x}{2x^2 - x - 3}$

49. $\frac{x}{x+4} - \frac{28}{x^2 + x - 12} = \frac{1}{x-3}$

50. $\frac{x}{2} + \frac{1}{x-1} < \frac{x}{4}$

51. $\frac{1}{2x} - \frac{4}{5x} > \frac{1}{3}$

52. **YARD WORK** Lana can plant a garden in 3 hours. Milo can plant the same garden in 4 hours. How long will it take them if they work together?

Example 8

Solve $\frac{3}{x+2} + \frac{1}{x} = 0$.

The LCD is $x(x+2)$.

$$\frac{3}{x+2} + \frac{1}{x} = 0$$

$$x(x+2)\left(\frac{3}{x+2} + \frac{1}{x}\right) = x(x+2)(0)$$

$$x(x+2)\left(\frac{3}{x+2}\right) + x(x+2)\left(\frac{1}{x}\right) = 0$$

$$3(x) + 1(x+2) = 0$$

$$3x + x + 2 = 0$$

$$4x + 2 = 0$$

$$4x = -2$$

$$x = -\frac{1}{2}$$

CHAPTER 8 Practice Test

Simplify each expression.

1. $\frac{r^2 + rt}{2r} \div \frac{r+t}{16r^2}$

2. $\frac{m^2 - 4}{3m^2} \cdot \frac{6m}{2 - m}$

3. $\frac{m^2 + m - 6}{n^2 - 9} \div \frac{m - 2}{n + 3}$

4. $\frac{x^2 + 4x + 3}{x^2 - 2x - 15} \cdot \frac{x^2 - 1}{x^2 - x - 20}$

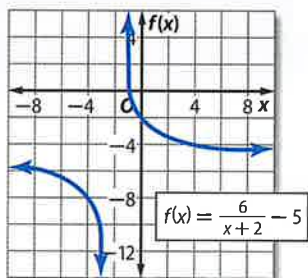
5. $\frac{x+4}{6x+3} + \frac{1}{2x+1}$

6. $\frac{x}{x^2 - 1} - \frac{3}{2x + 2}$

7. $\frac{1}{y} + \frac{2}{7} - \frac{3}{2y^2}$

8. $\frac{2 + \frac{1}{x}}{5 - \frac{1}{x}}$

9. Identify the asymptotes, domain, and range of the function graphed.



10. **MULTIPLE CHOICE** What is the equation for the vertical asymptote of the rational function

$$f(x) = \frac{x+1}{x^2+3x+2}?$$

- A $x = -2$
- B $x = -1$
- C $x = 1$
- D $x = 2$

Graph each function.

11. $f(x) = -\frac{8}{x} - 9$

12. $f(x) = \frac{2}{x+4}$

13. $f(x) = \frac{3}{x-1} + 8$

14. $f(x) = \frac{5x}{x+1}$

15. $f(x) = \frac{x}{x-5}$

16. $f(x) = \frac{x^2 + 5x - 6}{x-1}$

17. Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of the function $f(x) = \frac{x+5}{x^2 - 2x - 35}$.

18. Determine the equations of any oblique asymptotes in the graph of the function $f(x) = \frac{x^2 + x - 5}{x+3}$.

Solve each equation or inequality.

19. $\frac{-1}{x+4} = 6 - \frac{x}{x+4}$

20. $\frac{1}{3} = \frac{5}{m+3} + \frac{8}{21}$

21. $7 + \frac{2}{x} < -\frac{5}{x}$

22. $r + \frac{6}{r} - 5 = 0$

23. $\frac{6}{7} - \frac{3m}{2m-1} = \frac{11}{7}$

24. $\frac{r+2}{3r} = \frac{r+4}{r-2} - \frac{2}{3}$

25. If y varies inversely as x and $y = 18$ when $x = -\frac{1}{2}$, find x when $y = -10$.

26. If m varies directly as n and $m = 24$ when $n = -3$, find n when $m = 30$.

27. Suppose r varies jointly as s and t . If $s = 20$ when $r = 140$ and $t = -5$, find s when $r = 7$ and $t = 2.5$.

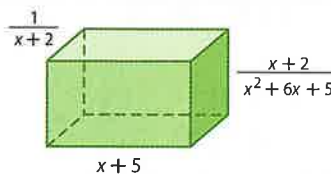
28. **BICYCLING** When Susan rides her bike, the distance that she travels varies directly with the amount of time she is biking. Suppose she bikes 50 miles in 2.5 hours. At this rate, how many hours would it take her to bike 80 miles?

29. **PAINTING** Peter can paint a house in 10 hours. Melanie can paint the same house in 9 hours. How long would it take if they worked together?

30. **MULTIPLE CHOICE** How many liters of a 25% acid solution must be added to 30 liters of an 80% acid solution to create a 50% acid solution?

- F 18
- G 30
- H 36
- J 66

31. What is the volume of the rectangular prism?



Guess and Check

It is very important to pace yourself and keep track of how much time you have when taking a standardized test. If time is running short, or if you are unsure how to solve a problem, the guess-and-check strategy may help you determine the correct answer quickly.

Strategies for Guessing and Checking

Step 1

Carefully look over each possible answer choice and evaluate for reasonableness. Eliminate unreasonable answers.

Ask yourself:

- Are there any answer choices that are clearly incorrect?
- Are there any answer choices that are not in the proper format?
- Are there any answer choices that do not have the proper units for the correct answer?

Step 2

For the remaining answer choices, use the guess-and-check method.

- **Equations:** If you are solving an equation, substitute the answer choice for the variable and see if this results in a true number sentence.
- **System of Equations:** For a system of equations, substitute the answer choice for all variables and make sure all equations result in a true number sentence.

Step 3

Choose an answer choice and see if it satisfies the constraints of the problem statement. Identify the correct answer.

- If the answer choice you are testing does not satisfy the problem, move on to the next reasonable guess and check it.
- When you find the correct answer choice, stop.



Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Solve: $\frac{2}{x-3} - \frac{4}{x+3} = \frac{8}{x^2-9}$.

A -1

C 5

B 1

D 7

The solution of the rational equation will be a real number. Since all four answer choices are real numbers, they are all possible correct answers and must be checked. Begin with the first answer choice and check it in the rational equation. Continue until you find the answer choice that results in a true number sentence.

	Check:
Guess: -1	$\frac{2}{(-1)-3} - \frac{4}{(-1)+3} = \frac{8}{(-1)^2-9}$ $-\frac{5}{2} \neq -1 \quad \times$

	Check:
Guess: 1	$\frac{2}{1-3} - \frac{4}{1+3} = \frac{8}{(1)^2-9}$ $-2 \neq -1 \quad \times$

	Check:
Guess: 5	$\frac{2}{5-3} - \frac{4}{5+3} = \frac{8}{(5)^2-9}$ $\frac{1}{2} = \frac{1}{2} \quad \checkmark$

If $x = 5$, the result is a true number sentence. So, the correct answer is C.

Exercises

Read each problem. Identify what you need to know. Then use the information in the problem to solve.

1. Solve: $\frac{2}{5x} - \frac{1}{2x} = -\frac{1}{2}$.

- A $\frac{1}{10}$
- B $\frac{1}{5}$
- C $\frac{1}{4}$
- D $\frac{1}{2}$

2. The sum of Kevin's, Anna's, and Tia's ages is 40. Anna is 1 year more than twice as old as Tia. Kevin is 3 years older than Anna. How old is Anna?

- F 7
- G 14
- H 15
- J 18

3. Determine the point(s) where the following rational function crosses the x -axis.

$$f(x) = \frac{2}{x-1} - \frac{x+4}{3}$$

- A -5
- B 4
- C 2 or 3
- D -5 or 2

4. Rafael's Theatre Company sells tickets for \$10. At this price, they sell 400 tickets. Rafael estimates that they would sell 40 fewer tickets for each \$2 price increase. What charge would give the most income?

- F 10
- G 13
- H 15
- J 20

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Greg's father can mow the lawn on his riding mower in 45 minutes. It takes Greg 1 hour 45 minutes to mow the lawn with a push mower. Which of the following rational equations can be solved for the number of minutes t it would take them to mow the lawn working together?
- A $\frac{t}{45} + \frac{t}{1.45} = 1$ C $\frac{t}{45} + \frac{t}{105} = 1$
 B $\frac{t}{150} = 1$ D $\frac{t+45}{t+105} = 1$
2. The total cost of reserving a campsite varies directly as the number of nights the site is rented, as shown in the table.

Days	Total Cost
1	\$24
2	\$48
3	\$72
4	\$96

Which equation represents the direct variation?

- F $y = x + 24$ H $y = \frac{24}{x}$
 G $y = 24x$ J $y = 96x$
3. In which direction must the graph of $y = \frac{1}{x}$ be shifted to produce the graph of $y = \frac{1}{x} + 2$?
- A up
 B down
 C right
 D left

Test-Taking Tip

Question 2 Check your answer by substituting 1, 2, 3, and 4 for x and making sure the values in the table are produced.

4. Which of the following is *not* an asymptote of the rational function $f(x) = \frac{1}{x^2 - 49}$?

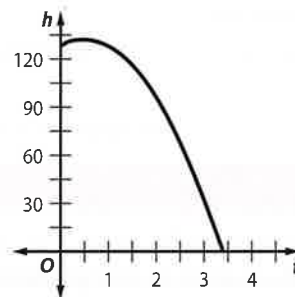
- F $f(x) = 0$ H $x = 7$
 G $x = -7$ J $f(x) = 1$

5. Simplify the complex fraction.

$$\frac{\frac{(x+3)^2}{x^2-16}}{\frac{x+3}{x+4}}$$

- A $\frac{x+3}{x+4}$ C $\frac{x+3}{x-4}$
 B $\frac{1}{x-4}$ D $\frac{x-4}{x+3}$

6. A ball was thrown upward with an initial velocity of 16 feet per second from the top of a building 128 feet high. Its height h in feet above the ground t seconds later will be $h = 128 + 16t - 16t^2$.



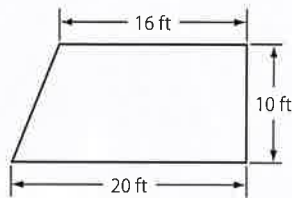
Which is the best conclusion about the ball's action?

- F The ball stayed above 128 feet for more than 3 seconds.
 G The ball returned to the ground in less than 4 seconds.
 H The ball traveled a greater distance going up than it did going down.
 J The ball traveled less than 128 feet in 3.4 seconds.
7. Which of these equations describes a relationship in which every negative real number x corresponds to a nonnegative real number y ?
- A $y = -x$ C $y = \sqrt{x}$
 B $y = x$ D $y = x^3$

Short Response/Gridded Response

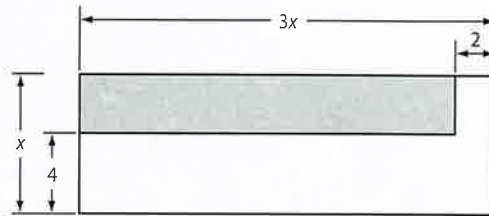
Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

8. Martha is putting a stone walkway around the garden pictured at the right. About how many feet of stone are needed?



9. **GRIDDED RESPONSE** Elaine had some money saved for a week-long vacation. The first day of the vacation she spent \$125 on food and a hotel. On the second day, she was given \$80 from her sister for expenses. Elaine then had \$635 left for the rest of the vacation. How much money, in dollars, did she begin the vacation with?
10. Carlos wants to print 800 one-page flyers for his landscaping business. He has a printer that is capable of printing 8 pages per minute. His business partner has another printer that prints 10 pages per minute.
- How long would it take Carlos' printer to print all the flyers? How long would it take his partner's printer?
 - Set up a rational equation that can be used to find the number of minutes t it would take to print all 800 flyers if both printers are used simultaneously.
 - Solve the equation you wrote in part **b**. How long would it take both printers to print all the flyers if they print simultaneously? Round to the nearest minute.
11. **GRIDDED RESPONSE** The population of a country can be modeled by the equation $P(t) = 40e^{0.02t}$, where P is the population in millions and t is the number of years since 2005. When will the population be 400 million?

12. What is the area of the shaded region of the rectangle expressed as a polynomial in simplest form?

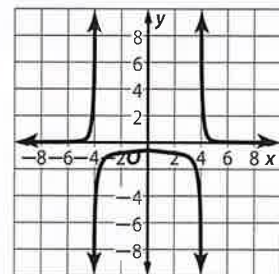


13. **GRIDDED RESPONSE** Suppose y varies inversely as x and $y = 4$ when $x = 12$. What is y when x is 5? Round to the nearest tenth.

Extended Response

Record your answers on a sheet of paper. Show your work.

14. Use the graph of the rational function at the right to answer each question.



- Describe the vertical and horizontal asymptotes of the graph.
 - Write the equation of the rational function. Explain how you found your answer.
15. Consider the polynomial function
- $$f(x) = 3x^4 + 19x^3 + 7x^2 - 11x - 2.$$
- What is the degree of the function?
 - What is the leading coefficient of the function?
 - Evaluate $f(1)$, $f(-2)$, and $f(2a)$. Show your work.

Need ExtraHelp?

If you missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Go to Lesson...	8-5	8-4	8-3	8-3	8-1	8-7	2-4	1-4	3-1	8-5	7-8	5-1	8-4	8-3	6-1

