

3 Systems of Equations and Inequalities



Then

- You graphed equations of lines, transformed functions, and solved equations.

Now

- You will:
 - Solve systems of linear equations and linear inequalities.
 - Solve problems by using linear programming.
 - Perform operations with matrices and determinants.

Why? ▲

- BUSINESS** Most of the time, being successful in business means that you have to have good math skills. In this chapter, you will learn how to maximize your profits and minimize your costs. By doing this you will earn the most money possible.



Systems of Equations and Inequalities

You decide to make candy covered chocolates and candy bars for your new business. You need to know how many candies you need to make each month to maximize your profits.

It takes 0.7 hours to chocolate coat the candies and 1.2 hours to package them. For the candy bar, it takes 1.1 hours to chocolate coat and 0.8 hours to package.

With the information, develop two linear inequalities if the maximum labor hours for the chocolate coating and packaging departments are 500 and 456 hours respectively.

	Chocolate Candies	Chocolate Bars	Total Hours
Coating	0.7	1.1	500
Packaging	1.2	0.8	456



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Animation



Vocabulary



eGlossary



Personal Tutor



Virtual Manipulatives



Graphing Calculator



Audio



Foldables



Self-Check Practice



Worksheets



Get Ready for the Chapter

Diagnose Readiness | You have two options for checking prerequisite skills.

1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Graph each equation.

1. $x = 4y$
2. $y = \frac{1}{3}x + 5$
3. $x + 2y = 4$
4. $y = -x + 6$
5. $3x + 5y = 15$
6. $3y - 2x = -12$

7. **BUSINESS** A museum charges \$8.50 for adult tickets and \$5.25 for children's tickets. On Friday they made \$650.

- a. Write an equation that can be used to model the ticket sales.
- b. Graph the equation.

Graph each inequality.

8. $y < 3$
9. $x + y \geq 1$
10. $3x - y > 6$
11. $x + 2y \leq 5$
12. $y > 4x - 1$
13. $5x - 4y < 12$

14. **FUNDRAISER** The student council is selling T-shirts for \$15 and sweatshirts for \$25. They must make \$2500 to cover expenses. Write and graph an inequality to show the number of T-shirts and sweatshirts that they must sell.

QuickReview

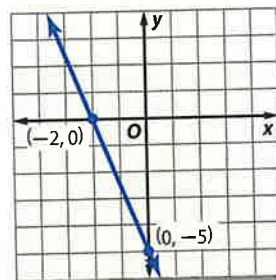
Example 1

Graph $2y + 5x = -10$.

Find the x - and y -intercepts.

$$\begin{array}{rcl} 2(0) + 5x = -10 & & 2y + 5(0) = -10 \\ 5x = -10 & & 2y = -10 \\ x = -2 & & y = -5 \end{array}$$

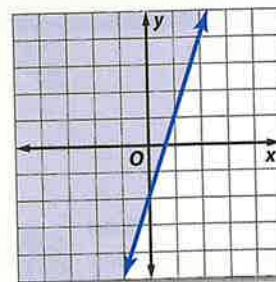
The graph crosses the x -axis at $(-2, 0)$ and the y -axis at $(0, -5)$. Use these ordered pairs to graph the equation.



Example 2

Graph $y \geq 3x - 2$.

The boundary is the graph of $y = 3x - 2$. Since the inequality symbol is \geq , the boundary will be solid.



Test the point $(0, 0)$.

$$\begin{array}{l} 0 \stackrel{?}{\geq} 3(0) - 2 \quad (x, y) = (0, 0) \\ 0 \geq -2 \quad \checkmark \end{array}$$

Shade the region that includes $(0, 0)$.

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.



Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 3. To get ready, identify important terms and organize your resources. You may refer to Chapter 0 to review prerequisite skills.

FOLDABLES Study Organizer



Systems of Equations and Inequalities Make this Foldable to help you organize your Chapter 3 notes about systems of equations and inequalities. Begin with a sheet of $8\frac{1}{2}$ " by 11" paper.

1 Fold in half along the height.



2 Cut along the fold.



3 Fold each sheet along the width into fourths.



4 Tape the ends of two sheets together.



5 Label the tabs with *Solve by Graphing*, *Substitution Method*, *Elimination Method*, *Optimization*, *Systems in Three Variables*, *Matrix Multiplication*, *Cramer's Rule*, and *Inverse Matrices*.



New Vocabulary



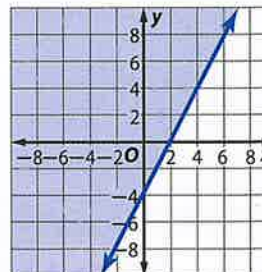
English		Español
break-even point	p. 136	punto de equilibrio
consistent	p. 137	consistente
inconsistent	p. 137	inconsistente
independent	p. 137	independiente
dependent	p. 137	dependiente
substitution method	p. 138	método de sustitución
elimination method	p. 139	método de eliminación
feasible region	p. 154	región viable
bounded	p. 154	acotada
unbounded	p. 154	no acotado
optimize	p. 156	optimizer
dimensions	p. 169	tamaño
scalar	p. 173	escalar
determinant	p. 189	determinante
Cramer's Rule	p. 192	regla de Crámer
coefficient matrix	p. 192	matriz coeficiente
identity matrix	p. 198	matriz identidad
square matrix	p. 198	matriz cuadrada
inverse matrix	p. 198	matriz inversa
variable matrix	p. 200	matriz variables
constant matrix	p. 200	matriz constante

Review Vocabulary



inequality desigualdad
an open sentence that contains the symbol $<$, \leq , $>$, or \geq

linear equation ecuación lineal
an equation that has no operations other than addition, subtraction, and multiplication of a variable by a constant



Graphing Technology Lab

Intersections of Graphs



You can use a TI-83/84 Plus graphing calculator to find where two graphs intersect. You can use the $Y=$ menu to graph each equation on the same set of axes.

CCSS Common Core State Standards Content Standards

A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

A.REI.11 Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.



Example Intersection of Two Graphs

Graph both equations in the standard viewing window.

$$3x + y = 9$$

$$x - y = -1$$

Step 1 Write each equation in the form $y = mx + b$.

$$\begin{array}{ll} 3x + y = 9 & x - y = 1 \\ y = -3x + 9 & -y = -x - 1 \\ & y = x + 1 \end{array}$$

Step 2 Enter $y = -3x + 9$ as Y_1 and $y = x + 1$ as Y_2 . Then graph the lines.

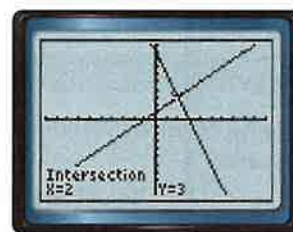
KEYSTROKES: $Y=$ $(-)$ 3 X,T,θ,n $+$ 9 $ENTER$

X,T,θ,n $+$ 1 $ENTER$ $ZOOM$ 6

Step 3 Find the intersection of the lines.

KEYSTROKES: $2nd$ $[CALC]$ 5 $ENTER$ $ENTER$ $ENTER$

The intersection is at (2, 3).



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

Exercises

Use a graphing calculator to find where each pair of graphs intersect.

1. $2x + 4y = 36$
 $10y - 5x = 0$

3. $4x - 2y = 16$
 $7x + 3y = 15$

5. $5x + y = 13$
 $3x = 15 - 3y$

7. $\frac{1}{4}x + y = \frac{11}{4}$
 $x - \frac{1}{2}y = 2$

9. $3x - 6y = 6$
 $2x - 4y = 4$

2. $2y - 3x = 7$
 $5x = 4y - 12$

4. $2x + 4y = 4$
 $x + 3y = 13$

6. $4y - 5 = 20 - 3x$
 $4x - 7y + 16 = 0$

8. $3x + 2y = -3$
 $x + \frac{1}{3}y = -4$

10. $6x + 8y = -16$
 $3x + 4y = 12$

Solving Systems of Equations

Then

- You graphed and solved linear equations.

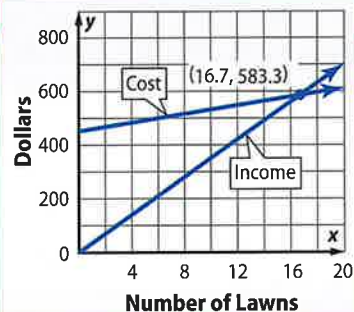
Now

- Solve systems of linear equations graphically.
- Solve systems of linear equations algebraically.

Why?

- Libby borrowed \$450 to start a lawn-mowing business. She charges \$35 per lawn and incurs \$8 in operating costs per lawn. A system of equations can be used to determine the break-even point. The **break-even point** is the point at which income equals cost.

Income Versus Cost



New Vocabulary

- break-even point
- system of equations
- consistent
- inconsistent
- independent
- dependent
- substitution method
- elimination method



Common Core State Standards

Content Standards
 A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

Mathematical Practices

- 2 Reason abstractly and quantitatively.
- 6 Attend to precision.

1 Solve Systems Graphically A **system of equations** is two or more equations with the same variables. To solve a system of equations with two variables, find the ordered pair that satisfies all of the equations.

To solve a system of equations by using a table, first write each equation in slope-intercept form. Then substitute different values for x and solve for the corresponding y -values. For ease of use, choose 0 and 1 as your first x -values.

$$y_1 = -2x + 8$$

$$y_2 = 4x - 7$$

x	y_1	y_2	Difference
0	8	-7	15
1	6	-3	9
2	4	1	3
3	2	5	-3

Because the difference between the y -values is closer to 0 for $x = 1$ than $x = 0$, a value greater than 1 should be tried next.

Because the difference between the y -values changed signs from $x = 2$ to $x = 3$, a value between these should be tried next.

The solution is between 2 and 3.

Example 1 Solve by Using a Table

Solve the system of equations.

$$3x + 2y = -2$$

$$-4x + 5y = -28$$

Write each equation in slope-intercept form.

$$3x + 2y = -2 \rightarrow y = -1.5x - 1$$

$$-4x + 5y = -28 \rightarrow y = 0.8x - 5.6$$

Use a table to find the solution that satisfies both equations.

x	y_1	y_2	(x, y_1)	(x, y_2)
0	-1	-5.6	(0, -1)	(0, -5.6)
1	-2.5	-4.8	(1, -2.5)	(1, -4.8)
2	-4	-4	(2, -4)	(2, -4)

The solution of the system is (2, -4).

Guided Practice

1A. $2x - 5y = 11$
 $-3x + 4y = -13$

1B. $4x + 3y = -17$
 $-7x - 2y = 20$



Another method for solving a system of equations is to graph the equations on the same coordinate plane. The point of intersection represents the solution.



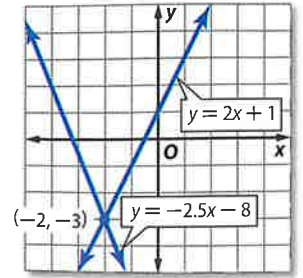
Example 2 Solve by Graphing

Solve the system of equations by graphing.

$$2x - y = -1 \qquad 2y + 5x = -16$$

Write each equation in slope-intercept form.

$$\begin{aligned} 2x - y = -1 &\rightarrow y = 2x + 1 \\ 2y + 5x = -16 &\rightarrow y = -2.5x - 8 \end{aligned}$$



The graphs of the lines appear to intersect at $(-2, -3)$.

CHECK Substitute the coordinates into each original equation.

$2x - y = -1$	$2y + 5x = -16$	Original equations
$2(-2) - (-3) \stackrel{?}{=} -1$	$2(-3) + 5(-2) \stackrel{?}{=} -16$	$x = -2$ and $y = -3$
$-1 = -1 \checkmark$	$-16 = -16 \checkmark$	Simplify.

The solution of the system is $(-2, -3)$.

StudyTip

Checking Solutions Always check to see if the values work for **both** of the original equations.

GuidedPractice

2A. $4x + 3y = 12$
 $-6x + 4y = -1$

2B. $-3y + 8x = 36$
 $6x + y = -21$

Systems of equations can be classified by the number of solutions. A system is **consistent** if it has at least one solution and **inconsistent** if it has no solutions. If it has exactly one solution, it is **independent**, and if it has an infinite number of solutions, it is **dependent**.

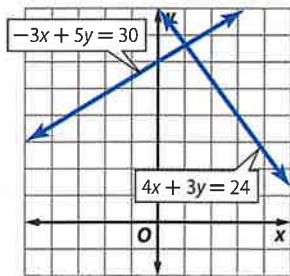
StudyTip

Slope and Classifying Systems If the equations have different slopes, then the system is consistent and independent.

Example 3 Classify Systems

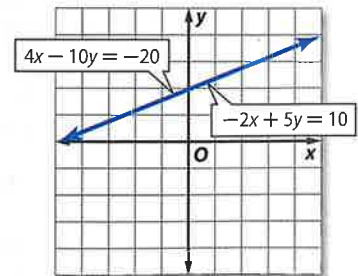
Graph each system of equations and describe them as *consistent and independent*, *consistent and dependent*, or *inconsistent*.

a. $4x + 3y = 24$
 $-3x + 5y = 30$



The graphs of the lines intersect at one point, so there is one solution. The system is consistent and independent.

b. $-2x + 5y = 10$
 $4x - 10y = -20$



Because the equations are equivalent, their graphs are the same line. The system is consistent and dependent.

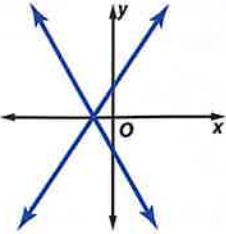
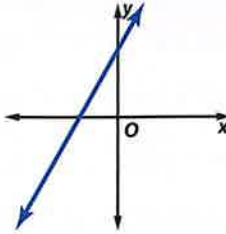
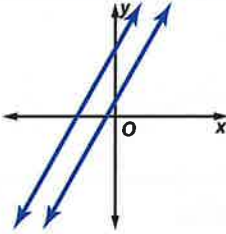
GuidedPractice

3A. $6x - 4y = 15$
 $-6x + 4y = 18$

3B. $-4x + 5y = -17$
 $-4x - 2y = 15$



The relationship between the graph and the solutions of a system is summarized below.

ConceptSummary Characteristics of Linear Systems		
Consistent and Independent	Consistent and Dependent	Inconsistent
 <p>intersecting lines; one solution</p>	 <p>same line; infinitely many solutions</p>	 <p>parallel lines; no solution</p>

2 Solve Systems Algebraically Algebraic methods are used to find exact solutions of systems of equations. One algebraic method is called the **substitution method**.

KeyConcept Substitution Method

- Step 1** Solve one equation for one of the variables.
- Step 2** Substitute the resulting expression into the other equation to replace the variable. Then solve the equation.
- Step 3** Substitute to solve for the other variable.

Systems of equations can be used to solve many real-world problems involving constraints modeled by two or more different functions.

Real-World Example 4 Use the Substitution Method

BUSINESS Alejandro has a computer support business. He estimates that the cost to run his business can be represented by $y = 48x + 500$, where x is the number of customers. He also estimates that his income can be represented by $y = 65x - 145$. How many customers will Alejandro need in order to break even? What will his profit be if he has 60 customers?

$$y = 65x - 145 \quad \text{Income equation}$$

$$48x + 500 = 65x - 145 \quad \text{Substitute } 48x + 500 \text{ for } y.$$

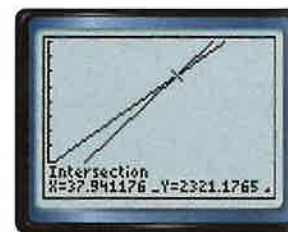
$$500 = 17x - 145 \quad \text{Subtract } 48x \text{ from each side.}$$

$$645 = 17x \quad \text{Add } 145 \text{ to each side.}$$

$$37.9 \approx x \quad \text{Divide each side by } 17.$$

Alejandro needs 38 customers to break even. If he has 60 customers, his income will be $65(60) - 145$ or \$3755, and his costs will be $48(60) + 500$ or \$3380, so his profit will be $3755 - 3380$ or \$375.

CHECK You can use a graphing calculator to check this solution. The break-even point is near $(37.9, 2321.2)$. Use the **CALC** function to find the cost and income for 60 customers.



[0, 65] scl: 5 by [0, 3000] scl: 300



GuidedPractice

Use substitution to solve each system of equations.

4A. $5x - 3y = 23$
 $2x + y = 7$

4B. $x - 7y = 11$
 $5x + 4y = -23$

4C. $-6x - y = 27$
 $3x + 8y = 9$

You can use the **elimination method** to solve a system when one of the variables has the same coefficient in both equations.

KeyConcept Elimination Method

Step 1 Multiply one or both equations by a number to result in two equations that contain opposite terms.

Step 2 Add the equations, eliminating one variable. Then solve the equation.

Step 3 Substitute to solve for the other variable.

Variables can be eliminated by addition or subtraction.

StudyTip

CCSS Perseverance

Remember when you add or subtract one equation from another to add or subtract every term, including the constant on the other side of the equal sign.

Example 5 Solve by Using Elimination



Use the elimination method to solve the system of equations.

$5x + 3y = -19$
 $8x + 3y = -25$

Notice that solving by substitution would involve fractions.

Step 1 Multiply one equation by -1 so the equations contain $3y$ and $-3y$.

$8x + 3y = -25$

Multiply by -1 .

$-8x - 3y = 25$

Step 2 Add the equations to eliminate one variable.

$$\begin{array}{r} 5x + 3y = -19 \\ (+) -8x - 3y = 25 \\ \hline -3x = 6 \\ x = -2 \end{array}$$

Equation 1

Equation 2 $\times (-1)$

Add the equations.

Divide each side by -3 .

Step 3 Substitute -2 for x into either original equation.

$$\begin{array}{r} 8x + 3y = -25 \\ 8(-2) + 3y = -25 \\ -16 + 3y = -25 \\ 3y = -9 \\ y = -3 \end{array}$$

Equation 2

$x = -2$

Multiply.

Add 16 to each side.

Divide each side by 3.

The solution is $(-2, -3)$.

GuidedPractice

5A. $4x - 3y = -22$
 $2x + 3y = 16$

5B. $6x - 5y = -8$
 $4x - 5y = -12$

5C. $2x - 9y = 34$
 $-2x + 6y = -28$

Review Vocabulary

Least Common Multiple
the least number that is a common multiple of two or more numbers

Sometimes, adding or subtracting equations will not eliminate either variable. You can use multiplication and least common multiples to find a common coefficient.



Standardized Test Example 6 No Solution and Infinite Solutions

Solve the system of equations.

$$5x + 3y = 52$$

$$15x + 9y = 54$$

- A (3, 1) B (8, 4) C no solution D infinite solutions

Read the Test Item

You are given a system of two linear equations and are asked to find the solution.

Solve the Test Item

Neither variable has a common coefficient. The coefficients of the y -variables are 3 and 9 and their least common multiple is 9, so multiply each equation by the value that will make the y -coefficient 9.

$$5x + 3y = 52$$

Multiply by 4.

$$15x + 9y = 156$$

$$15x + 9y = 54$$

$$(-) 15x + 9y = 54$$

$$0 = 102$$

Subtract the equations.

Because $0 = 102$ is not true, this system has no solution.

The correct answer is C.

Guided Practice

6. Solve the system of equations.

$$2x + 3y = 5$$

$$6x + 9y = 15$$

- F (-2, 3) G (7, 3) H no solution J infinite solutions

StudyTip

Adding and Subtracting Equations If you add or subtract two equations in a system and the result is an equation that is never true, then the system is inconsistent. When you add or subtract two equations in a system and the result is an equation that is always true, then the system is dependent.



Math HistoryLink

Nina Karlovna Bari (1901–1961) Russian mathematician Nina Karlovna Bari was considered the principal leader of mathematics at Moscow State University, shown above. She is best known for her textbooks *Higher Algebra* and *The Theory of Series*.

The following summarizes the various methods for solving systems.

ConceptSummary Solving Systems of Equations

Method	The Best Time to Use
Table	to estimate the solution, since a table may not provide an exact solution
Graphing	to estimate the solution, since graphing usually does not give an exact solution
Substitution	if one of the variables in either equation has a coefficient of 1 or -1
Elimination Using Addition	if one of the variables has opposite coefficients in the two equations
Elimination Using Subtraction	if one of the variables has the same coefficient in the two equations
Elimination Using Multiplication	if none of the coefficients are 1 or -1 and neither of the variables can be eliminated by simply adding or subtracting the equations





Example 1

Solve each system of equations by using a table.

1. $y = 3x - 4$
 $y = -2x + 11$

2. $4x - y = 1$
 $5x + 2y = 24$

Example 2

Solve each system of equations by graphing.

3. $y = -3x + 6$
 $2y = 10x - 36$

4. $y = -x - 9$
 $3y = 5x + 5$

5. $y = 0.5x + 4$
 $3y = 4x - 3$

6. $-3y = 4x + 11$
 $2x + 3y = -7$

7. $4x + 5y = -41$
 $3y - 5x = 5$

8. $8x - y = 50$
 $x + 4y = -2$

9. **CCSS MODELING** Refer to the table at the right.

- a. Write equations that represent the cost of printing digital photos at each lab.
- b. Under what conditions is the cost to print digital photos the same at both stores?
- c. When is it best to use the online store and when is it best to use the local store?

Digital Photos	
Online Store	\$0.15 per photo + \$2.70 shipping
Local Store	\$0.25 per photo



Example 3

Graph each system of equations and describe it as *consistent and independent*, *consistent and dependent*, or *inconsistent*.

10. $y + 4x = 12$
 $3y = 8 - 12x$

11. $-2x - 3y = 9$
 $4x + 6y = -18$

12. $9x - 2y = 11$
 $5x + 4y = 13$

Example 4

Solve each system of equations by using substitution.

13. $x + 5y = 3$
 $3x - 2y = -8$

14. $y = 2x - 10$
 $y = -4x + 8$

15. $2a + 8b = -8$
 $3a - 5b = 22$

16. $a - 3b = -22$
 $4a + 2b = -4$

17. $6x - 7y = 23$
 $8x + 4y = 44$

18. $9c - 3d = -33$
 $6c + 5d = -8$

Examples 5-6 Solve each system of equations by using elimination.

19. $-6w - 8z = -44$
 $3w + 6z = 36$

20. $4x - 3y = 29$
 $4x + 3y = 35$

21. $3a + 5b = -27$
 $4a + 10b = -46$

22. $8a - 3b = -11$
 $+(-5a + 2b = -3)$
 $3a - 5b = -8$

23. $5a + 15b = -24$
 $-2a - 6b = 28$

24. $6x - 4y = 30$
 $12x + 5y = -18$

25. **MULTIPLE CHOICE** What is the solution of the linear system?

$4x + 3y = 2$
 $4x - 2y = 12$

A (8, -10)

B (2, -2)

C (-10, 14)

D no solution



Example 1 Solve each system of equations by using a table.

$$\begin{aligned} 26. \quad y &= 5x + 3 \\ y &= x - 9 \end{aligned}$$

$$\begin{aligned} 27. \quad 3x - 4y &= 16 \\ -6x + 5y &= -29 \end{aligned}$$

$$\begin{aligned} 28. \quad 2x - 5 &= y \\ -3x + 4y &= 0 \end{aligned}$$

29. **FUNDRAISER** To raise money for new uniforms, the band boosters sell T-shirts and hats. The cost and sale price of each item is shown. The boosters spend a total of \$2000 on T-shirts and hats. They sell all of the merchandise, and make \$3375. How many T-shirts did they sell?

Item	Cost	Sale Price
T-Shirt	\$6	\$10
Hat	\$4	\$7



Example 2 Solve each system of equations by graphing.

$$\begin{aligned} 30. \quad -3x + 2y &= -6 \\ -5x + 10y &= 30 \end{aligned}$$

$$\begin{aligned} 31. \quad 4x + 3y &= -24 \\ 8x - 2y &= -16 \end{aligned}$$

$$\begin{aligned} 32. \quad 6x - 5y &= 17 \\ 6x + 2y &= 31 \end{aligned}$$

$$\begin{aligned} 33. \quad -3x - 8y &= 12 \\ 12x + 32y &= -48 \end{aligned}$$

$$\begin{aligned} 34. \quad y - 3x &= -29 \\ 9x - 6y &= 102 \end{aligned}$$

$$\begin{aligned} 35. \quad -10x + 4y &= 7 \\ 2x - 5y &= 7 \end{aligned}$$

36. **CCSS MODELING** Jerilyn has a \$10 coupon and a 15% discount coupon for her favorite store. The store has a policy that only one coupon may be used per purchase. When is it best for Jerilyn to use the \$10 coupon, and when is it best for her to use the 15% discount coupon?

Example 3 Graph each system of equations and describe it as *consistent and independent*, *consistent and dependent*, or *inconsistent*.

$$\begin{aligned} 37. \quad y &= 3x - 4 \\ y &= 6x - 8 \end{aligned}$$

$$\begin{aligned} 38. \quad y &= 2x - 1 \\ y &= 2x + 6 \end{aligned}$$

$$\begin{aligned} 39. \quad 2x + 5y &= 10 \\ -4x - 10y &= 20 \end{aligned}$$

$$\begin{aligned} 40. \quad x - 6y &= 12 \\ 3x + 18y &= 14 \end{aligned}$$

$$\begin{aligned} 41. \quad -5x - 6y &= 13 \\ 12y + 10x &= -26 \end{aligned}$$

$$\begin{aligned} 42. \quad 8y - 3x &= 15 \\ -16y + 6x &= -30 \end{aligned}$$

Example 4 Solve each system of equations by using substitution.

$$\begin{aligned} 43. \quad 9y + 3x &= 18 \\ -3y - x &= -6 \end{aligned}$$

$$\begin{aligned} 44. \quad 5x - 20y &= 70 \\ 6x + 5y &= -32 \end{aligned}$$

$$\begin{aligned} 45. \quad -4x - 16y &= -96 \\ 7x + 3y &= 68 \end{aligned}$$

$$\begin{aligned} 46. \quad -4a - 5b &= 14 \\ 9a + 3b &= -48 \end{aligned}$$

$$\begin{aligned} 47. \quad -9c - 4d &= 31 \\ 6c + 6d &= -24 \end{aligned}$$

$$\begin{aligned} 48. \quad 8f + 3g &= 12 \\ -32f - 12g &= 48 \end{aligned}$$

49. **TENNIS** At a park, there are 38 people playing tennis. Some are playing doubles, and some are playing singles. There are 13 matches in progress. A doubles match requires 4 players, and a singles match requires 2 players.

- Write a system of two equations that represents the number of singles and doubles matches going on.
- How many matches of each kind are in progress?

Examples 5–6 Solve each system of equations by using elimination.

$$\begin{aligned} 50. \quad 8x + y &= 27 \\ -3x + 4y &= 3 \end{aligned}$$

$$\begin{aligned} 51. \quad 2a - 5b &= -20 \\ 2a + 5b &= 20 \end{aligned}$$

$$\begin{aligned} 52. \quad 6j + 4k &= -46 \\ 2j + 4k &= -26 \end{aligned}$$

$$\begin{aligned} 53. \quad 3x - 8y &= 24 \\ -12x + 32y &= 96 \end{aligned}$$

$$\begin{aligned} 54. \quad 5a - 2b &= -19 \\ 8a + 5b &= -55 \end{aligned}$$

$$\begin{aligned} 55. \quad r - 6t &= 44 \\ 9r + 12t &= 0 \end{aligned}$$

$$\begin{aligned} 56. \quad 6d + 5f &= -32 \\ 5d - 9f &= 26 \end{aligned}$$

$$\begin{aligned} 57. \quad 11u &= 5v + 35 \\ 8v &= -6u + 62 \end{aligned}$$

$$\begin{aligned} 58. \quad -1.2c + 3.4d &= 6 \\ 6c &= -30 + 17d \end{aligned}$$



Use a graphing calculator to solve each system of equations. Round the coordinates of the intersection to the nearest hundredth.

59. $12y = 5x - 15$
 $4.2y + 6.1x = 11$

60. $-3.8x + 2.9y = 19$
 $6.6x - 5.4y = -23$

61. $5.8x - 6.3y = 18$
 $-4.3x + 8.8y = 32$

Solve each system of equations.

62. $11p + 3q = 6$
 $-0.75q - 2.75p = -1.5$

63. $8r - 5t = -60$
 $6r + 3t = -18$

64. $10t + 4v = 13$
 $-4t - 7v = 11$

65. $6w = 12 - 4x$
 $6x = -9w + 18$

66. $\frac{3}{2}y + z = 3$
 $-y - \frac{2}{3}z = -2$

67. $\frac{5}{2}a - \frac{3}{4}b = 46$
 $-\frac{7}{8}a - 3b = 10$

68. **ROWING** Allison can row a boat 1 mile upstream (against the current) in 24 minutes. She can row the same distance downstream in 13 minutes. Assume that both the rowing speed and the speed of the current are constant.

- Find the speed at which Allison is rowing and the speed of the current.
- If Allison plans to meet her friends 3 miles upstream one hour from now, will she be on time? Explain.

69. **CCSS MODELING** The table shows the winning times in seconds for the 100-meter dash at the Olympics between 1964 and 2008.

Years Since 1964, x	Men's Gold Medal Time	Women's Gold Medal Time
0	10.0	11.4
4	9.90	11.0
8	10.14	11.07
12	10.06	11.08
16	10.25	11.06
20	9.99	10.97
24	9.92	10.54
28	9.96	10.82
32	9.84	10.94
36	9.87	10.75
40	9.85	10.93
44	9.69	10.78



- Write equations that represent the winning times for men and women since 1964. Assume that both times continue along the same trend.
- Graph both equations. Estimate when the women's performance will catch up to the men's performance. Do you think that your prediction is reasonable? Explain.

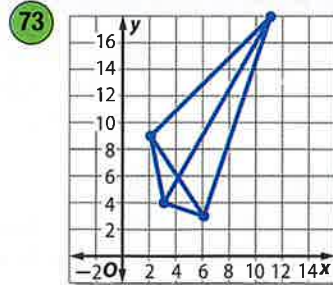
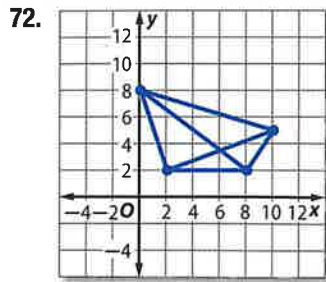
70. **JOBS** Levi has a job offer in which he will receive \$800 per month plus a commission of 2% of the total price of the cars that he sells. At his current job, he receives \$1200 per month plus a commission of 1.5% of his total sales. How much must he sell per month to make the new job a better deal?

71. **TRAVEL** A youth group went on a trip to an amusement park, travelling in two vans. The number of people in each van and the total cost of admission are shown in the table. Find the adult price and student price of admission.

Van	Adults	Students	Total Cost
A	2	5	\$77
B	2	7	\$95



GEOMETRY Find the point at which the diagonals of the quadrilaterals intersect.



74. **ELECTIONS** In the election for student council, Candidate A received 55% of the total votes, while Candidate B received 1541 votes. If Candidate C received 40% of the votes that Candidate A received, how many total votes were cast?

75. **MULTIPLE REPRESENTATIONS** In this problem, you will explore systems of three linear equations and two variables.

$$\begin{aligned} 3y + x &= 16 \\ y - 2x &= -4 \\ y + 5x &= 10 \end{aligned}$$

- Tabular** Make a table of x - and y -values for each equation.
- Analytical** Which values from the table indicate intersections? Is there a solution that satisfies all three equations?
- Graphical** Graph the three equations on a single coordinate plane.
- Verbal** What conditions must be met for a system of three equations with two variables to have a solution? What conditions result in no solution?

H.O.T. Problems Use Higher-Order Thinking Skills

76. **CCSS CRITIQUE** Gloria and Syreeta are solving the system $6x - 4y = 26$ and $-3x + 4y = -17$. Is either of them correct? Explain your reasoning.

<i>Gloria</i>	
$6x - 4y = 26$	$6(3) - 4y = 26$
$-3x + 4y = -17$	$18 - 4y = 26$
$3x = 9$	$-4y = 8$
$x = 3$	$y = -2$
The solution is $(3, -2)$.	

<i>Syreeta</i>	
$6x - 4y = 26$	$6(-3) - 4y = 26$
$-3x + 4y = -17$	$-18 - 4y = 26$
$3x = -9$	$-4y = 44$
$x = -3$	$y = -11$
The solution is $(-3, -11)$.	

77. **CHALLENGE** Find values of a and b for which the following system has a solution of $(b - 1, b - 2)$.

$$\begin{aligned} -8ax + 4ay &= -12a \\ 2bx - by &= 9 \end{aligned}$$

78. **REASONING** If a is consistent and dependent with b , b is inconsistent with c , and c is consistent and independent with d , then a will *sometimes*, *always*, or *never* be consistent and independent with d . Explain your reasoning.

79. **OPEN ENDED** Write a system of equations in which one equation needs to be multiplied by 3 and the other needs to be multiplied by 4 in order to solve the system with elimination. Then solve your system.

80. **WRITING IN MATH** Why is substitution sometimes more helpful than elimination, and vice versa?



Standardized Test Practice

81. **SHORT RESPONSE** Simplify $3y(4x + 6y - 5)$.

82. **SAT/ACT** Which of the following best describes the graph of the equations?

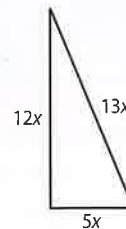
$$4y = 3x + 8$$

$$-6x = -8y + 24$$

- A The lines are parallel.
- B The lines are perpendicular.
- C The lines have the same x -intercept.
- D The lines have the same y -intercept.
- E The lines are the same.

83. **GEOMETRY** Which set of dimensions corresponds to a triangle similar to the one shown at the right?

- F 1 unit, 2 units, 3 units
- G 7 units, 11 units, 12 units
- H 10 units, 23 units, 24 units
- J 20 units, 48 units, 52 units



84. Move-A-Lot Rentals will rent a moving truck for \$100 plus \$0.10 for every mile it is driven. Which equation can be used to find C , the cost of renting a moving truck, and driving it for m miles?

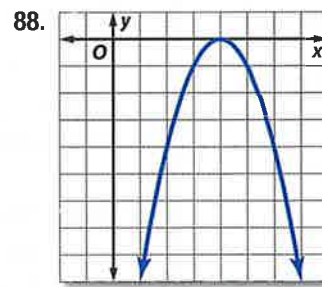
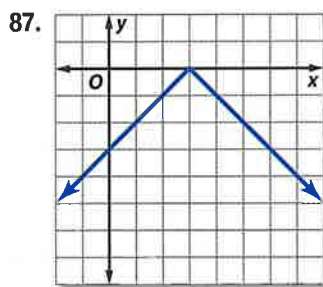
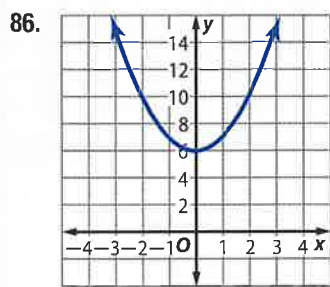
- A $C = 0.1(100 + m)$
- B $C = 100 + 0.1m$
- C $C = 100m + 0.1$
- D $C = 100(m + 0.1)$

Spiral Review

85. **CRAFTS** Priscilla sells stuffed animals at a local craft show. She charges \$10 for the small ones and \$15 for the large ones. To cover her expenses, she needs to sell at least \$350 worth of animals. (Lesson 2-8)

- a. Write an inequality for this situation.
- b. Graph the inequality.
- c. If she sells 10 small and 15 large animals, will she cover her expenses?

Write an equation for each function. (Lesson 2-7)



Solve each equation. Check your solution. (Lesson 1-3)

89. $2p = 14$

90. $-14 + n = -6$

91. $7a - 3a + 2a - a = 16$

92. $x + 9x - 6x + 4x = 20$

93. $27 = -9(y + 5) + 6(y + 8)$

94. $-7(p + 7) + 3(p - 4) = -17$

Skills Review

Determine whether the given point satisfies each inequality.

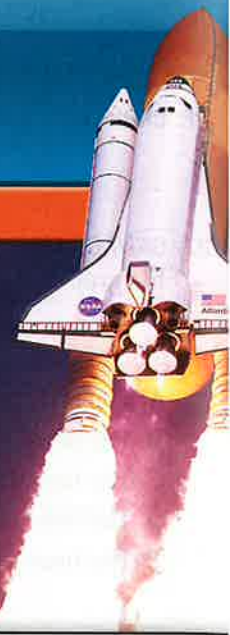
95. $4x + 5y \leq 15$; $(2, -2)$

96. $3x + 5y \geq 8$; $(1, 1)$

97. $6x + 9y < -1$; $(0, 0)$



Solving Systems of Inequalities by Graphing



Then

- You solved systems of linear equations graphically and algebraically.

Now

- Solve systems of inequalities by graphing.
- Determine the coordinates of the vertices of a region formed by the graph of a system of inequalities.

Why?

- Many weather conditions need to be met before a space shuttle can launch. The temperature must be greater than 35°F and less than 100°F, and the wind speed cannot exceed 30 knots. A system of inequalities can be used to show these three conditions.



New Vocabulary

system of inequalities



Common Core State Standards

Content Standards
A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

Mathematical Practices

- 1 Make sense of problems and persevere in solving them.

1 Systems of Inequalities Solving a **system of inequalities** means finding the ordered pairs that satisfy all of the inequalities in the system.

KeyConcept Solving Systems of Inequalities

Step 1 Graph each inequality, shading the correct area.

Step 2 Identify the region that is shaded for all of the inequalities. This is the solution of the system.

Example 1 Intersecting Regions

Solve the system of inequalities.

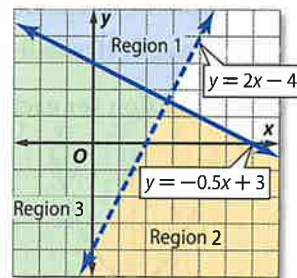
$$y > 2x - 4$$

$$y \leq -0.5x + 3$$

Solution of $y > 2x - 4 \rightarrow$ Regions 1 and 3

Solution of $y \leq -0.5x + 3 \rightarrow$ Regions 2 and 3

Region 3 is part of the solution of both inequalities, so it is the solution of the system.



CHECK Notice that the origin is part of the solution of the system. The origin can be used as a test point. You can test the solution by substituting (0, 0) for x and y in each equation.

$$\begin{aligned} y &> 2x - 4 \\ 0 &\stackrel{?}{>} 2(0) - 4 \\ 0 &\stackrel{?}{>} 0 - 4 \\ 0 &> -4 \quad \checkmark \end{aligned}$$

$$\begin{aligned} y &\leq -0.5x + 3 \\ 0 &\stackrel{?}{\leq} -0.5(0) + 3 \\ 0 &\stackrel{?}{\leq} 0 + 3 \\ 0 &\leq 3 \quad \checkmark \end{aligned}$$

Guided Practice

1A. $y \leq -2x + 5$
 $y > -\frac{1}{4}x - 6$

1B. $y \geq |x|$
 $y < \frac{4}{3}x + 5$



ReadingMath

Empty Set The empty set is also called the *null set*. It can be represented by \emptyset or $\{\}$.

It is possible that the regions do not intersect. When this occurs, there is no solution of the system or the solution set is the *empty set*.



Example 2 Separate Regions

Solve the system of inequalities by graphing.

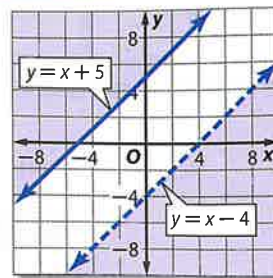
$$y \geq x + 5$$

$$y < x - 4$$

Graph both inequalities.

Since the graphs of the inequalities do not overlap, there are no points in common and there is no solution to the system.

The solution set is the empty set.



GuidedPractice

2A. $y \geq -4x + 8$
 $y < -4x + 4$

2B. $y \geq |2x|$
 $y < 2x - 24$



Real-WorldLink

Typical incoming freshmen will spend more than 3 times as many hours studying in college as in high school.

Source: National Survey of Student Engagement

Real-World Example 3 Write and Use a System of Inequalities



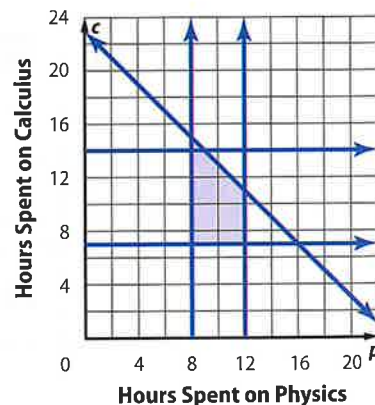
TIME MANAGEMENT Chelsea has final exams in calculus, physics, and history. She has up to 25 hours to study for the exams. She plans to study history for 2 hours. She needs to spend at least 7 hours studying for calculus, but over 14 is too much. She hopes to spend between 8 and 12 hours on physics. Write and graph a system of inequalities to represent the situation.

Calculus: at least 7 hours, but no more than 14
 $7 \leq c \leq 14$

Physics: at least 8 hours, but no more than 12
 $8 \leq p \leq 12$

Chelsea has 25 hours available, and 2 of those will be spent on history. She has up to 23 hours left for calculus and physics.
 $c + p \leq 23$

Graph all of the inequalities. Any ordered pair in the intersection is a solution of the system. One solution is 10 hours on physics and 12 hours on calculus.



GuidedPractice

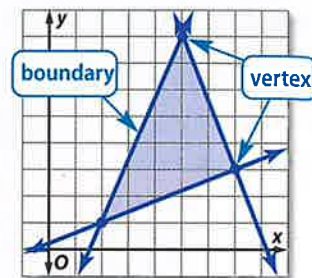
3. **TRAVEL** Mr. and Mrs. Rodriguez are driving across the country with their two children. They plan on driving a maximum of 10 hours each day. Mr. Rodriguez wants to drive at least 4 hours a day but no more than 8 hours a day. Mrs. Rodriguez can drive between 2 and 5 hours per day. Write and graph a system of inequalities that represents this information.



StudyTip

Boundaries If the inequality that forms the boundary is $<$ or $>$, then the boundary is not included in the solution, and the line should be dashed.

2 Find Vertices of an Enclosed Region Sometimes the graph of a system of inequalities produces an enclosed region in the form of a polygon. To find the vertices of the region, determine the coordinates of the points at which the boundaries intersect.

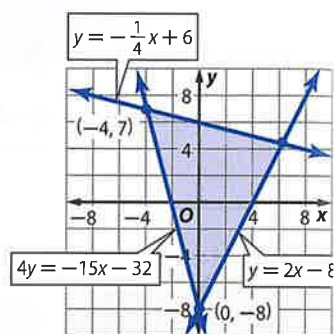


Example 4 Find Vertices

Find the coordinates of the vertices of the triangle formed by $y \geq 2x - 8$, $y \leq -\frac{1}{4}x + 6$, and $4y \geq -15x - 32$.

Step 1 Graph each inequality.

The coordinates $(-4, 7)$ and $(0, -8)$ can be determined from the graph. To find the coordinates of the third vertex, solve the system of equations $y = 2x - 8$ and $y = -\frac{1}{4}x + 6$.



Step 2 Substitute for y in the second equation.

$$2x - 8 = -\frac{1}{4}x + 6 \quad \text{Replace } y \text{ with } 2x - 8.$$

$$2x = -\frac{1}{4}x + 14 \quad \text{Add 8 to each side.}$$

$$\frac{9}{4}x = 14 \quad \text{Add } \frac{1}{4}x \text{ to each side.}$$

$$x = \frac{56}{9} \text{ or } 6\frac{2}{9} \quad \text{Divide each side by } \frac{9}{4}.$$

Step 3 Find y .

$$y = 2\left(6\frac{2}{9}\right) - 8 \quad \text{Replace } x \text{ with } 6\frac{2}{9}.$$

$$= 12\frac{4}{9} - 8 \quad \text{Distributive Property}$$

$$= 4\frac{4}{9} \quad \text{Simplify.}$$

CHECK Compare the coordinates to the coordinates on the graph. The x -coordinate of the third vertex is between 6 and 7, so $6\frac{2}{9}$ is reasonable. The y -coordinate of the third vertex is between 4 and 5, so $4\frac{4}{9}$ is reasonable.

The vertices of the triangle are at $(-4, 7)$, $(0, -8)$, and $\left(6\frac{2}{9}, 4\frac{4}{9}\right)$.

GuidedPractice

Find the coordinates of the vertices of the triangle formed by each system of inequalities.

4A. $y \geq -3x - 6$
 $2y \geq x - 16$
 $11y + 7x \leq 12$

4B. $5y \leq 2x + 9$
 $y \leq -x + 6$
 $9y \geq -2x + 5$





Examples 1–2 Solve each system of inequalities by graphing.

1. $y \leq 6$
 $y > -3 + x$
2. $y \leq -3x + 4$
 $y \geq 2x - 1$
3. $y > -2x + 4$
 $y \leq -3x - 3$

Example 3

4. **CCSS REASONING** The most Kala can spend on hot dogs and buns for her cookout is \$35. A package of 10 hot dogs costs \$3.50. A package of buns costs \$2.50 and contains 8 buns. She needs to buy at least 40 hot dogs and 40 buns.
- a. Graph the region that shows how many packages of each item she can purchase.
- b. Give an example of three different purchases she can make.

Example 4

Find the coordinates of the vertices of the triangle formed by each system of inequalities.

5. $y \geq 2x + 1$
 $y \leq 8$
 $4x + 3y \geq 8$
6. $3y \geq -7x - 16$
 $7y \leq x + 32$
 $y \geq 15x - 40$

Practice and Problem Solving

Extra Practice is on page R3.

Examples 1–2 Solve each system of inequalities by graphing.

7. $x < 3$
 $y \geq -4$
8. $y > 3x - 5$
 $y \leq 4$
9. $y < -3x + 4$
 $3y + x > -6$
10. $y \geq 0$
 $y < x$
11. $6x - 2y \geq 12$
 $3x + 4y > 12$
12. $-8x > -2y - 1$
 $-4y \geq 2x - 5$
13. $5y < 2x + 10$
 $y - 4x > 8$
14. $3y - 2x \leq -24$
 $y \geq \frac{2}{3}x - 1$
15. $y > -\frac{2}{5}x + 2$
 $5y \leq -2x - 15$

Example 3

16. **RECORDING** Jane's band wants to spend no more than \$575 recording their first CD. The studio charges at least \$35 per hour to record. Graph a system of inequalities to represent this situation.

17. **SUMMER TRIP** Rondell has to save at least \$925 to go to Rome with his Latin class in 8 weeks. He earns \$9 an hour working at the Pizza Palace and \$12 an hour working at a car wash. By law, he cannot work more than 25 hours per week. Graph two inequalities that Rondell can use to determine the number of hours he needs to work at each job if he wants to make the trip.

Example 4

Find the coordinates of the vertices of the triangle formed by each system of inequalities.

18. $x \geq 0$
 $y \geq 0$
 $x + 2y < 4$
19. $y \geq 3x - 7$
 $y \leq 8$
 $x + y > 1$
20. $x \leq 4$
 $y > -3x + 12$
 $y \leq 9$
21. $-3x + 4y \leq 15$
 $2y + 5x > -12$
 $10y + 60 \geq 27x$
22. $8y - 19x < 74$
 $38y + 26x \leq 119$
 $54y - 12x \geq -198$
23. $6y - 24x \geq -168$
 $8y + 7x > 10$
 $20y - 2x \leq 64$

24. **BAKING** Rebecca wants to bake cookies and cupcakes for a bake sale. She can bake 15 cookies at a time and 12 cupcakes at a time. She needs to make at least 120 baked goods, but no more than 360, and she wants to have at least three times as many cookies as cupcakes. What combination of batches of each could Rebecca make?



- 25. CELL PHONES** Dale has a maximum of 800 minutes on his cell phone plan that he can use each month. Daytime minutes cost \$0.15, and nighttime minutes cost \$0.10. Dale plans to use at least twice as many daytime minutes as nighttime minutes. If Dale uses at least 200 nighttime minutes and does not go over his limit, what is his maximum bill? his minimum bill?

- 26. TREES** Trees are divided into four categories according to height and trunk circumference. Data for the trees in a forest are described in the table.

Crown Class	dominant	co-dominant	intermediate	suppressed
Height (in feet)	over 72	56–72	40–55	under 39
Trunk Circumference (in inches)	over 60	48–60	34–48	under 33

Source: USDA Forest Service

- a. Write and graph the system of inequalities that represents the range of heights h and circumferences c for a co-dominant tree.
- b. Determine the crown class of a basswood that is 48 feet tall. Find the expected trunk circumference.
- 27. CCSS REASONING** On a camping trip, Jessica needs at least 3 pounds of food and 0.5 gallon of water per day. Marc needs at least 5 pounds of food and 0.5 gallon of water per day. Jessica's equipment weighs 10 pounds, and Marc's equipment weighs 20 pounds. A gallon of water weighs approximately 8 pounds. Each of them carries their own supplies, and Jessica is capable of carrying 35 pounds while Marc can carry 50 pounds.
- a. Graph the inequalities that represent how much they can carry.
- b. How many days can they camp, assuming that they bring all their supplies in at once?
- c. Who will run out of supplies first?

Solve each system of inequalities by graphing.

28. $y \geq |2x + 4| - 2$
 $3y + x \leq 15$

29. $y \geq |6 - x|$
 $|y| \leq 4$

30. $|y| \geq x$
 $y < 2x$

31. $y > -3x + 1$
 $4y \leq x - 8$
 $3x - 5y < 20$

32. $6y + 2x \leq 9$
 $2y + 18 \geq 5x$
 $y > -4x - 9$

33. $|x| > y$
 $y \leq 6$
 $y \geq -2$

34. $2x + 3y \geq 6$
 $y \leq |x - 6|$

35. $8x + 4y < 10$
 $y > |2x - 1|$

36. $y \geq |x - 2| + 4$
 $y \leq \lceil x \rceil - 3$

- 37. MUSIC** Steve is trying to decide what to put on his MP3 player. Audio books are 3 hours long and songs are 2.5 minutes long. Steve wants no more than 4 audio books on his MP3 player, but at least ten songs and one audio book. Each book costs \$15.00 and each song costs \$0.95. Steve has \$63 to spend on books and music. Graph the inequalities to show possible combinations of books and songs that Steve can have.

- 38. JOBS** Louie has two jobs and can work no more than 25 total hours per week. He wants to earn at least \$150 per week. Graph the inequalities to show possible combinations of hours worked at each job that will help him reach his goal.

Job	Pay
Busboy	\$6.50
Clerk	\$8.00



39. **TIME MANAGEMENT** Ramir uses his spare time to write a novel and to exercise. He has budgeted 35 hours per week. He wants to exercise at least 7 hours a week but no more than 15. He also hopes to write between 20 and 25 hours per week. Write and graph a system of inequalities that represents this situation.

Find the coordinates of the vertices of the figure formed by each system of inequalities.

40. $y \geq 2x - 12$
 $y \leq -4x + 20$
 $4y - x \leq 8$
 $y \geq -3x + 2$

41. $y \geq -x - 8$
 $2y \geq 3x - 20$
 $4y + x \leq 24$
 $y \leq 4x + 22$

42. $2y - x \geq -20$
 $y \geq -3x - 6$
 $y \leq -2x + 2$
 $y \leq 2x + 14$

43. **FINANCIAL LITERACY** Mr. Hoffman is investing \$10,000 in two funds. One fund will pay 6% interest, and a riskier second fund will pay 10% interest. What is the least amount he can invest in the risky fund and still earn at least \$740 after one year?

44. **DODGEBALL** A high school is selecting a dodgeball team to play in a fund-raising exhibition against their rival. There can be between 10 and 15 players on the team and there must be more girls than boys on the team.
- Write and graph a system of inequalities to represent the situation.
 - List all of the possible combinations of boys and girls for the team.
 - Explain why there is not an infinite number of possibilities.

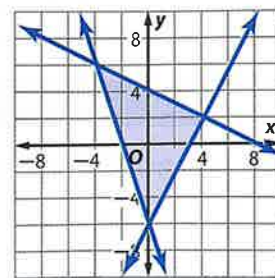
H.O.T. Problems Use Higher-Order Thinking Skills

45. **CHALLENGE** Find the area of the region defined by the following inequalities.

$$\begin{aligned} y &\geq -4x - 16 \\ 4y &\leq 26 - x \\ 3y + 6x &\leq 30 \\ 4y - 2x &\geq -10 \end{aligned}$$

46. **OPEN ENDED** Write a system of two inequalities in which the solution:
- lies only in the third quadrant.
 - does not exist.
 - lies only on a line.
 - lies on exactly one point.

47. **CHALLENGE** Write a system of inequalities to represent the solution shown at the right. How many points with integer coordinates are solutions of the system?



48. **CCSS ARGUMENTS** Determine whether the statement is *true* or *false*. If false, give a counterexample.

A system of two linear inequalities has either no points or infinitely many points in its solution.

49. **WRITING IN MATH** Write a how-to manual for determining where to shade when graphing a system of inequalities.
50. **WRITING IN MATH** Explain how you would test to see whether $(-4, 6)$ is a solution of a system of inequalities.



Standardized Test Practice

51. To be a member of the marching band, a student must have a grade-point average of at least 2.0 and must have attended at least five after-school practices. Choose the system of inequalities that best represents this situation.

- A $x \geq 2$ C $x < 2$
 $y \geq 5$ $y < 5$
 B $x \leq 2$ D $x > 2$
 $y \leq 5$ $y > 5$

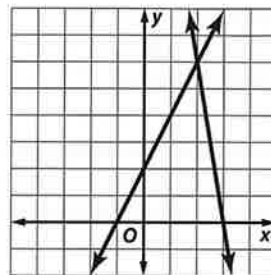
52. **SAT/ACT** The table at the right shows a relationship between x and y . Which equation represents this relationship?

x	y
1	5
2	8
3	11
4	14
5	17
6	20

- F $y = 3x - 2$
 G $y = 3x + 2$
 H $y = 4x + 1$
 J $y = 4x + 2$
 K $y = 4x - 1$

53. **SHORT RESPONSE** If $3x = 2y$ and $5y = 6z$, what is the value of x in terms of z ?

54. **GEOMETRY** Look at the graph below. Which of these statements describes the relationship between the two lines?



- A They intersect at $(6, 2)$.
 B They intersect at $(0, 2)$.
 C They intersect at $(3.5, 0)$.
 D They intersect at $(2, 6)$.

Spiral Review

55. **GEOMETRY** Find the coordinates of the vertices of the parallelogram with sides that are contained in the lines with equations $y = 3$, $y = 7$, $y = 2x$, and $y = 2x - 13$. (Lesson 3-1)

Graph each inequality. (Lesson 2-8)

56. $x + y \geq 6$

57. $4x - 3y < 10$

58. $5x + 7y \geq -20$

Graph each function. Identify the domain and range. (Lesson 2-6)

59. $g(x) = \begin{cases} 0 & \text{if } x < 0 \\ -x + 2 & \text{if } x \geq 0 \end{cases}$

60. $h(x) = \begin{cases} x + 3 & \text{if } x \leq -1 \\ 2x & \text{if } x > -1 \end{cases}$

61. $h(x) = \begin{cases} -1 & \text{if } x < -2 \\ 1 & \text{if } x > 2 \end{cases}$

62. **BOOK CLUB** For each meeting of the Putnam High School book club, \$25 is taken from the activities account to buy snacks and materials. After their sixth meeting, there will be \$350 left in the activities account. (Lesson 2-4)

- a. If no money is put back into the account, what equation can be used to show how much money is left in the activities account after having x number of meetings?
 b. How much money was originally in the account?
 c. After how many meetings will there be no money left in the activities account?

Skills Review

Find each value if $f(x) = 2x + 5$ and $g(x) = 3x - 4$.

63. $f(-3)$

64. $g(-2)$

65. $f(-1)$

66. $g(-0.5)$

67. $f(-0.25)$

68. $g(-0.75)$



Graphing Technology Lab Systems of Linear Inequalities



You can graph systems of linear inequalities with a graphing calculator by using the $Y=$ menu. You can choose different graphing styles to shade above or below a line.



Example Intersection of Two Graphs

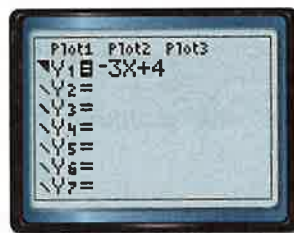
Graph the system of inequalities in the standard viewing window.

$$y \geq -3x + 4$$

$$y \leq 2x - 1$$

Step 1 Enter $-3x + 4$ as Y_1 . Because y is greater than $-3x + 4$, shade above the line.

KEYSTROKES: $Y=$ \leftarrow \leftarrow ENTER ENTER \rightarrow \rightarrow
 (\leftarrow) 3 $[X,T,\theta,n]$ $+$ 4 ENTER

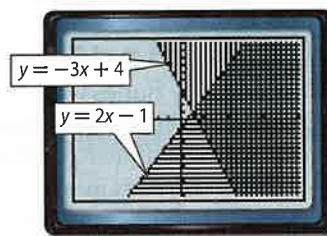


Step 2 Enter $2x - 1$ as Y_2 . Because y is less than $2x - 1$, shade below the line.

KEYSTROKES: \leftarrow \leftarrow ENTER ENTER ENTER \rightarrow \rightarrow 2
 $[X,T,\theta,n]$ $-$ 1 ENTER

Step 3 Display the graphs in the standard viewing window.

KEYSTROKES: ZOOM 6



$[-10, 10]$ scl: 1 by $[-10, 10]$ scl: 1

Notice the shading pattern above the line $y = -3x + 4$ and the shading pattern below the line $y = 2x - 1$. The intersection of the graphs is the region where the patterns overlap. This region includes all the points that satisfy the system $y \geq -3x + 4$ and $y \leq 2x - 1$.

Exercises

Use a graphing calculator to solve each system of inequalities.

1. $y \geq 3$
 $y \leq -x + 1$

2. $y \geq -4x$
 $y \leq -5$

3. $y \geq 2 - x$
 $y \leq x + 3$

4. $y \geq 2x + 1$
 $y \leq -x - 1$

5. $2y \geq 3x - 1$
 $3y \leq -x + 7$

6. $y + 5x \geq 12$
 $y - 3 \leq 10$

7. $5y + 3x \geq 11$
 $3y - x \leq -8$

8. $10y - 7x \geq -19$
 $7y - 5x \leq 11$

9. $\frac{1}{6}y - x \geq -3$
 $\frac{1}{5}y + x \leq 7$

Optimization with Linear Programming

Then

- You solved systems of linear inequalities by graphing.

Now

- Find the maximum and minimum values of a function over a region.
- Solve real-world optimization problems using linear programming.

Why?

- An electronics company produces digital audio players and phones. A sign on the company bulletin board is shown.

If at least 2000 items must be produced per shift, how many of each type should be made to minimize costs?

The company is experiencing limitations, or constraints, on production caused by customer demand, shipping, and the productivity of their factory. A system of inequalities can be used to represent these constraints.



Keeping Costs Down: We Can Do It!

Our Goal: Production per Shift			
Unit	Minimum	Maximum	Cost per Unit
audio	600	1500	\$55
phone	800	1700	\$95



New Vocabulary

- linear programming
- feasible region
- bounded
- unbounded
- optimize



Common Core State Standards

Content Standards

A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

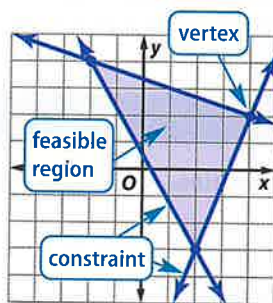
Mathematical Practices

- Model with mathematics.
- Look for and express regularity in repeated reasoning.

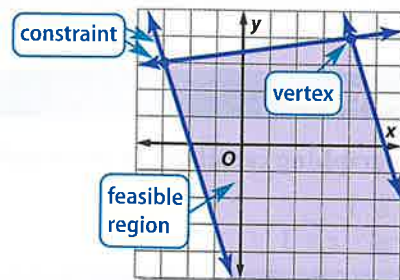
1 Maximum and Minimum Values Situations often occur in business in which a company hopes to either maximize profits or minimize costs, and many constraints need to be considered. These issues can often be addressed by the use of systems of inequalities in linear programming.

Linear programming is a method for finding maximum or minimum values of a function over a given system of inequalities with each inequality representing a constraint. After the system is graphed and the vertices of the solution set, called the **feasible region**, are substituted into the function, you can determine the maximum or minimum value.

KeyConcept Feasible Regions



The feasible region is enclosed, or **bounded**, by the constraints. The maximum or minimum value of the related function *always* occurs at a vertex of the feasible region.



The feasible region is open and can go on forever. It is **unbounded**. Unbounded regions have either a maximum or a minimum.





Example 1 Bounded Region

Graph the system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function for this region.

$$\begin{aligned}
 3 &\leq y \leq 6 \\
 y &\leq 3x + 12 \\
 y &\leq -2x + 6 \\
 f(x, y) &= 4x - 2y
 \end{aligned}$$

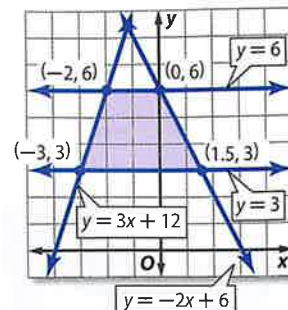
Step 1 Graph the inequalities and locate the vertices.

Step 2 Evaluate the function at each vertex.

(x, y)	$4x - 2y$	$f(x, y)$
$(-3, 3)$	$4(-3) - 2(3)$	-18
$(1.5, 3)$	$4(1.5) - 2(3)$	0
$(0, 6)$	$4(0) - 2(6)$	-12
$(-2, 6)$	$4(-2) - 2(6)$	-20

← maximum

← minimum



The maximum value is 0 at $(1.5, 3)$. The minimum value is -20 at $(-2, 6)$.

Guided Practice

1A. $-2 \leq x \leq 6$
 $1 \leq y \leq 5$
 $y \leq x + 3$
 $f(x, y) = -5x + 2y$

1B. $-6 \leq y \leq -2$
 $y \leq -x + 2$
 $y \leq 2x + 2$
 $f(x, y) = 6x + 4y$

When a system of inequalities does not form a closed region, it is unbounded.



Example 2 Unbounded Region

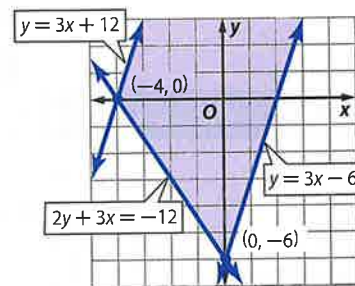
Graph the system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function for this region.

$$\begin{aligned}
 2y + 3x &\geq -12 \\
 y &\leq 3x + 12 \\
 y &\geq 3x - 6 \\
 f(x, y) &= 9x - 6y
 \end{aligned}$$

Evaluate the function at each vertex.

(x, y)	$9x - 6y$	$f(x, y)$
$(-4, 0)$	$9(-4) - 6(0)$	-36
$(0, -6)$	$9(0) - 6(-6)$	36

The maximum value is 36 at $(0, -6)$. There is no minimum value. Notice that another point in the feasible region, $(0, 8)$, yields a value of -48 , which is less than -36 .



Guided Practice

2A. $y \leq 8$
 $y \geq -x + 4$
 $y \geq -x + 10$
 $f(x, y) = -6x + 8y$

2B. $y \geq x - 9$
 $y \leq -4x + 16$
 $y \geq -4x - 4$
 $f(x, y) = 10x + 7y$

ReadingMath

Function Notation

The notation $f(x, y)$ is used to represent a function with two variables, x and y . It is read f of x and y .

WatchOut!

CCSS Precision Do not assume that there is no maximum if the feasible region is unbounded above the vertices. Test points are needed to determine if there is a minimum or maximum.



2 Optimization To **optimize** means to seek the best price or amount to minimize costs or maximize profits. This is often obtained with the use of linear programming.

KeyConcept Optimization with Linear Programming

- Step 1** Define the variables.
- Step 2** Write a system of inequalities.
- Step 3** Graph the system of inequalities.
- Step 4** Find the coordinates of the vertices of the feasible region.
- Step 5** Write a linear function to be maximized or minimized.
- Step 6** Substitute the coordinates of the vertices into the function.
- Step 7** Select the greatest or least result. Answer the problem.



Real-World Career

Operations Manager

Operations management is an area of business that is concerned with the production of goods and services, and involves the responsibility of ensuring that business operations are efficient and effective. A master's degree in business and experience in operations are preferred.

StudyTip

Reasonableness Check your solutions for reasonableness by thinking of the context of the problem.

When using a system of inequalities to describe constraints in real-world problems, often only whole-number solutions will make sense.

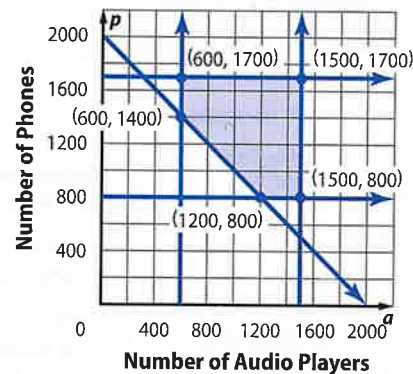
Real-World Example 3 Optimization with Linear Programming

BUSINESS Refer to the application at the beginning of the lesson. Determine how many of each type of device should be made per shift.

Step 1 Let a = number of audio players produced.
Let p = number of phones produced.

Step 2 $600 \leq a \leq 1500$
 $800 \leq p \leq 1700$
 $a + p \geq 2000$

Steps 3 and 4 The system is graphed at the right. Note the vertices of the feasible region.



Step 5 The function to be minimized is $f(a, p) = 55a + 95p$.

Step 6

(a, p)	$55a + 95p$	$f(a, p)$
(600, 1700)	$55(600) + 95(1700)$	194,500
(600, 1400)	$55(600) + 95(1400)$	166,000
(1500, 1700)	$55(1500) + 95(1700)$	244,000
(1500, 800)	$55(1500) + 95(800)$	158,500
(1200, 800)	$55(1200) + 95(800)$	142,000

← maximum

← minimum

Step 7 Produce 1200 audio players and 800 phones to minimize costs.

Guided Practice

3. JEWELRY Each week, Mackenzie can make 10 to 25 necklaces and 15 to 40 pairs of earrings. If she earns profits of \$3 on each pair of earrings and \$5 on each necklace, and she plans to sell at least 30 pieces of jewelry, how can she maximize profit?





Examples 1–2 Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

- | | | |
|---|--|--|
| <p>1. $y \leq 5$
 $x \leq 4$
 $y \geq -x$
 $f(x, y) = 5x - 2y$</p> | <p>2. $y \leq -3x + 6$
 $-y \leq x$
 $y \leq 3$
 $f(x, y) = 8x + 4y$</p> | <p>3. $y \geq -3x + 2$
 $9x + 3y \leq 24$
 $y \geq -4$
 $f(x, y) = 2x + 14y$</p> |
| <p>4. $-2 \leq y \leq 6$
 $3y \leq 4x + 26$
 $y \leq -2x + 2$
 $f(x, y) = -3x - 6y$</p> | <p>5. $-3 \leq y \leq 7$
 $4y \geq 4x - 8$
 $6y + 3x \leq 24$
 $f(x, y) = -12x + 9y$</p> | <p>6. $y \leq 2x + 6$
 $y \geq 2x - 8$
 $y \geq -2x - 18$
 $f(x, y) = 5x - 4y$</p> |

Example 3

7. **CCSS PRECISION** The total number of workers' hours per day available for production in a skateboard factory is 85 hours. There are 40 hours available for finishing decks and quality control each day. The table shows the number of hours needed in each department for two different types of skateboards.

Skateboard Manufacturing Time		
Board Type	Production Time	Deck Finishing/Quality control
Pro Boards	1.5 hours	2 hours
Specialty Boards	1 hour	0.5 hour



- Write a system of inequalities to represent the situation.
- Draw the graph showing the feasible region.
- List the coordinates of the vertices of the feasible region.
- If the profit on a pro board is \$50 and the profit on a specialty board is \$65, write a function for the total profit on the skateboards.
- Determine the number of each type of skateboard that needs to be made to have a maximum profit. What is the maximum profit?

Practice and Problem Solving

Extra Practice is on page R3.

Examples 1–2 Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

- | | | |
|---|---|---|
| <p>8. $1 \leq y \leq 4$
 $4y - 6x \geq -32$
 $2y \geq -x + 4$
 $f(x, y) = -6x + 3y$</p> | <p>9 $2 \geq x \geq -3$
 $y \geq -2x - 6$
 $4y \leq 2x + 32$
 $f(x, y) = -4x - 9y$</p> | <p>10. $-2 \leq x \leq 4$
 $5 \leq y \leq 8$
 $2x + 3y \leq 26$
 $f(x, y) = 8x - 10y$</p> |
| <p>11. $-8 \leq y \leq -2$
 $y \leq x$
 $y \leq -3x + 10$
 $f(x, y) = 5x + 14y$</p> | <p>12. $x + 4y \geq 2$
 $2x + 4y \leq 24$
 $2 \leq x \leq 6$
 $f(x, y) = 6x + 7y$</p> | <p>13. $3 \leq y \leq 7$
 $2y + x \leq 8$
 $y - 2x \leq 23$
 $f(x, y) = -3x + 5y$</p> |



Examples 1–2 Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

14. $-9 \leq x \leq -3$
 $-9 \leq y \leq -5$
 $3y + 12x \leq -75$
 $f(x, y) = 20x + 8y$

15. $x \geq -8$
 $3x + 6y \leq 36$
 $2y + 12 \geq 3x$
 $f(x, y) = 10x - 6y$

16. $y \geq |x - 2|$
 $y \leq 8$
 $8y + 5x \leq 49$
 $f(x, y) = -5x - 15y$

17. $x \geq -6$
 $y + x \leq -1$
 $2x + 3y \geq -9$
 $f(x, y) = -10x - 12y$

18. $-5 \geq y \geq -17$
 $y \leq 3x + 19$
 $y \leq -4x + 15$
 $f(x, y) = 8x - 3y$

19. $-8 \leq x \leq 16$
 $y \geq 2x - 10$
 $2y + x \leq 80$
 $f(x, y) = 12x + 15y$

20. $y \leq x + 4$
 $y \geq x - 4$
 $y \leq -x + 10$
 $y \geq -x - 10$
 $f(x, y) = -10x + 9y$

21. $-4 \leq x \leq 8$
 $-8 \leq y \leq 6$
 $y \geq x - 6$
 $4y + 7x \leq 31$
 $f(x, y) = 12x + 8y$

22. $y \geq |x + 1| - 2$
 $0 \leq y \leq 6$
 $-6 \leq x \leq 2$
 $x + 3y \leq 14$
 $f(x, y) = 5x + 4y$

Example 3

23. **COOKING** Jenny's Bakery makes two types of birthday cakes: yellow cake, which sells for \$25, and strawberry cake, which sells for \$35. Both cakes are the same size, but the decorating and assembly time required for the yellow cake is 2 hours, while the time is 3 hours for the strawberry cake. There are 450 hours of labor available for production. How many of each type of cake should be made to maximize revenue?
24. **BUSINESS** The manager of a travel agency is printing brochures and fliers to advertise special discounts on vacation spots during the summer months. Each brochure costs \$0.08 to print, and each flier costs \$0.04 to print. A brochure requires 3 pages, and a flier requires 2 pages. The manager does not want to use more than 600 pages, and she needs at least 50 brochures and 150 fliers. How many of each should she print to minimize the cost?
25. **CCSS PRECISION** Sean has 20 days to paint as many play houses and sheds as he is able. The sheds can be painted at a rate of 2.5 per day, and the play houses can be painted at a rate of 2 per day. He has 45 structures that need to be painted.
- Write a system of inequalities to represent the possible ways Sean can paint the structures.
 - Draw a graph showing the feasible region and list the coordinates of the vertices of the feasible region.
 - If the profit is \$26 per shed and \$30 per play house, how many of each should he paint?
 - What is the maximum profit?
26. **MOVIES** Employees at a local movie theater work 8-hour shifts from noon to 8 P.M. or from 4 P.M. to midnight. The table below shows the number of employees needed and their corresponding pay. Find the numbers of day-shift workers and night-shift workers that should be scheduled to minimize the cost. What is the minimal cost?

Time	noon to 4 P.M.	4 P.M. to 8 P.M.	8 P.M. to midnight
Number of Employees Needed	at least 5	at least 14	6
Rate per Hour	\$5.50	\$7.50	\$7.50



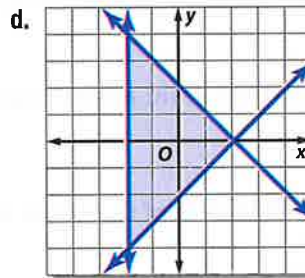
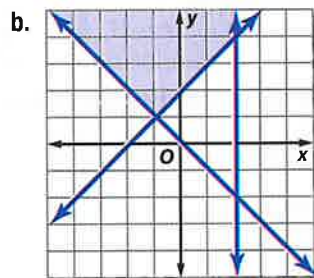
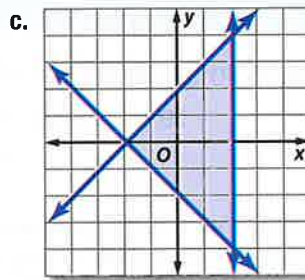
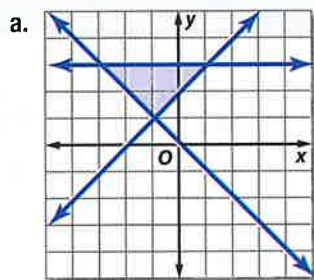
- 27. BUSINESS** Each car on a freight train can hold 4200 pounds of cargo and has a capacity of 480 cubic feet. The freight service handles two types of packages: small—which weigh 25 pounds and are 3 cubic feet each, and large—which are 50 pounds and are 5 cubic feet each. The freight service charges \$5 for each small package and \$8 for each large package.

- Find the number of each type of package that should be placed on a train car to maximize revenue.
- What is the maximum revenue per train car?
- In this situation, is maximizing the revenue necessarily the best thing for the company to do? Explain.

- 28. RECYCLING** A recycling plant processes used plastic into food or drink containers. The plant processes up to 1200 tons of plastic per week. At least 300 tons must be processed for food containers, while at least 450 tons must be processed for drink containers. The profit is \$17.50 per ton for processing food containers and \$20 per ton for processing drink containers. What is the profit if the plant maximizes processing?

H.O.T. Problems Use Higher-Order Thinking Skills

- 29. OPEN ENDED** Create a set of inequalities that forms a bounded region with an area of 20 units² and lies only in the fourth quadrant.
- 30. CHALLENGE** Find the area of the bounded region formed by the following constraints: $y \geq |x| - 3$, $y \leq -|x| + 3$, and $x \geq |y|$.
- 31. CCSS ARGUMENTS** Identify the system of inequalities that is not the same as the other three. Explain your reasoning.



- 32. REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

An unbounded region will not have both a maximum and minimum value.

- 33. WRITING IN MATH** Upon determining a bounded feasible region, Ayumi noticed that vertices $A(-3, 4)$ and $B(5, 2)$ yielded the same maximum value for $f(x, y) = 16y + 4x$. Kelvin confirmed that his constraints were graphed correctly and his vertices were correct. Then he said that those two points were not the only maximum values in the feasible region. Explain how this could have happened.



Systems of Equations in Three Variables

Then

- You solved systems of linear equations in two variables.

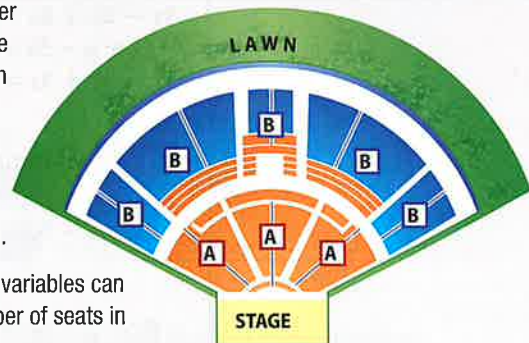
Now

- Solve systems of linear equations in three variables.
- Solve real-world problems using systems of linear equations in three variables.

Why?

- Seats closest to an amphitheater stage cost \$30. The seats in the next section cost \$25, and lawn seats are \$20. There are twice as many seats in section B as in section A. When all 19,200 seats are sold, the amphitheater makes \$456,000.

A system of equations in three variables can be used to determine the number of seats in each section.



New Vocabulary

ordered triple



Common Core State Standards

Content Standards
A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

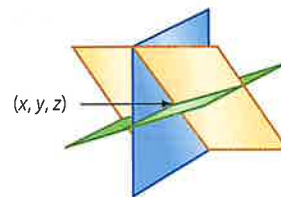
Mathematical Practices
3 Construct viable arguments and critique the reasoning of others.

1 Systems in Three Variables Like systems of equations in two variables, systems in three variables can have one solution, infinite solutions, or no solution. A solution of such a system is an **ordered triple** (x, y, z) .

The graph of an equation in three variables is a three-dimensional graph in the shape of a plane. The graphs of a system of equations in three variables form a system of planes.

One Solution

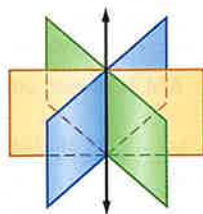
The three individual planes intersect at a specific point.



Infinitely Many Solutions

The planes intersect in a line.

Every coordinate on the line represents a solution of the system.

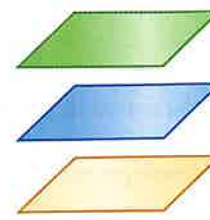
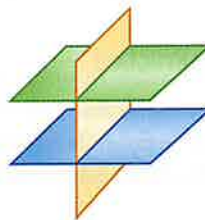
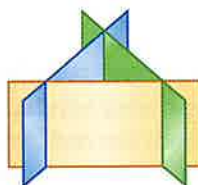


The planes intersect in the same plane.

Every equation is equivalent. Every coordinate in the plane represents a solution of the system.



No Solution There are no points in common with all three planes.



Solving systems of equations in three variables is similar to solving systems of equations in two variables. Use the strategies of substitution and elimination to find the ordered triple that represents the solution of the system.



Example 1 A System with One Solution

Solve the system of equations.

$$\begin{aligned} 3x - 2y + 4z &= 35 \\ -4x + y - 5z &= -36 \\ 5x - 3y + 3z &= 31 \end{aligned}$$

The coefficient of 1 in the second equation makes y a good choice for elimination.

Step 1 Eliminate one variable by using two pairs of equations.

$$\begin{array}{r} 3x - 2y + 4z = 35 \quad \text{Equation 1} \\ -4x + y - 5z = -36 \quad \text{Equation 2} \times 2 \end{array} \quad \begin{array}{l} \text{Multiply by 2.} \\ (+) \end{array} \quad \begin{array}{r} 3x - 2y + 4z = 35 \\ -8x + 2y - 10z = -72 \\ \hline -5x \quad -6z = -37 \end{array}$$

$$\begin{array}{r} -4x + y - 5z = -36 \quad \text{Equation 2} \times 3 \\ 5x - 3y + 3z = 31 \quad \text{Equation 3} \end{array} \quad \begin{array}{l} \text{Multiply by 3.} \\ (+) \end{array} \quad \begin{array}{r} -12x + 3y - 15z = -108 \\ (+) 5x - 3y + 3z = 31 \\ \hline -7x \quad -12z = -77 \end{array}$$

The y -terms in each equation have been eliminated. We now have a system of two equations and two variables, x and z .

Step 2 Solve the system of two equations.

$$\begin{array}{r} -5x - 6z = -37 \\ -7x - 12z = -77 \end{array} \quad \begin{array}{l} \text{Multiply by } -2. \\ (+) \end{array} \quad \begin{array}{r} 10x + 12z = 74 \\ (-) -7x - 12z = -77 \\ \hline 3x = -3 \\ x = -1 \end{array} \quad \begin{array}{l} \text{Eliminate } z. \\ \text{Divide by 3.} \end{array}$$

Use substitution to solve for z .

$$\begin{aligned} -5x - 6z &= -37 && \text{Equation with two variables} \\ -5(-1) - 6z &= -37 && \text{Substitution} \\ 5 - 6z &= -37 && \text{Multiply.} \\ -6z &= -42 && \text{Subtract 5 from each side.} \\ z &= 7 && \text{Divide each side by } -6. \end{aligned}$$

The result is $x = -1$ and $z = 7$.

Step 3 Substitute the two values into one of the original equations to find y .

$$\begin{aligned} -4x + y - 5z &= -36 && \text{Equation 2} \\ -4(-1) + y - 5(7) &= -36 && \text{Substitution} \\ 4 + y - 35 &= -36 && \text{Multiply.} \\ y &= -5 && \text{Add 31 to each side.} \end{aligned}$$

CHECK

$$\begin{aligned} -4x + y - 5z &= -36 && \text{Equation 2} \\ -4(-1) + (-5) - 5(7) &\stackrel{?}{=} -36 && x = -1, y = -5, z = 7 \\ 4 + (-5) - 35 &\stackrel{?}{=} -36 && \text{Simplify.} \\ -36 &= -36 && \checkmark \end{aligned}$$

The solution is $(-1, -5, 7)$.

Guided Practice

1A.
$$\begin{aligned} 2x + 4y - 5z &= 18 \\ -3x + 5y + 2z &= -27 \\ -5x + 3y - z &= -17 \end{aligned}$$

1B.
$$\begin{aligned} 4x - 3y + 6z &= 18 \\ -x + 5y + 4z &= 48 \\ 6x - 2y + 5z &= 0 \end{aligned}$$

StudyTip

Checking Solutions Always substitute your answer into all of the original equations to confirm your answer.



When solving a system of three linear equations with three variables, it is important to check your answer using all three of the original equations. This is necessary because it is possible for a solution to work for two of the equations but not the third.



Example 2 No Solution and Infinite Solutions

Solve each system of equations.

a. $5x + 4y - 5z = -10$
 $-4x - 10y - 8z = -16$
 $6x + 15y + 12z = 24$

Eliminate x in the second two equations.

$$\begin{array}{r} -4x - 10y - 8z = -16 \quad \text{Multiply by 3.} \quad -12x - 30y - 24z = -48 \\ 6x + 15y + 12z = 24 \quad \text{Multiply by 2.} \quad (+) 12x + 30y + 24z = 48 \\ \hline 0 = 0 \end{array}$$

The equation $0 = 0$ is always true. This indicates that the last two equations represent the same plane. Check to see if this plane intersects the first plane.

$$\begin{array}{r} 5x + 4y - 5z = -10 \quad \text{Multiply by 4.} \quad 20x + 16y - 20z = -40 \\ -4x - 10y - 8z = -16 \quad \text{Multiply by 5.} \quad (+) -20x - 50y - 40z = -80 \\ \hline -34y - 60z = -120 \end{array}$$

The planes intersect in a line. So, there are an infinite number of solutions.

b. $-6a + 9b - 12c = 21$
 $-2a + 3b - 4c = 7$
 $10a - 15b + 20c = -30$

Eliminate a in the first two equations.

$$\begin{array}{r} -6a + 9b - 12c = 21 \\ -2a + 3b - 4c = 7 \quad \text{Multiply by } -3. \quad (-) 6a - 9b + 12c = -21 \\ \hline 0 = 0 \end{array}$$

The equation $0 = 0$ is always true. This indicates that the first two equations represent the same plane. Check to see if this plane intersects the last plane.

$$\begin{array}{r} -2a + 3b - 4c = 7 \quad \text{Multiply by 5.} \quad -10a + 15b - 20c = 35 \\ 10a - 15b + 20c = -30 \quad (+) 10a - 15b + 20c = -30 \\ \hline 0 = 5 \end{array}$$

The equation $0 = 5$ is never true. So, there is no solution of this system.

Guided Practice

2A. $-4x - 2y - z = 15$
 $12x + 6y + 3z = 45$
 $2x + 5y + 7z = -29$

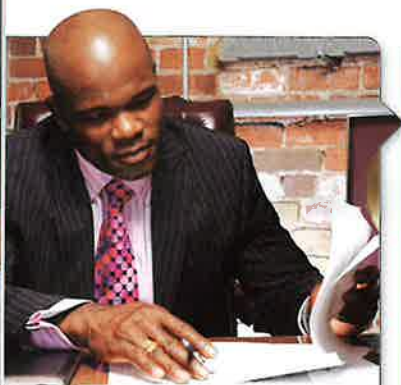
2B. $3x + 5y - 2z = 13$
 $-5x - 2y - 4z = 20$
 $-14x - 17y + 2z = -19$

StudyTip

Infinite and No Solutions
 When solving a system of more than two equations, $0 = 5$ will always yield no solution, while $0 = 0$ may not always yield infinite solutions.

2 Real-World Problems When solving problems involving three variables, use the four-step plan to help organize the information. Identify the three variables and what they represent. Then use the information in the problem to form equations using the variables. Once you have three equations and all three variables are represented, you can solve the problem.





Real-World Career

Music Management

Music management involves acting as the talent manager to the artists. Other duties include negotiating with record labels, music promoters, and tour promoters. Managers usually earn a percentage of the artist's income. A bachelor's degree is usually required.

Real-World Example 3 Write and Solve a System of Equations

CONCERTS Refer to the beginning of the lesson. Write and solve a system of equations to determine how many seats are in each section of the amphitheater.

Understand Define the variables. x = seats in section A
 y = seats in section B
 z = lawn seats

Plan There are 19,200 seats.

$$x + y + z = 19,200$$

The total revenue is \$456,000.

Equation 1

$$30x + 25y + 20z = 456,000$$

Equation 2

There are twice as many seats in section B as in section A.

$$y = 2x$$

Equation 3

Solve Solve the system.

Step 1 Substitute $y = 2x$ in the first two equations.

$$x + y + z = 19,200 \quad \text{Equation 1}$$

$$x + 2x + z = 19,200 \quad y = 2x$$

$$3x + z = 19,200 \quad \text{Add.}$$

$$30x + 25y + 20z = 456,000 \quad \text{Equation 2}$$

$$30x + 25(2x) + 20z = 456,000 \quad y = 2x$$

$$80x + 20z = 456,000 \quad \text{Simplify.}$$

Step 2 Solve the system of two equations in two variables.

$$\begin{array}{r} 3x + z = 19,200 \\ 80x + 20z = 456,000 \end{array} \quad \begin{array}{l} \text{Multiply by } -20. \\ \rightarrow \end{array} \quad \begin{array}{r} -60x - 20z = -384,000 \\ (+) 80x + 20z = 456,000 \\ \hline 20x \qquad \qquad = 72,000 \\ x = 3600 \end{array}$$

Step 3 Substitute to find z .

$$3x + z = 19,200 \quad \text{Remaining equation in two variables}$$

$$3(3600) + z = 19,200 \quad x = 3600$$

$$10,800 + z = 19,200 \quad \text{Multiply.}$$

$$z = 8400 \quad \text{Simplify.}$$

Step 4 Substitute to find y .

$$y = 2x \quad \text{Equation 3}$$

$$y = 2(3600) \text{ or } 7200 \quad x = 3600$$

The solution is (3600, 7200, 8400). There are 3600 seats in section A, 7200 in section B, and 8400 lawn seats.

Check Substitute the values into either of the first two equations.

Guided Practice

3. Ms. Garza invested \$50,000 in three different accounts. She invested three times as much money in an account that paid 8% interest than an account that paid 10% interest. The third account earned 12% interest. If she earned a total of \$5160 in interest in a year, how much did she invest in each account?





Examples 1–2 Solve each system of equations.

$$\begin{aligned} 1. \quad & -3a - 4b + 2c = 28 \\ & a + 3b - 4c = -31 \\ & 2a + 3c = 11 \end{aligned}$$

$$\begin{aligned} 2. \quad & 3y - 5z = -23 \\ & 4x + 2y + 3z = 7 \\ & -2x - y - z = -3 \end{aligned}$$

$$\begin{aligned} 3. \quad & 3x + 6y - 2z = -6 \\ & 2x + y + 4z = 19 \\ & -5x - 2y + 8z = 62 \end{aligned}$$

$$\begin{aligned} 4. \quad & -4r - s + 3t = -9 \\ & 3r + 2s - t = 3 \\ & r + 3s - 5t = 29 \end{aligned}$$

$$\begin{aligned} 5. \quad & 3x + 5y - z = 12 \\ & -2x - 3y + 5z = 14 \\ & 4x + 7y + 3z = 38 \end{aligned}$$

$$\begin{aligned} 6. \quad & 2a - 3b + 5c = 58 \\ & -5a + b - 4c = -51 \\ & -6a - 8b + c = 22 \end{aligned}$$

Example 3

7. **DOWNLOADING** Heather downloaded some television shows. A sitcom uses 0.3 gigabyte of memory; a drama, 0.6 gigabyte; and a talk show, 0.6 gigabyte. She downloaded 7 programs totaling 3.6 gigabytes. There were twice as many episodes of the drama as the sitcom.

- Write a system of equations for the number of episodes of each type of show.
- How many episodes of each show did she download?

Practice and Problem Solving

Extra Practice is on page R3.

Examples 1–2 Solve each system of equations.

$$\begin{aligned} 8. \quad & -5x + y - 4z = 60 \\ & 2x + 4y + 3z = -12 \\ & 6x - 3y - 2z = -52 \end{aligned}$$

$$\begin{aligned} 9. \quad & 4a + 5b - 6c = 2 \\ & -3a - 2b + 7c = -15 \\ & -a + 4b + 2c = -13 \end{aligned}$$

$$\begin{aligned} 10. \quad & -2x + 5y + 3z = -25 \\ & -4x - 3y - 8z = -39 \\ & 6x + 8y - 5z = 14 \end{aligned}$$

$$\begin{aligned} 11. \quad & 4r + 6s - t = -18 \\ & 3r + 2s - 4t = -24 \\ & -5r + 3s + 2t = 15 \end{aligned}$$

$$\begin{aligned} 12. \quad & -2x + 15y + z = 44 \\ & 4x + 3y + 3z = 18 \\ & -3x + 6y - z = 8 \end{aligned}$$

$$\begin{aligned} 13. \quad & 4x + 2y + 6z = 13 \\ & -12x + 3y - 5z = 8 \\ & -4x + 7y + 7z = 34 \end{aligned}$$

$$\begin{aligned} 14. \quad & 8x + 3y + 6z = 43 \\ & -3x + 5y + 2z = 32 \\ & 5x - 2y + 5z = 24 \end{aligned}$$

$$\begin{aligned} 15. \quad & -6x - 5y + 4z = 53 \\ & 5x + 3y + 2z = -11 \\ & 8x - 6y + 5z = 4 \end{aligned}$$

$$\begin{aligned} 16. \quad & -9a + 3b - 2c = 61 \\ & 8a + 7b + 5c = -138 \\ & 5a - 5b + 8c = -45 \end{aligned}$$

$$\begin{aligned} 17. \quad & 2x - y + z = 1 \\ & x + 2y - 4z = 3 \\ & 4x + 3y - 7z = -8 \end{aligned}$$

$$\begin{aligned} 18. \quad & x + 2y = 12 \\ & 3y - 4z = 25 \\ & x + 6y + z = 20 \end{aligned}$$

$$\begin{aligned} 19. \quad & r - 3s + t = 4 \\ & 3r - 6s + 9t = 5 \\ & 4r - 9s + 10t = 9 \end{aligned}$$

Example 3

20. **CCSS SENSE-MAKING** A friend e-mails you the results of a recent high school swim meet. The e-mail states that 24 individuals placed, earning a combined total of 53 points. First place earned 3 points, second place earned 2 points, and third place earned 1 point. There were as many first-place finishers as second- and third-place finishers combined.

- Write a system of three equations that represents how many people finished in each place.
- How many swimmers finished in first place, in second place, and in third place?
- Suppose the e-mail had said that the athletes scored a combined total of 47 points. Explain why this statement is false and the solution is unreasonable.

21. **AMUSEMENT PARKS** Nick goes to the amusement park to ride roller coasters, bumper cars, and water slides. The wait for the roller coasters is 1 hour, the wait for the bumper cars is 20 minutes long, and the wait for the water slides is only 15 minutes long. Nick rode 10 total rides during his visit. Because he enjoys roller coasters the most, the number of times he rode the roller coasters was the sum of the times he rode the other two rides. If Nick waited in line for a total of 6 hours and 20 minutes, how many of each ride did he go on?



22. **BUSINESS** Ramón usually gets one of the routine maintenance options at Annie's Garage. Today however, he needs a different combination of work than what is listed.

a. Assume that the price of an option is the same price as purchasing each item separately. Find the prices for an oil change, a radiator flush, and a brake pad replacement.

b. If Ramón wants his brake pads replaced and his radiator flushed, how much should he plan to spend?

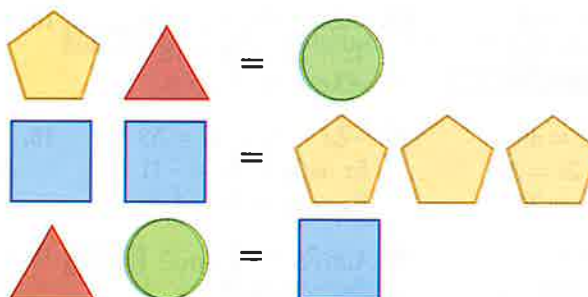


23. **FINANCIAL LITERACY** Kate invested \$100,000 in three different accounts. If she invested \$30,000 more in account A than account C and is expected to earn \$6300 in interest, how much did she invest in each account?

Account	Expected Interest
A	4%
B	8%
C	10%

H.O.T. Problems Use Higher-Order Thinking Skills

24. **CCSS REASONING** Write a system of equations to represent the three rows of figures below. Use the system to find the number of red triangles that will balance one green circle.



25. **CHALLENGE** The general form of an equation for a parabola is $y = ax^2 + bx + c$, where (x, y) is a point on the parabola. If three points on a parabola are $(2, -10)$, $(-5, -101)$, and $(6, -90)$, determine the values of a , b , and c and write the general form of the equation.

26. **PROOF** Consider the following system and prove that if $b = c = -a$, then $ty = a$.

$$\begin{aligned} rx + ty + vz &= a \\ rx - ty + vz &= b \\ rx + ty - vz &= c \end{aligned}$$

27. **OPEN ENDED** Write a system of three linear equations that has a solution of $(-5, -2, 6)$. Show that the ordered triple satisfies all three equations.

28. **REASONING** Use the diagrams of solutions of systems of equations on page 167 to consider a system of inequalities in three variables. Describe the solution of such a system.

29. **WRITING IN MATH** Use your knowledge of solving a system of three linear equations with three variables to explain how to solve a system of four equations with four variables.



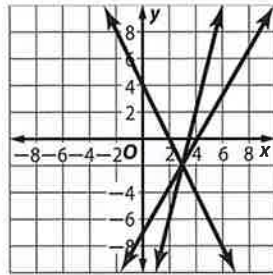
Standardized Test Practice

30. Solve the system of equations shown below.

$$\begin{cases} x - y + z = 0 \\ -5x + 3y - 2z = -1 \\ 2x - y + 4z = 11 \end{cases}$$

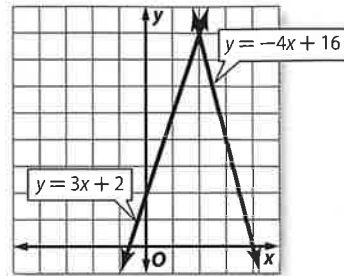
- A (0, 3, 3)
- B (2, 5, 3)
- C no solution
- D infinitely many solutions

31. **SAT/ACT** The graph shows which system of equations?



- | | |
|---|---|
| A $y + 14 = 4x$
$y = 4 - 2x$
$-7 = y - \frac{5}{3}x$ | D $y - 14x = 4$
$2x = 4 + y$
$7 = y - \frac{5}{3}x$ |
| B $y - 14 = 4x$
$y = 4 + 2x$
$-7 = y + \frac{5}{3}x$ | E $y - 4x = 14$
$y = 2x + 4$
$7 = y + \frac{5}{3}x$ |
| C $y + 14x = 4$
$-2y = 4 + y$
$-7 = y - \frac{5}{3}x$ | |

32. **EXTENDED RESPONSE** Use the graph to find the solution of the systems of equations. Describe one way to check the solution.



33. Which of the following represents a correct procedure for solving each equation?

- | | |
|--|---|
| F $-3(x - 7) = -16$
$-3x - 21 = -16$
$-3x = 5$
$x = -\frac{5}{3}$ | H $2(x - 4) = 20$
$2x - 8 = 20$
$2x = 12$
$x = 6$ |
| G $7 - 4x = 3x + 27$
$7 - 7x = 27$
$-7x = -\frac{20}{7}$
$x = 20$ | J $6(2x + 1) = 30$
$12x + 6 = 30$
$12x = 24$
$x = 2$ |

Spiral Review

A feasible region has vertices at $(-3, 2)$, $(1, 3)$, $(6, 1)$, and $(2, -2)$. Find the maximum and minimum values of each function. (Lesson 3-3)

34. $f(x, y) = 2x - y$ 35. $f(x, y) = x + 5y$ 36. $f(x, y) = y - 4x$ 37. $f(x, y) = -x + 3y$

38. **SKI CLUB** The ski club's budget for the year is \$4250. They are able to find skis for \$75 per pair and boots for \$40 per pair. They know they should buy more boots than skis because the skis are adjustable to several sizes of boots. (Lesson 3-2)

- a. Give an example of three different purchases that the ski club can make.
- b. Suppose the ski club wants to spend all of its budget. What combination of skis and boots should they buy? Explain.

Skills Review

Solve each system of equations.

- | | | | |
|----------------------------------|-------------------------------------|--------------------------------------|----------------------------------|
| 39. $x = y + 5$
$3x + y = 19$ | 40. $3x - 2y = 1$
$4x + 2y = 20$ | 41. $5x + 3y = 25$
$4x + 7y = -3$ | 42. $y = x - 7$
$2x - 8y = 2$ |
|----------------------------------|-------------------------------------|--------------------------------------|----------------------------------|



3 Mid-Chapter Quiz

Lessons 3-1 through 3-4

Solve each system of equations. (Lesson 3-1)

$$\begin{aligned} 1. \quad & 2x - 3y = 9 \\ & 4x + 3y = 9 \end{aligned}$$

$$\begin{aligned} 2. \quad & x + 2y = 7 \\ & y = 5x - 2 \end{aligned}$$

$$\begin{aligned} 3. \quad & -x + y = 2 \\ & 4x - 3y = -3 \end{aligned}$$

$$\begin{aligned} 4. \quad & \frac{1}{2}x + \frac{1}{3}y = 7 \\ & \frac{1}{5}x - \frac{2}{3}y = -2 \end{aligned}$$

Solve each system of inequalities by graphing. (Lesson 3-2)

$$\begin{aligned} 5. \quad & x + y \leq 4 \\ & y \geq x \end{aligned}$$

$$\begin{aligned} 6. \quad & 2x + 3y > 12 \\ & 3x - y < 21 \end{aligned}$$

$$\begin{aligned} 7. \quad & x - y > 0 \\ & 4 + y \leq 2x \end{aligned}$$

$$\begin{aligned} 8. \quad & 2y - 5x \leq 6 \\ & 4x + y < -4 \end{aligned}$$

9. **MULTIPLE CHOICE** Which statement best describes the graphs of the two equations? (Lesson 3-1)

$$\begin{aligned} & x + 4y = 8 \\ & 3x + 12y = 2 \end{aligned}$$

- A The lines are parallel.
- B The lines are the same.
- C The lines intersect in only one point.
- D The lines intersect in more than one point, but are not the same.

Solve each system of equations. (Lesson 3-4)

$$\begin{aligned} 10. \quad & x - 2y + 3z = 1 \\ & 4y - 4z = 12 \\ & 8y - 14z = 0 \end{aligned}$$

$$\begin{aligned} 11. \quad & x + y + z = 4 \\ & x + 3y + 3z = 10 \\ & 2x + y - z = 3 \end{aligned}$$

$$\begin{aligned} 12. \quad & 2x - y - 2z = 5 \\ & 10x + 8z = -4 \\ & 3x - y = 1 \end{aligned}$$

$$\begin{aligned} 13. \quad & 2x + 3y + z = 0 \\ & 3x + y = 1 \\ & x - 2y + z = 9 \end{aligned}$$

14. **MULTIPLE CHOICE** Seela rented a raft from River Rafter's Inc. She paid \$100 to rent the raft and \$25 for each hour. Martin rented a raft from Oscar's Outdoor Shop. He paid \$50 to rent the raft and \$35 per hour. For what number of hours will both rafting companies charge the same amount? (Lesson 3-1)

- F 0
- G 4
- H 5
- J 10

15. **CARPENTRY** Cal's Carpentry makes tables and chairs. The process involves some carpentry time and some finishing time. The carpentry times and finishing times are listed in the table below.

Product	Carpentry Time (hr)	Finishing Time (hr)
chair	3	0.5
table	2	1

Cal's Carpentry can work for a maximum of 108 carpentry hours and 20 finishing hours per day. The profit is \$35 for a table and \$25 for a chair. How many tables and chairs should be made each day to maximize profit? (Lesson 3-3)

- a. Using c for the number of chairs and t for the number of tables, write a system of inequalities to represent this situation.
- b. Draw the graph showing the feasible region.
- c. Determine the number of tables and chairs that need to be made to maximize profit. What is the maximum profit?

16. **DRAMA** On opening night of the drama club's play, they made \$1366. They sold a total of 199 tickets. They charged \$8.50 for each adult ticket and \$5.00 for each child's ticket. Write a system of equations that can be used to find the number of adult tickets and the number of children's tickets sold. (Lesson 3-1)

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and the minimum values of the given function. (Lesson 3-3)

$$\begin{aligned} 17. \quad & 5 \geq y \geq -3 \\ & 4x + y \leq 5 \\ & -2x + y \leq 5 \\ & f(x, y) = 4x - 3y \end{aligned}$$

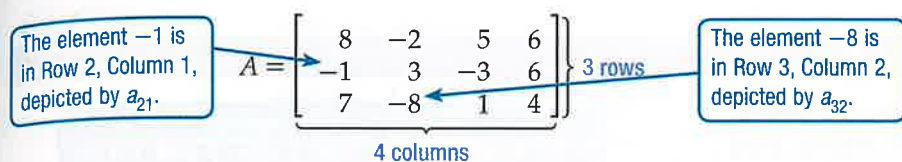
$$\begin{aligned} 18. \quad & x \geq -10 \\ & 1 \geq y \geq -6 \\ & 3x + 4y \leq -8 \\ & 2y \geq x - 10 \\ & f(x, y) = 2x + y \end{aligned}$$

19. **GEOMETRY** An isosceles trapezoid has shorter base of measure a , longer base of measure c , and congruent legs of measure b . The perimeter of the trapezoid is 58 inches. The average of the bases is 19 inches and the longer base is twice the leg plus 7. (Lesson 3-4)

- a. Find the lengths of the sides of the trapezoid.
- b. Find the area of the trapezoid.



A **matrix** is a rectangular array of variables or constants in rows and columns, usually enclosed in brackets. In a matrix, the numbers or data are organized so that each position in the matrix has a purpose. Each value in the matrix is called an **element**. A matrix is usually named using an uppercase letter.



A matrix can be described by its **dimensions**. A matrix with m rows and n columns is an $m \times n$ matrix (read “ m by n ”). Matrix A above is a 3×4 matrix because it has 3 rows and 4 columns. a_{12} refers to an element of A , whereas b_{12} refers to an element of B .

Activity 1 Organize Data in a Matrix



FOOTBALL The West High School football team used five running backs throughout its season. Coach Williams wanted to compare the statistics of each player.

Joey: 11 games,
72 attempts, 439 yards,
6.10 average, 8 TDs

DeShawn: 9 games,
143 attempts, 1024 yards,
7.16 average, 12 TDs

Dario: 11 games,
164 attempts, 885 yards,
5.40 average, 15 TDs

Leo: 11 games,
84 attempts, 542 yards,
6.45 average, 7 TDs

Alex: 10 games,
151 attempts, 966 yards,
6.40 average, 11 TDs

a. Organize the data in a matrix from greatest to least attempts.

Player	Games	Attempts	Yards	Average	TDs
Dario	11	164	885	5.40	15
Alex	10	151	966	6.40	11
DeShawn	9	143	1024	7.16	12
Leo	11	84	542	6.45	7
Joey	11	72	439	6.10	8

b. What are the dimensions of the matrix? What value is a_{34} ?

There are five rows and five columns, so the dimensions are 5×5 . The value a_{34} , which is in the third row and fourth column, is 7.16.

11	164	885	5.40	15
10	151	966	6.40	11
9	143	1024	7.16	12
11	84	542	6.45	7
11	72	439	6.10	8

Exercise

- Julie is shopping for a new smartphone and discovers that many different options are available.
 - Organize the data in a matrix. List the options in descending order, and label the columns options, price, memory, color, and interface respectively.
 - What are the dimensions of the matrix? What is the value of a_{23} ?

Option 1

\$420
Memory: 512, Color: 24,
Interface: infrared

Option 2

\$399
Memory: 512, Color: 24,
Interface: Bluetooth

Option 3

\$315
Memory: 256, Color: 24,
Interface: infrared

Option 4

\$289
Memory: 128, Color: 18,
Interface: wi-fi

(continued on the next page)

Spreadsheet Lab

Organizing Data with Matrices *Continued*

People in the workforce often use computer **spreadsheets** to organize, display, and analyze data. Similar to a matrix, data in a spreadsheet are entered into rows and columns. Then the data can be used to create graphs or perform calculations.

Activity 2 Organize Data in a Spreadsheet

The manager of a gourmet food store has gathered data on the number of pounds of bulk coffees they have sold each week in January. Enter the data into a spreadsheet.

Weekly Sales for January				
Coffee	1/5	1/12	1/19	1/26
Hawaiian Kona	17	22	11	23
Mocha Java	31	34	22	29
House Blend	55	61	44	71
Espresso	41	36	60	77
Decaf Espresso	23	29	19	44
Breakfast Blend	8	18	19	31
Decaf Breakfast Blend	22	18	30	32
Organic Italian Roast	26	16	31	39

Use Column A for the types of coffee, Column B for the sales in the week starting 1/5, Column C for sales in the week starting 1/12, and Columns D and E for the sales in the weeks starting 1/19 and 1/26.

	A	B	C	D	E
1	Hawaiian Kona	17	22	11	23
2	Mocha Java	31	34	22	29
3	House Blend	55	61	44	71
4	Espresso	41	36	60	77
5	Decaf Espresso	23	29	19	44
6	Breakfast Blend	8	18	19	31
7	Decaf Breakfast Blend	22	18	30	32
8	Organic Italian Roast	26	16	31	39

Each **row** contains data for the same type of coffee. Row 2 represents Mocha Java.

Each **cell** of the spreadsheet contains one piece of data. Cell D7 contains the value 30, representing the number of pounds of Decaf Breakfast Blend sold the week of 1/19.

Exercises

- Refer to Activity 2. A SUM formula allows you to find the sum of the entries in a column or row.
 - The formula $\text{=SUM}(B1:B8)$ finds the sum of column B. Enter formulas in cells B9, C9, D9, and E9 to find the sums of those columns. What do the sums of the columns represent in the situation?
 - Enter formulas in cells F1 through F8 to find the sums of rows 1 through 8. What do the sums of the rows represent in the situation?
 - Find the sum of row 9 and the sum of column F. What do you observe? Explain.
- Enter the data on smartphones from Exercise 1 into a spreadsheet.
- Compare and contrast how data are organized in a spreadsheet and in a matrix.

Then

You organized data into matrices.

Now

- 1 Analyze data in matrices.
- 2 Perform algebraic operations with matrices.

Why?

Coastal Sales Company has three locations in Florida. The matrices below show the average daily wages and annual sales of all of the representatives.

	Miami		Tampa		Tallahassee	
	Wages	Sales	Wages	Sales	Wages	Sales
Entry	900	145,000	900	122,000	1050	109,500
Assistant	2400	225,000	1800	145,500	1800	135,000
Associate	2700	290,000	1800	160,000	1800	150,500



New Vocabulary
 scalar
 scalar multiplication

1 Analyze Data Data that are organized in a matrix can be analyzed and interpreted. Sometimes, further analysis is needed. Other times, the data are meaningless.

Real-World Example 1 Analyze Data with Matrices



BUSINESS The manager at the Miami location would like to use their matrix to further analyze the representatives.

	Wages	Sales
Entry	900	145,000
Assistant	2400	225,000
Associate	2700	290,000

a. Add the elements in each column and interpret the results.

The sum of the first column is 6000. This is the total average daily wages of the three types of employees. The sum of the second column is 660,000. This is the total average annual sales of the employees.

b. The manager wants to determine the average wages for all of the employees at the Miami location. He decides to divide the total of the first column by three, the number of different positions. What is the average?

The average is $6000 \div 3$ or 2000.

c. Is this an accurate average? Explain.

If there is the same number of each type of representative, then the average is accurate. If there is more of one type of representative than the others, then the average is not accurate and would need to be weighted accordingly.

d. Would adding the rows provide any meaningful data for the manger?

No. The sum of a row includes two different forms of data, wages and sales.

Guided Practice

1. **POPULATION** The table displays U.S. Census data.

- A. Organize the data in a matrix.
- B. Add the elements in the columns and interpret the results.
- C. Add the elements in the rows and interpret the results.
- D. Would finding the average of the rows or columns provide any meaningful data?

Latino Population in the U.S. (millions)		
Age	Male	Female
0-19	7.1	6.6
20-39	6.8	5.9
40-59	3.2	2.2
60+	1.1	1.4



2 Algebraic Operations Several algebraic operations can be performed on data that are organized in matrices. Matrices can be added or subtracted if and only if they have the same dimensions.

Key Concept Adding and Subtracting Matrices

Words To add or subtract two matrices with the same dimensions, add or subtract their corresponding elements.

$$A + B = A + B$$

$$\text{Symbols} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$A - B = A - B$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Example $\begin{bmatrix} 3 & -5 \\ 1 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -9 & 10 \end{bmatrix} = \begin{bmatrix} 3+2 & -5+0 \\ 1+(-9) & 7+10 \end{bmatrix}$

StudyTip

Corresponding Elements Elements are *corresponding* if they are in the exact same position in each matrix.

Example 2 Add and Subtract Matrices

Find each of the following for $A = \begin{bmatrix} 16 & 2 \\ -9 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -4 & -1 \\ -3 & -7 \end{bmatrix}$, and $C = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$.

a. $A + B$

$$A + B = \begin{bmatrix} 16 & 2 \\ -9 & 8 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ -3 & -7 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 16 + (-4) & 2 + (-1) \\ -9 + (-3) & 8 + (-7) \end{bmatrix} \quad \text{Add corresponding elements.}$$

$$= \begin{bmatrix} 12 & 1 \\ -12 & 1 \end{bmatrix} \quad \text{Simplify.}$$

b. $B - C$

$$B - C = \begin{bmatrix} -4 & -1 \\ -3 & -7 \end{bmatrix} - \begin{bmatrix} 8 \\ 6 \end{bmatrix} \quad \text{Substitution}$$

Since the dimensions of B and C are different, you cannot subtract the matrices.

c. $B - A$

$$B - A = \begin{bmatrix} -4 & -1 \\ -3 & -7 \end{bmatrix} - \begin{bmatrix} 16 & 2 \\ -9 & 8 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} -4 - 16 & -1 - 2 \\ -3 - (-9) & -7 - 8 \end{bmatrix} \quad \text{Subtract corresponding elements.}$$

$$= \begin{bmatrix} -20 & -3 \\ 6 & -15 \end{bmatrix} \quad \text{Simplify.}$$

Guided Practice

2A. $\begin{bmatrix} -3 & 4 \\ -9 & -5 \end{bmatrix} - \begin{bmatrix} -4 & 12 \\ 8 & -7 \end{bmatrix}$

2B. $\begin{bmatrix} -9 & 8 & 3 \\ -2 & 4 & -7 \end{bmatrix} + \begin{bmatrix} -4 & -3 & 6 \\ -9 & -5 & 18 \end{bmatrix}$

2C. $\begin{bmatrix} 8 & -3 \\ -2 & 0 \\ 1 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 1 & -4 & 2 \\ 10 & -6 & 9 & 0 \end{bmatrix}$



You can multiply any matrix by a constant called a **scalar**. When you do this, you multiply each individual element by the value of the scalar. This operation is called **scalar multiplication**.

ReadingMath

Scalar Think of a scalar as a coefficient for a variable, but instead it is for a matrix.

KeyConcept Multiplying by a Scalar

Words To multiply a matrix by a scalar k , multiply each element by k .

$$k \cdot A = kA$$

Symbols $k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

Example $-3 \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} -3(4) & -3(1) \\ -3(7) & -3(-2) \end{bmatrix}$

StudyTip

Scalar Multiplication

The bracket of a matrix is treated just like a regular grouping symbol. So when multiplying by a scalar, distribute the same way as with a grouping symbol.

Example 3 Multiply a Matrix by a Scalar

If $R = \begin{bmatrix} -12 & 8 & 6 \\ -16 & 4 & 19 \end{bmatrix}$, find $5R$.

$$5R = 5 \begin{bmatrix} -12 & 8 & 6 \\ -16 & 4 & 19 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 5(-12) & 5(8) & 5(6) \\ 5(-16) & 5(4) & 5(19) \end{bmatrix} \quad \text{Distribute the scalar.}$$

$$= \begin{bmatrix} -60 & 40 & 30 \\ -80 & 20 & 95 \end{bmatrix} \quad \text{Multiply.}$$

GuidedPractice

3. If $T = \begin{bmatrix} 8 & 0 & 3 & -2 \\ -1 & -4 & -2 & 9 \end{bmatrix}$, find $-4T$.

Many properties of real numbers also hold true for matrices. A summary of these properties is listed below.

KeyConcept Properties of Matrix Operations

For any matrices A , B , and C for which the matrix sum and product are defined and any scalar k , the following properties are true.

Commutative Property of Addition $A + B = B + A$

Associative Property of Addition $(A + B) + C = A + (B + C)$

Left Scalar Distributive Property $k(A + B) = kA + kB$

Right Scalar Distributive Property $(A + B)k = kA + kB$

Multi-step operations can be performed on matrices. The order of these operations is the same as with real numbers.



Example 4 Multi-Step Operations

If $A = \begin{bmatrix} -9 & 12 \\ 2 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -8 \\ 2 & -3 \end{bmatrix}$, find $-4B - 3A$.

$$-4B - 3A = -4 \begin{bmatrix} -4 & -8 \\ 2 & -3 \end{bmatrix} - 3 \begin{bmatrix} -9 & 12 \\ 2 & -6 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} -4(-4) & -4(-8) \\ -4(2) & -4(-3) \end{bmatrix} - \begin{bmatrix} 3(-9) & 3(12) \\ 3(2) & 3(-6) \end{bmatrix} \quad \text{Distribute the scalars in each matrix.}$$

$$= \begin{bmatrix} 16 & 32 \\ -8 & 12 \end{bmatrix} - \begin{bmatrix} -27 & 36 \\ 6 & -18 \end{bmatrix} \quad \text{Multiply.}$$

$$= \begin{bmatrix} 16 - (-27) & 32 - 36 \\ -8 - 6 & 12 - (-18) \end{bmatrix} \quad \text{Subtract corresponding elements.}$$

$$= \begin{bmatrix} 43 & -4 \\ -14 & 30 \end{bmatrix} \quad \text{Simplify.}$$

Guided Practice

4. If $A = \begin{bmatrix} -5 & 3 \\ 6 & -8 \\ 2 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 12 & 5 \\ 5 & -4 \\ 4 & -7 \end{bmatrix}$, find $-6B + 7A$.



Real-World Career

Financial Planner

Financial planners often use matrices to organize and describe the data they use. Financial planners need a bachelor's degree. Usually they have degrees in accounting, finance, economics, business, marketing, or commerce.

Study Tip

Corresponding Elements

When representing quantities with multiple matrices, make sure the corresponding elements represent corresponding quantities.

Matrices can be used in many business applications.

Real-World Example 5 Use Multi-Step Operations with Matrices

BUSINESS Refer to the application at the beginning of the lesson. Express the average wages and sales for the entire company for a 5-day week.

To calculate the 5-day sales for the entire company, each matrix needs to be multiplied by 5 and the totals added together.

$$5 \begin{bmatrix} 900 & 145,000 \\ 2400 & 225,000 \\ 2700 & 290,000 \end{bmatrix} + 5 \begin{bmatrix} 900 & 122,000 \\ 1800 & 145,500 \\ 1800 & 160,000 \end{bmatrix} + 5 \begin{bmatrix} 1050 & 109,500 \\ 1800 & 135,000 \\ 1800 & 150,500 \end{bmatrix} \quad \text{Write matrices.}$$

$$= \begin{bmatrix} 4500 & 725,000 \\ 12,000 & 1,125,000 \\ 13,500 & 1,450,000 \end{bmatrix} + \begin{bmatrix} 4500 & 610,000 \\ 9000 & 727,500 \\ 9000 & 800,000 \end{bmatrix} + \begin{bmatrix} 5250 & 547,500 \\ 9000 & 675,000 \\ 9000 & 752,500 \end{bmatrix} \quad \text{Multiply scalars.}$$

$$= \begin{matrix} & \text{Wages} & \text{Sales} \\ \text{Entry} & \begin{bmatrix} 14,250 & 1,882,500 \\ 30,000 & 2,527,500 \\ 31,500 & 3,002,500 \end{bmatrix} \end{matrix} \quad \text{Add matrices.}$$

The final matrix indicates the average weekly sales and wages for all of the representatives of the company.

Guided Practice

5. Use the data above to calculate the average yearly sales and wages for the company assuming 260 working days.



Example 1

1. **CCSS MODELING** Use the table that shows the city and highway gas mileage of five different types of vehicles.

Vehicle	SUV	Mini-van	Sedan	Compact	APV
City	23	21	21	42	61
Highway	25	24	32	49	70

Source: Auto Hoppers

- Organize the gas mileages in a matrix.
- Add the elements of each row and interpret the results.
- Add the elements of each column and interpret the results.

Example 2

Perform the indicated operations. If the matrix does not exist, write *impossible*.

- $[-8 \ 2 \ 6] + [11 \ -7 \ 1]$
- $[9 \ -8 \ 4] + [12 \ 2]$
- $\begin{bmatrix} 7 & -12 \\ 15 & 4 \end{bmatrix} - \begin{bmatrix} 9 & 6 \\ 4 & -9 \end{bmatrix}$
- $\begin{bmatrix} 5 & 13 & -6 \\ 3 & -17 & 2 \end{bmatrix} - \begin{bmatrix} -2 & -18 & 8 \\ 2 & -11 & 0 \end{bmatrix}$

Example 3

Perform the indicated operations. If the matrix does not exist, write *impossible*.

- $3 \begin{bmatrix} 6 & 4 & 0 \\ -2 & 14 & -8 \\ -4 & -6 & 7 \end{bmatrix}$
- $-6 \begin{bmatrix} 15 & -9 & 2 & 3 \\ 6 & -11 & 14 & -2 \\ 4 & -8 & -10 & 27 \end{bmatrix}$

Example 4

Use matrices A , B , C , and D to find the following.

$$A = \begin{bmatrix} 6 & -4 \\ 3 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 8 & -1 \\ -2 & 7 \end{bmatrix} \quad C = \begin{bmatrix} -4 & -6 \\ 12 & -7 \end{bmatrix} \quad D = \begin{bmatrix} 9 & 6 & 0 \\ -2 & 8 & 0 \end{bmatrix}$$

- $4B - 2A$
- $-5B - 2D$
- $-8C + 3A$
- $-4C - 5B$

Example 5

12. **GRADES** Geraldo, Olivia, and Nikki have had two tests in their math class. The table shows the test grades for each student.

- Write a matrix for the information from each test.
- Find the sum of the scores from the two tests expressed as a matrix.
- Express the difference in scores from test 1 to test 2 as a matrix.

Student	Test 1	Test 2
Geraldo	85	72
Olivia	75	74
Nikki	96	83

Practice and Problem Solving

Extra Practice is on page R3.

Example 1

13. **SHOES** A consumer service company rated several pairs of shoes by cost, level of comfort, look, and longevity using a scale of 1–5, with 1 being low and 5 being high.

- Write a 4×4 matrix to organize this information.
- Which shoe would you buy based on this information, and why?
- Would finding the sum of the rows or columns provide any useful information? Explain your reasoning.

Brand	Cost	Comfort	Look	Longevity
A	3	2	2	1
B	4	3	2	3
C	5	5	4	4
D	1	5	5	2

Example 2

Perform the indicated operations. If the matrix does not exist, write *impossible*.

- $\begin{bmatrix} 12 & -5 \\ -8 & -3 \end{bmatrix} + \begin{bmatrix} -6 & 11 \\ -7 & 2 \end{bmatrix}$
- $\begin{bmatrix} 9 & 5 \\ -2 & 16 \end{bmatrix} + \begin{bmatrix} -6 & -3 & 7 \\ 12 & 2 & -4 \end{bmatrix}$



Examples 3–4 16. BUSINESS The drink menu from a fast-food restaurant is shown at the right. The store owner has decided that all of the prices must be increased by 10%.

Drink	Small	Medium	Large
Soda	\$0.95	\$1.00	\$1.05
Iced tea	\$0.75	\$0.80	\$0.85
Lemonade	\$0.75	\$0.80	\$0.85
Coffee	\$1.00	\$1.10	\$1.20

- Write matrix C to represent the current prices.
- What scalar can be used to determine a matrix N to represent the new prices?
- Find N .
- What is $N - C$? What does this represent in this situation?

Use matrices A , B , C , and D to find the following.

$$A = \begin{bmatrix} 0 & -7 \\ 8 & 12 \end{bmatrix} \quad B = \begin{bmatrix} 11 & 4 \\ -3 & -17 \end{bmatrix} \quad C = \begin{bmatrix} 8 & 2 & -2 \\ 1 & -9 & 15 \end{bmatrix} \quad D = \begin{bmatrix} -2 & -8 & 0 \\ 4 & 13 & 1 \end{bmatrix}$$

- $-3B + 2A$
- $9C - 4D$
- $2C + 11A$
- $7A - 2B$

Example 5

21. CCSS MODELING Library A has 10,000 novels, 5000 biographies, and 5000 children's books. Library B has 15,000 novels, 10,000 biographies, and 2500 children's books. Library C has 4000 novels, 700 biographies, and 800 children's books.

- Express each library's number of books as a matrix. Label the matrices A , B , and C .
- Find the total number of each type of book in all 3 libraries. Express as a matrix.
- How many more books of each type does Library A have than Library C?
- Find $A + B$. Does the matrix have meaning in this situation? Explain.

Perform the indicated operations. If the matrix does not exist, write *impossible*.

$$22. -2 \begin{bmatrix} -9.2 & -8.4 \\ 5.6 & -4.3 \end{bmatrix} - 4 \begin{bmatrix} 4.1 & -2.9 \\ 7.2 & -8.2 \end{bmatrix} \quad 23. -\frac{3}{4} \begin{bmatrix} 12 & -16 \\ 15 & 8 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 21 & 18 \\ -4 & -6 \end{bmatrix}$$

$$24. -3 \begin{bmatrix} 18 & -6 & -8 \\ -5 & -3 & 12 \\ 0 & 3x & -y \end{bmatrix} \quad 25. 8 \begin{bmatrix} -a & 4b & c - b \\ -13 & 10 & -5c \end{bmatrix}$$

$$26. -4 \begin{bmatrix} -7 \\ 4 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} -8 \\ 3x \\ -9 \end{bmatrix} - 5 \begin{bmatrix} x - 4 \\ x - 6 \\ 12 \end{bmatrix} \quad 27. -5 \left(\begin{bmatrix} 4 & -8 \\ 8 & -9 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ -3 & -6 \end{bmatrix} \right)$$

28. WEATHER The table shows snowfall in inches.

City	Normal Snowfall			2007 Snowfall		
	Jan	Feb	Mar	Jan	Feb	Mar
Grand Rapids, MI	21.1	12.2	9.0	15.4	33.6	13.0
Boston, MA	13.3	11.3	8.3	1.0	4.6	10.2
Buffalo, NY	26.1	17.8	12.4	15.5	33.5	5.4
Pittsburgh, PA	12.3	8.5	7.9	11.3	14.0	9.2

Source: National Weather Service

- Express the normal snowfall data and the 2007 data in two 4×3 matrices.
- Subtract the matrix of normal data from the matrix of 2007 data. What does the difference represent in the context of the situation?
- Explain the meaning of positive and negative numbers in the difference matrix. What trends do you see in the data?



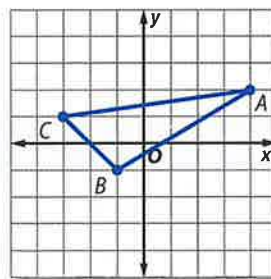
- 29** **CCSS MODELING** The table shows some of the world, Olympic, and American women's freestyle swimming records.

Distance (m)	World	Olympic	American
50	24.13 s	24.13 s	24.63 s
100	53.52 s	53.52 s	53.99 s
200	1:56.54 min	1:57.65 min	1:57.41 min
800	8:16.22 min	8:19.67 min	8:16.22 min

Source: USA Swimming

- Find the difference between the American and World records expressed as a matrix.
 - What is the meaning of each row in the column?
 - In which events were the fastest times set at the Olympics?
- 30.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate using matrices to represent transformations.

- Algebraic** The matrix $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 6 & 0 \end{bmatrix}$ represents a triangle with vertices at $(-3, 8)$, $(-4, 6)$, and $(1, 0)$. Write a matrix to represent $\triangle ABC$.



- Geometric** Multiply the matrix you wrote by 2. Then graph the figure represented by the new matrix.
- Analytical** How do the figures compare? Make a conjecture about the result of multiplying the matrix by 0.5. Verify your conjecture.

H.O.T. Problems Use Higher-Order Thinking Skills

- PROOF** Prove that matrix addition is commutative for 2×2 matrices.
- PROOF** Prove that matrix addition is associative for 2×2 matrices.
- CHALLENGE** Find the elements of C if:

$$A = \begin{bmatrix} -3 & -4 \\ 8 & 6 \end{bmatrix}, B = \begin{bmatrix} 5 & -1 \\ 2 & -4 \end{bmatrix}, \text{ and } 3A - 4B + 6C = \begin{bmatrix} 13 & 22 \\ 10 & 4 \end{bmatrix}.$$
- REASONING** Determine whether each statement is *sometimes*, *always*, or *never* true for matrices A and B . Explain your reasoning.
 - If $A + B$ exists, then $A - B$ exists.
 - If k is a real number, then kA and kB exist.
 - If $A - B$ does not exist, then $B - A$ does not exist.
 - If A and B have the same number of elements, then $A + B$ exists.
 - If kA exists and kB exists, then $kA + kB$ exists.
- OPEN ENDED** Give an example of matrices A and B if $4B - 3A = \begin{bmatrix} -6 & 5 \\ -2 & -1 \end{bmatrix}$.
- WRITING IN MATH** Explain how to find $4D - 3C$ for two given matrices, C and D with the same dimensions.



Standardized Test Practice

37. What is the solution of the system of equations?

$$\begin{aligned} 0.06p + 4q &= 0.88 \\ p - q &= -2.25 \end{aligned}$$

- A $(-0.912, -1.338)$
 B $(0.912, -3.162)$
 C $(-2, 0.25)$
 D $(-2, -4.25)$

38. **SHORT RESPONSE** Find $A + B$ if $A = \begin{bmatrix} -7 & 3 \\ 2 & 6 \end{bmatrix}$
 and $B = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}$.

39. **SAT/ACT** Solve for x and y .

$$\begin{aligned} x + 3y &= 16 \\ 7 - x &= 12 \end{aligned}$$

- F $x = -5, y = 7$ J $x = 5, y = 7$
 G $x = 7, y = 3$ K $x = -5, y = 3$
 H $x = 7, y = 5$

40. **PROBABILITY** A local pizzeria offers 5 different meat toppings and 6 different vegetable toppings. You decide to get two vegetable toppings and one meat topping. How many different types of pizzas can you order?

- A 60 C 120
 B 75 D 150

Spiral Review

Solve each system of equations. (Lesson 3-4)

41. $\begin{cases} 2x + 3y - z = -1 \\ 5x + y + 4z = 30 \\ -8x - 2y + 5z = -2 \end{cases}$

42. $\begin{cases} 3x - 4y + 6z = 26 \\ 5x + 3y + 2z = 5 \\ -2x + 5y - 3z = -9 \end{cases}$

43. $\begin{cases} 5x + 2y - 4z = 22 \\ 6x + 3y + 5z = 5 \\ -2x - 4y + z = 2 \end{cases}$

44. **PACKAGING** The Cookie Factory sells chocolate chip and peanut butter cookies in combination packages that contain between six and twelve cookies. At least three of each type of cookie should be in each package. How many of each type of cookie should be in each package to maximize the profit? (Lesson 3-3)

Cookie	chocolate chip	peanut butter
Cost	\$0.19	\$0.13
Price	\$0.44	\$0.39

Solve each system of inequalities by graphing. (Lesson 3-2)

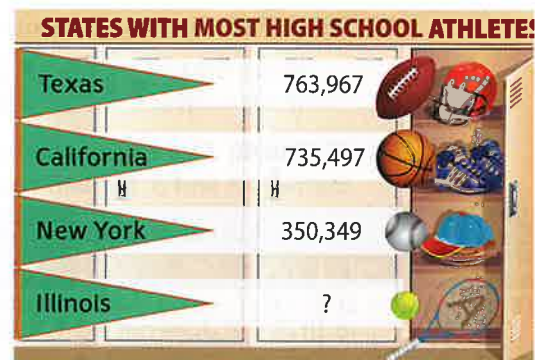
45. $\begin{cases} x - 2y > -4 \\ y < -2x - 3 \end{cases}$

46. $\begin{cases} y \geq -4x + 6 \\ 3y < 2x + 9 \end{cases}$

47. $\begin{cases} 4x + 2y > 8 \\ 4y - 3x \leq 12 \end{cases}$

48. **RAKING LEAVES** A student can earn \$20 plus an extra \$5 for each trash bag he or she completely fills with leaves. Write and solve an equation to determine how many bags the student will need to fill in order to earn \$100. (Lesson 2-4)

49. **SPORTS** There are 15,991 more student athletes in New York than in Illinois. Write and solve an equation to find the number of student athletes in Illinois. (Lesson 1-3)



Skills Review

Simplify each expression.

50. $4(2x - 3y) + 2(5x - 6y)$

51. $-3(2a - 5b) - 4(4b + a)$

52. $-7(x - y) + 5(y - x)$



LESSON 3-6 Multiplying Matrices



Then

Now

Why?

You multiplied matrices by a scalar.

1 Multiply matrices.

2 Use the properties of matrix multiplication.

The table shows the scoring summary for Lisa Leslie, the WNBA's all-time scoring leader, during her highest scoring seasons. Her total baskets can be summarized in the baskets matrix B . The point values for each type of basket made can be organized in the point value matrix P .

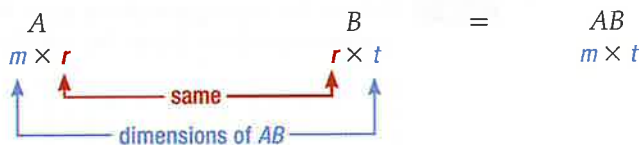
Lisa Leslie Regular Season Scoring				
Type	2005	2006	2008	2009
Field Goal	197	249	184	143
3-Point Field Goal	7	8	4	1
Free Throw	102	158	117	65

Source: WNBA

You can use matrix multiplication to find the points scored during each season.

$$B = \begin{bmatrix} 197 & 249 & 184 & 143 \\ 7 & 8 & 4 & 1 \\ 102 & 158 & 117 & 65 \end{bmatrix} \quad P = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}$$

1 Multiply Matrices You can multiply two matrices A and B if and only if the number of columns in A is equal to the number of rows in B . When you multiply two matrices $A_{m \times r}$ and $B_{r \times t}$ the resulting matrix AB is an $m \times t$ matrix.



Example 1 Dimensions of Matrix Products



Determine whether each matrix product is defined. If so, state the dimensions of the product.

a. $A_{3 \times 4}$ and $B_{4 \times 2}$

$$A \cdot B = AB$$

The inner dimensions are equal, so the product is defined. Its dimensions are 3×2 .

b. $A_{5 \times 3}$ and $B_{5 \times 4}$

$$A \cdot B$$

The inner dimensions are not equal, so the matrix product is not defined.

Guided Practice

1A. $A_{4 \times 6}$ and $B_{6 \times 2}$

1B. $A_{3 \times 2}$ and $B_{3 \times 2}$



The product of two matrices is found by multiplying columns and rows.

KeyConcept Multiplying Matrices

Words The element in the m th row and r th column of matrix AB is the sum of the products of the corresponding elements in row m of matrix A and column r of matrix B .

Symbols

$$\begin{matrix} A & \cdot & B & = & AB \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \cdot & \begin{bmatrix} e & f \\ g & h \end{bmatrix} & = & \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \end{matrix}$$

WatchOut!

Saving Your Place It is easy to lose your place as you multiply matrices. It may help to cover rows or columns not being multiplied as you find elements of the product matrix.

Example 2 Multiply Square Matrices

Find XY if $X = \begin{bmatrix} 6 & -3 \\ -10 & -2 \end{bmatrix}$ and $Y = \begin{bmatrix} -5 & -4 \\ 3 & 3 \end{bmatrix}$.

$$XY = \begin{bmatrix} 6 & -3 \\ -10 & -2 \end{bmatrix} \cdot \begin{bmatrix} -5 & -4 \\ 3 & 3 \end{bmatrix}$$

Step 1 Multiply the numbers in the first row of X by the numbers in the first column of Y , add the products, and put the result in the first row, first column of XY .

$$\begin{bmatrix} 6 & -3 \\ -10 & -2 \end{bmatrix} \cdot \begin{bmatrix} -5 & -4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 6(-5) + (-3)(3) & \\ & \end{bmatrix}$$

Step 2 Follow the same procedure as in Step 1 using the first row and the second column numbers. Write the result in the first row, second column.

$$\begin{bmatrix} 6 & -3 \\ -10 & -2 \end{bmatrix} \cdot \begin{bmatrix} -5 & -4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 6(-5) + (-3)(3) & 6(-4) + (-3)(3) \\ & \end{bmatrix}$$

Step 3 Follow the same procedure with the second row and the first column numbers. Write the result in the second row, first column.

$$\begin{bmatrix} 6 & -3 \\ -10 & -2 \end{bmatrix} \cdot \begin{bmatrix} -5 & -4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 6(-5) + (-3)(3) & 6(-4) + (-3)(3) \\ -10(-5) + (-2)(3) & \end{bmatrix}$$

Step 4 The procedure is the same for the numbers in the second row, second column.

$$\begin{bmatrix} 6 & -3 \\ -10 & -2 \end{bmatrix} \cdot \begin{bmatrix} -5 & -4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 6(-5) + (-3)(3) & 6(-4) + (-3)(3) \\ -10(-5) + (-2)(3) & -10(-4) + (-2)(3) \end{bmatrix}$$

Step 5 Simplify the product matrix.

$$\begin{bmatrix} 6(-5) + (-3)(3) & 6(-4) + (-3)(3) \\ -10(-5) + (-2)(3) & -10(-4) + (-2)(3) \end{bmatrix} = \begin{bmatrix} -39 & -33 \\ 44 & 34 \end{bmatrix}$$

GuidedPractice

2. Find UV if $U = \begin{bmatrix} 5 & 9 \\ -3 & -2 \end{bmatrix}$ and $V = \begin{bmatrix} 2 & -1 \\ 6 & -5 \end{bmatrix}$.





Real-WorldLink

Swim meets consist of racing and diving competitions. There are more than 241,000 high schools that participate each year.

Source: National Federation of State High School Associations

Real-World Example 3 Multiply Matrices

SWIM MEET At a particular swim meet, 7 points were awarded for each first-place finish, 4 points for second, and 2 points for third. Find the total number of points for each school. Which school won the meet?

School	First Place	Second Place	Third Place
Central	4	7	3
Franklin	8	9	1
Hayes	10	5	3
Lincoln	3	3	6

Understand The final scores can be found by multiplying the swim results for each school by the points awarded for each first-, second-, and third-place finish.

Plan Write the results of the races and the points awarded in matrix form. Set up the matrices so that the number of rows in the points matrix equals the number of columns in the results matrix.

$$R = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix} \quad \begin{matrix} \text{Results} \\ \\ \\ \end{matrix} \quad P = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} \quad \begin{matrix} \text{Points} \\ \\ \end{matrix}$$

Solve Multiply the matrices.

$$RP = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 4(7) + 7(4) + 3(2) \\ 8(7) + 9(4) + 1(2) \\ 10(7) + 5(4) + 3(2) \\ 3(7) + 3(4) + 6(2) \end{bmatrix} \quad \text{Multiply columns by rows.}$$

$$= \begin{bmatrix} 62 \\ 94 \\ 96 \\ 45 \end{bmatrix} \quad \text{Simplify.}$$

The product matrix shows the scores for Central, Franklin, Hayes, and Lincoln, respectively. Hayes won the swim meet with a total of 96 points.

Check R is a 4×3 matrix and P is a 3×1 matrix, so their product should be a 4×1 matrix.

Guided Practice

3. BASKETBALL Refer to the beginning of the lesson. Use matrix multiplication to determine in which season Lisa Leslie scored the most points. How many points did she score that season?

2 Multiplicative Properties Recall that the properties of real numbers also held true for matrix addition. However, some of these properties do *not* always hold true for matrix multiplication.



Example 4 Test of the Commutative Property

Find each product if $G = \begin{bmatrix} 1 & 3 & -5 \\ 4 & -2 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 2 & 3 \\ -2 & -8 \\ 1 & 7 \end{bmatrix}$.

a. GH

$$GH = \begin{bmatrix} 1 & 3 & -5 \\ 4 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -2 & -8 \\ 1 & 7 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 2 - 6 - 5 & 3 - 24 - 35 \\ 8 + 4 + 0 & 12 + 16 + 0 \end{bmatrix} \text{ or } \begin{bmatrix} -9 & -56 \\ 12 & 28 \end{bmatrix}$$

b. HG

$$HG = \begin{bmatrix} 2 & 3 \\ -2 & -8 \\ 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -5 \\ 4 & -2 & 0 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 2 + 12 & 6 - 6 & -10 + 0 \\ -2 - 32 & -6 + 16 & 10 + 0 \\ 1 + 28 & 3 - 14 & -5 + 0 \end{bmatrix} \text{ or } \begin{bmatrix} 14 & 0 & -10 \\ -34 & 10 & 10 \\ 29 & -11 & -5 \end{bmatrix} \quad \text{Notice that } GH \neq HG.$$

StudyTip**Proof and Counterexamples**

To show that a property is *not* always true, you need to find only one counterexample.

GuidedPractice

4. Determine if $AB = BA$ is true for $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ -4 & 5 \end{bmatrix}$.

Example 4 demonstrates that the Commutative Property of Multiplication does not hold for matrix multiplication. The order in which you multiply matrices is very important.

Example 5 Test of the Distributive Property

Find each product if $J = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix}$, $K = \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$, and $L = \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix}$.

a. $J(K + L)$

$$J(K + L) = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix} \right) \quad \text{Substitution}$$

$$= \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{Add.}$$

$$= \begin{bmatrix} -2 + 8 & 2 + 12 \\ 5 - 4 & -5 - 6 \end{bmatrix} \text{ or } \begin{bmatrix} 6 & 14 \\ 1 & -11 \end{bmatrix} \quad \text{Multiply.}$$

b. $JK + JL$

$$JK + JL = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(3) + 4(-1) & 2(2) + 4(3) \\ -5(3) + (-2)(-1) & -5(2) + (-2)(3) \end{bmatrix} + \begin{bmatrix} 2(-4) + 4(3) & 2(-1) + 4(0) \\ -5(-4) + (-2)(3) & -5(-1) + (-2)(0) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 16 \\ -13 & -16 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 14 & 5 \end{bmatrix} \text{ or } \begin{bmatrix} 6 & 14 \\ 1 & -11 \end{bmatrix} \quad \text{Notice that } J(K + L) = JK + JL.$$

GuidedPractice

5. Use the matrices $R = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 4 & 6 \\ -2 & 5 \end{bmatrix}$, and $T = \begin{bmatrix} -3 & 7 \\ -4 & 8 \end{bmatrix}$ to determine if $(S + T)R = SR + TR$.



The previous example suggests that the Distributive Property is true for matrix multiplication. Some properties of matrix multiplication are shown below.

KeyConcept Properties of Matrix Multiplication

For any matrices A , B , and C for which the matrix product is defined and any scalar k , the following properties are true.

Associative Property of Matrix Multiplication	$(AB)C = A(BC)$
Associative Property of Scalar Multiplication	$k(AB) = (kA)B = A(kB)$
Left Distributive Property	$C(A + B) = CA + CB$
Right Distributive Property	$(A + B)C = AC + BC$

Check Your Understanding

 = Step-by-Step Solutions begin on page R14.



Example 1 Determine whether each matrix product is defined. If so, state the dimensions of the product.

1. $A_{2 \times 4} \cdot B_{4 \times 3}$

2. $C_{5 \times 4} \cdot D_{5 \times 4}$

3. $E_{8 \times 6} \cdot F_{6 \times 10}$

Examples 2–3 Find each product, if possible.

4. $\begin{bmatrix} 2 & 1 \\ 7 & -5 \end{bmatrix} \cdot \begin{bmatrix} -6 & 3 \\ -2 & -4 \end{bmatrix}$

5. $\begin{bmatrix} 10 & -2 \\ -7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$

6. $[9 \quad -2] \cdot \begin{bmatrix} -2 & 4 \\ 6 & -7 \end{bmatrix}$

7. $\begin{bmatrix} -9 \\ 6 \end{bmatrix} \cdot [-1 \quad -10 \quad 1]$

8. $\begin{bmatrix} -8 & 7 & 4 \\ -5 & -3 & 8 \end{bmatrix} \cdot \begin{bmatrix} 10 & 6 \\ 8 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 8 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -7 \end{bmatrix}$

10. $\begin{bmatrix} -4 & 3 & 2 \\ -1 & -5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 6 \\ 8 & 4 & -1 \\ 5 & 3 & -2 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 5 & 3 & -1 \\ -3 & 1 & 8 & -3 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \\ -7 & 1 \\ 2 & 0 \\ -1 & 0 \end{bmatrix}$

12. **CCSS SENSE-MAKING** The table shows the number of people registered for aerobics for the first quarter.

Quinn's Gym charges the following registration fees: class-by-class, \$165; 11-class pass, \$110; unlimited pass, \$239.

Quinn's Gym		
Payment	Aerobics	Step Aerobics
class-by-class	35	28
11-class pass	32	17
unlimited pass	18	12

- Write a matrix for the registration fees and a matrix for the number of students.
- Find the total amount of money the gym received from aerobics and step aerobic registrations.

Examples 4–5 Use $X = \begin{bmatrix} -10 & -3 \\ 2 & -8 \end{bmatrix}$, $Y = \begin{bmatrix} -5 & 6 \\ -1 & 9 \end{bmatrix}$, and $Z = \begin{bmatrix} -5 & -1 \\ -8 & -4 \end{bmatrix}$ to determine whether the

following equations are true for the given matrices.

13. $XY = YX$

14. $X(YZ) = (XY)Z$



Example 1 Determine whether each matrix product is defined. If so, state the dimensions of the product.

15. $P_{2 \times 3} \cdot Q_{3 \times 4}$

16. $A_{5 \times 5} \cdot B_{5 \times 5}$

17. $M_{3 \times 1} \cdot N_{2 \times 3}$

18. $X_{2 \times 6} \cdot Y_{6 \times 3}$

19. $J_{2 \times 1} \cdot K_{2 \times 1}$

20. $S_{5 \times 2} \cdot T_{2 \times 4}$

Examples 2–3 Find each product, if possible.

21. $\begin{bmatrix} 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} -10 \\ 6 \end{bmatrix}$

22. $\begin{bmatrix} 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -7 \end{bmatrix}$

23. $\begin{bmatrix} -3 & -7 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ 9 & -3 \end{bmatrix}$

24. $\begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \\ 7 & -2 \end{bmatrix}$

25. $\begin{bmatrix} -1 & 0 & 6 \\ -4 & -10 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & -7 \\ -2 & -9 \end{bmatrix}$

26. $\begin{bmatrix} -6 & 4 & -9 \\ 2 & 8 & 7 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}$

27. $\begin{bmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{bmatrix}$

28. $\begin{bmatrix} -4 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 \end{bmatrix}$

29. **TRAVEL** The Wolf family owns three bed and breakfasts in a vacation spot. A room with a single bed is \$220 per night, a room with two beds is \$250 per night, and a suite is \$360.

a. Write a matrix for the number of each type of room at each bed and breakfast. Then write a room-cost matrix.

B & B	Single	Double	Suite
1	3	2	2
2	2	3	1
3	4	3	0

b. Write a matrix for total daily income, assuming that all the rooms are rented.

c. What is the total daily income from all three bed and breakfasts, assuming that all the rooms are rented?

Examples 4–5 Use $P = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 6 & 4 \\ -2 & -5 \end{bmatrix}$, $R = \begin{bmatrix} 4 & 6 \\ -6 & 4 \end{bmatrix}$, and $k = 2$ to determine whether the following equations are true for the given matrices.

30. $k(PQ) = P(kQ)$

31. $PQR = RQP$

32. $PR + QR = (P + Q)R$

33. $R(P + Q) = PR + QR$

34. **CCSS SENSE-MAKING** Student Council is selling flowers for Mother's Day. They bought 200 roses, 150 daffodils, and 100 orchids for the purchase prices shown. They sold all of the flowers for the sales prices shown.

Flower	Purchase Price	Sales Price
rose	\$1.67	\$3.00
daffodil	\$1.03	\$2.25
orchid	\$2.59	\$4.50

a. Organize the data in two matrices, and use matrix multiplication to find the total amount that was spent on the flowers.

b. Write two matrices, and use matrix multiplication to find the total amount the student council received for the flower sale.

c. Use matrix operations to find how much money the student council made on their project.



- 35. AUTO SALES** A car lot has four sales associates. At the end of the year, each sales associate gets a bonus of \$1000 for every new car they have sold and \$500 for every used car they have sold.

Cars Sold by Each Associate		
Sales Associate	New Cars	Used Cars
Mason	27	49
Westin	35	36
Gallagher	9	56
Stadler	15	62

- Use a matrix to determine which sales associate earned the most money.
- What is the total amount of money the car lot spent on bonuses for the sales associates this year?

Use matrices $X = \begin{bmatrix} 2 & -6 \\ 3y & -4.5 \end{bmatrix}$, $Y = \begin{bmatrix} -5 & -1.5 \\ x+2 & y \\ 13 & 1.2 \end{bmatrix}$, and $Z = \begin{bmatrix} -3 \\ x+y \end{bmatrix}$ to find each of

the following. If the matrix does not exist, write *undefined*.

36. XY 37. YX 38. ZY 39. YZ
 40. $(YX)Z$ 41. $(XZ)X$ 42. $X(ZZ)$ 43. $(XX)Z$

- 44. CAMERAS** Prices of digital cameras depend on features like optical zoom, digital zoom, and megapixels.

Optical Zoom	6 MP	7 MP	10 MP
3 to 4	\$189.99	\$249.99	\$349.99
5 to 6	\$199.99	\$289.99	\$399.99
10 to 12	\$299.99	\$399.99	\$499.99

- The 10-mp cameras are on sale for 20% off, and the other models are 10% off. Write a cost matrix.
- Write a new matrix allowing for a 6.25% sales tax on the discounted prices.
- Describe what the differences in these two matrices represent.

- 45. BUSINESS** The Kangy Studio has packages available for senior portraits.

Size (price)	Packages			
	A	B	C	D
4 × 5 (\$7)	10	10	8	0
5 × 7 (\$10)	4	4	4	4
8 × 10 (\$14)	2	2	2	2
11 × 14 (\$45)	1	1	0	0
16 × 20 (\$95)	1	0	0	0
Wallets (8 for \$13)	88	56	16	0

- Use matrices to determine the total cost of each package.
- The studio offers an early bird discount of 15% off any package. Find the early bird price for each package.

H.O.T. Problems Use Higher-Order Thinking Skills

- 46. REASONING** If the product matrix AB has dimensions 5×8 , and A has dimensions 5×6 , what are the dimensions of matrix B ?
- 47. CCSS ARGUMENTS** Show that each property of matrices is true for all 2×2 matrices.
- Scalar Distributive Property
 - Matrix Distributive Property
 - Associative Property of Multiplication
 - Associative Property of Scalar Multiplication
- 48. OPEN ENDED** Write two matrices A and B such that $AB = BA$.
- 49. CHALLENGE** Find the missing values in $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 20 & 29 \end{bmatrix}$.
- 50. WRITING IN MATH** Use the data on Lisa Leslie found at the beginning of the lesson to explain how matrices can be used in sports statistics. Describe a matrix that represents the total number of points she has scored during her career and an example of a sport in which different point values are used in scoring.



Standardized Test Practice

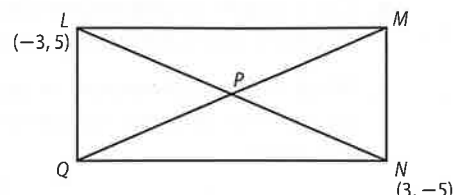
51. **GRIDDED RESPONSE** The average (arithmetic mean) of r , w , x , and y is 8, and the average of x and y is 4. What is the average of r and w ?
52. Carla, Meiko, and Kayla went shopping to get ready for college. Their purchases and total amounts spent are shown in the table below.

Person	Shirts	Pants	Shoes	Total Spent
Carla	3	4	2	\$149.79
Meiko	5	3	3	\$183.19
Kayla	6	5	1	\$181.14

Assume that all of the shirts were the same price, all of the pants were the same price, and all of the shoes were the same price. What was the price of each item?

- A shirt, \$12.95; pants, \$15.99; shoes, \$23.49
 B shirt, \$15.99; pants, \$12.95; shoes, \$23.49
 C shirt, \$15.99; pants, \$23.49; shoes, \$12.95
 D shirt, \$23.49; pants, \$15.99; shoes, \$12.95

53. **GEOMETRY** Rectangle $LMNQ$ has diagonals that intersect at point P .



Which of the following represents point P ?

- F (2, 2) H (0, 0)
 G (1, 1) J (-1, -1)

54. **SAT/ACT** What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$$

- A 1×4 D 4×1
 B 3×3 E 4×3
 C 3×1

Spiral Review

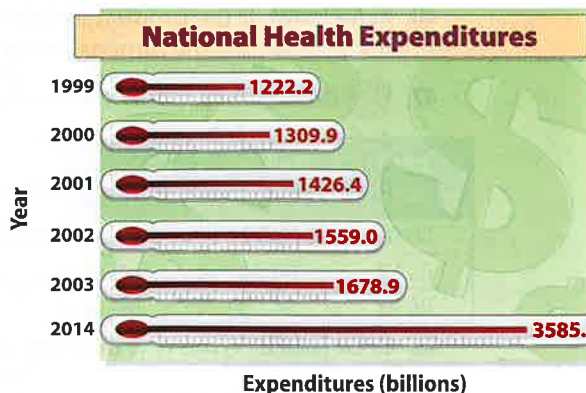
Perform the indicated operations. If the matrix does not exist, write *impossible*. (Lesson 3-5)

55. $4 \begin{bmatrix} 8 & -1 \\ -3 & -4 \end{bmatrix} - 5 \begin{bmatrix} -2 & 4 \\ 6 & 3 \end{bmatrix}$ 56. $5 \left(2 \begin{bmatrix} -2 & -5 \\ -1 & 3 \end{bmatrix} - 3 \begin{bmatrix} -1 & -2 \\ 6 & 4 \end{bmatrix} \right)$ 57. $-4 \left(\begin{bmatrix} 8 & 9 \\ -5 & 5 \end{bmatrix} - 2 \begin{bmatrix} -6 & -1 \\ 6 & 3 \end{bmatrix} \right)$

Solve each system of equations. (Lesson 3-4)

58. $2x - 4y + 3z = -3$ 59. $-4x - 2y + 9z = -29$ 60. $-7x + 8y - z = 43$
 $-7x + 5y - 4z = 11$ $10x - 12y + 7z = 51$ $3x - 2y + 5z = -43$
 $x - y - 2z = -21$ $3x + 5y - 14z = 25$ $2x - 4y + 6z = -50$

61. **MEDICINE** The graph shows how much Americans spent on doctors' visits in some recent years and a prediction for 2014. (Lesson 2-5)
- Find a regression equation for the data without the predicted value.
 - Use your equation to predict the expenditures for 2014.
 - Compare your prediction to the one given in the graph.



62. How many different ways can the letters of the word *SAMPLING* be arranged? (Lesson 0-4)

Skills Review

Describe the transformation in each function. Then graph the function.

63. $f(x) = |x - 4| + 3$ 64. $f(x) = 2|x + 3| - 5$ 65. $f(x) = (x + 2)^2 - 6$



EXERCISE 3-6

Graphing Technology Lab Operations with Matrices



A graphing calculator can be used to perform operations with matrices.

Activity 1 Perform Operations

Use A , B , and C to find the following.

$$A = \begin{bmatrix} -3.2 & 1.7 \\ 0.4 & -5.8 \end{bmatrix}$$

$$B = \begin{bmatrix} 4.9 & 0.3 \\ -7.1 & 2.6 \end{bmatrix}$$

$$C = \begin{bmatrix} 5.6 & -6.1 & 2.1 \\ -8.2 & 7.6 & 0.2 \end{bmatrix}$$

a. $3A + 2B$

Begin by entering matrix A into a graphing calculator.

KEYSTROKES: 2^{nd} [MATRIX] \blacktriangleright \blacktriangleright [ENTER] 2 [ENTER] 2
 [ENTER] (\leftarrow) 3.2 [ENTER] 1.7 [ENTER]
 0.4 [ENTER] (\leftarrow) 5.8 [ENTER] 2^{nd} [QUIT]



Enter matrix B into the graphing calculator using similar keystrokes. Then, perform the indicated operations.

KEYSTROKES: 3 2^{nd} [MATRIX] [ENTER] + 2 2^{nd} [MATRIX] \blacktriangledown
 [ENTER] [ENTER]

$3A + 2B$ is equal to $\begin{bmatrix} 0.2 & 5.7 \\ -13 & -12.2 \end{bmatrix}$.



b. $4C + 3A$

Enter matrix C into the graphing calculator. Perform the indicated operations. Notice that the calculator displays an error message when the dimensions do not allow the operations to be performed.



Exercises

Use A , B , C , and D to find the following. If the matrix does not exist, write *impossible*.

$$A = \begin{bmatrix} 4.5 & -9.0 \\ -7.4 & 9.4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.9 & -5.9 \\ 2.9 & 5.0 \end{bmatrix}$$

$$C = \begin{bmatrix} 7.0 & 5.5 & -1.9 \\ 7.6 & -9.9 & 0.5 \end{bmatrix}$$

$$D = \begin{bmatrix} 8.5 & 8.0 \\ -1.2 & -5.9 \\ 0.7 & 8.9 \end{bmatrix}$$

1. $CD + 4A$
2. $-2B + 7A$
3. $4(DC)$
4. $6B + DC$
5. $2(AB) - 3B + 5A$
6. $-3(CD) + 4(BA) + 7A$

(continued on the next page)

Activity 2 Explore Properties of Matrix Operations

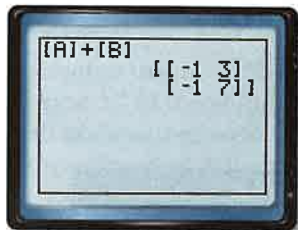
Use A , B , and C to find the following.

$$A = \begin{bmatrix} -6 & 2 \\ 7 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 1 \\ -8 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 11 & -3 \\ 4 & -1 \end{bmatrix}$$

a. $A + B$ and $B + A$



Find $A + B$.
Then find $B + A$.

$A + B$ and $B + A$ are both equal to $\begin{bmatrix} -1 & 3 \\ -1 & 7 \end{bmatrix}$.

b. $(A + B) + C$ and $A + (B + C)$



Find $(A + B) + C$.
Then find $A + (B + C)$.

$(A + B) + C$ and $A + (B + C)$ are both equal to $\begin{bmatrix} 10 & 0 \\ 3 & 6 \end{bmatrix}$.

Analyze the Results

What property is illustrated in each part of Activity 2?

7. part a

8. part b

Find each set of products. How are they similar or different? What property is illustrated?

9. $(AB)C, A(BC)$

10. $3(AB), (3A)B, A(3B)$

11. The zero matrix O is an $m \times n$ matrix with all elements equal to 0. If $A + O$ is defined, verify the Additive Identity Property for Matrices, $A + O = A$.

12. Two matrices are additive inverses if their sum is the zero matrix. Find matrix E so that $A + E$ are additive inverses. Then verify that $A + E = O$.

13. Find the additive inverses for matrix B and matrix C .

14. **CHALLENGE** What observations can be made about a matrix and its additive inverse?

Find the additive inverse for $\begin{bmatrix} w & -x \\ -y & z \end{bmatrix}$.

LESSON 3-7 Solving Systems of Equations Using Cramer's Rule

Then

- You solved systems of equations algebraically.

Now

- Evaluate determinants.
- Solve systems of linear equations by using Cramer's Rule.

Why?

- A zoologist tagged a tiger with a GPS tracker so that she could determine the tiger's territory. After several days, the zoologist determined that the tiger's territory was a triangular region. By using the coordinates of the vertices of this triangle, she could use matrices and determinants to determine the size of the tiger's territory.



New Vocabulary

- determinant
- second-order determinant
- third-order determinant
- diagonal rule
- Cramer's Rule
- coefficient matrix



Common Core State Standards

Content Standards
 A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context

Mathematical Practices

7 Look for and make use of structure.

1 Determinants Every square matrix has a **determinant**. The determinant of a 2×2 matrix is called a **second-order determinant**.

Key Concept Second-Order Determinant

Words The value of a second-order determinant is the difference of the products of the two diagonals.

Symbols $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example $\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 4(6) - (5)(-3) = 39$

Example 1 Second-Order Determinant

Evaluate each determinant.

a. $\begin{vmatrix} 5 & -4 \\ 8 & 9 \end{vmatrix}$
 $\begin{vmatrix} 5 & -4 \\ 8 & 9 \end{vmatrix} = 5(9) - (-4)(8)$ Definition of determinant
 $= 45 + 32$ Simplify.
 $= 77$

b. $\begin{vmatrix} 0 & 6 \\ 4 & -11 \end{vmatrix}$
 $\begin{vmatrix} 0 & 6 \\ 4 & -11 \end{vmatrix} = 0(-11) - 6(4)$ Definition of determinant
 $= 0 - 24$ Simplify.
 $= -24$

Guided Practice

1A. $\begin{vmatrix} -6 & -7 \\ 10 & 8 \end{vmatrix}$

1B. $\begin{vmatrix} 7 & 5 \\ 9 & -4 \end{vmatrix}$

MORITZSCH, MARC (VA HORNOCKER, MAURICE)/National Geographic Stock



StudyTip

Diagonal Rule The diagonal rule can only be used for 3×3 matrices.

Determinants of 3×3 matrices are called **third-order determinants**. They can be evaluated by using the **diagonal rule**.

KeyConcept Diagonal Rule

- Step 1** Rewrite the first two columns to the right of the determinant.
- Step 2** Draw diagonals, beginning with the upper left-hand element. Multiply the elements in each diagonal. Repeat the process, beginning with the upper right-hand element.
- Step 3** Find the sum of the products of the elements in each set of diagonals.
- Step 4** Subtract the second sum from the first sum.

Example 2 Use Diagonals

Evaluate $\begin{vmatrix} 4 & -8 & 3 \\ -3 & 2 & 6 \\ -4 & 5 & 9 \end{vmatrix}$ using diagonals.

- Step 1** Rewrite the first two columns to the right of the determinant.

$$\begin{vmatrix} 4 & -8 & 3 & 4 & -8 \\ -3 & 2 & 6 & -3 & 2 \\ -4 & 5 & 9 & -4 & 5 \end{vmatrix}$$

- Step 2** Find the products of the elements of the diagonals.

$$\begin{vmatrix} \cancel{4} & \cancel{-8} & \cancel{3} & 4 & -8 \\ -3 & 2 & 6 & -3 & 2 \\ -4 & 5 & 9 & -4 & 5 \end{vmatrix}$$

$$4(2)(9) = 72$$

$$-8(6)(-4) = 192$$

$$3(-3)(5) = -45$$

$$\begin{vmatrix} 4 & -8 & 3 & \cancel{4} & \cancel{-8} \\ -3 & 2 & 6 & -3 & 2 \\ -4 & 5 & 9 & -4 & 5 \end{vmatrix}$$

$$-4(2)(3) = -24$$

$$5(6)(4) = 120$$

$$9(-3)(-8) = 216$$

- Step 3** Find the sum of each group.

$$72 + 192 + (-45) = 219$$

$$-24 + 120 + 216 = 312$$

- Step 4** Subtract the sum of the second group from the sum of the first group.

$$219 - 312 = -93$$

The value of the determinant is -93 .

GuidedPractice

Evaluate each determinant.

2A. $\begin{vmatrix} -5 & 9 & 4 \\ -2 & -1 & 5 \\ -4 & 6 & 2 \end{vmatrix}$

2B. $\begin{vmatrix} -8 & -4 & 4 \\ 0 & -5 & -8 \\ 3 & 4 & 1 \end{vmatrix}$



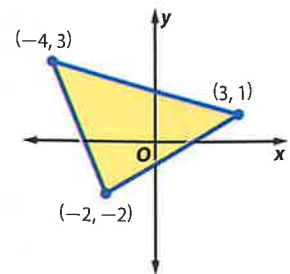
Determinants can also be used to find the areas of triangles. If the coordinates of the vertices of the triangle are known, the formula below can be used to calculate the area of the triangle. Because area cannot be negative, use the absolute value of the determinant.

KeyConcept Area of a Triangle

Words The area of a triangle with vertices (a, b) , (c, d) , and (e, f) is $|A|$, where

$$A = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

Example $A = \frac{1}{2} \begin{vmatrix} -4 & 3 & 1 \\ 3 & 1 & 1 \\ -2 & -2 & 1 \end{vmatrix}$



Real-WorldLink

Tigers are very territorial, solitary animals. Their territories can measure up to 100 square kilometers.

Source: National Geographic

Real-World Example 3 Use Determinants

ZOOLOGY Refer to the application at the beginning of the lesson. The coordinates of the vertices of the tiger's territory are shown to the right. Use determinants to find the area of the tiger's territory.

$$A = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 4 & 12 & 1 \\ -2 & 8 & 1 \end{vmatrix} \quad \begin{array}{l} (a, b) = (0, 0) \\ (c, d) = (4, 12) \\ (e, f) = (-2, 8) \end{array}$$

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 12 & 1 & 4 & 12 & 4 & 12 & 4 & 12 \\ -2 & 8 & 1 & -2 & 8 & -2 & 8 & -2 & 8 \end{vmatrix}$$

$$0 + 0 + 32 = 32$$

$$-24 + 0 + 0 = -24$$

$$A = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 4 & 12 & 1 \\ -2 & 8 & 1 \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)[32 - (-24)] \text{ or } 28$$

Diagonal Rule

Sum of products of diagonals

Area of a Triangle

Simplify.

The area of the tiger's territory is 28 square kilometers.

GuidedPractice

3. **CAR WASH** To raise money for their rowing club, Hannah, Christina, and Dario are advertising a car wash at three different street corners in a neighborhood. On a map, the coordinates for the corners are $(3, 15)$, $(6, 4)$, and $(11, 9)$. Each unit represents 0.5 kilometer. What is the area of the neighborhood in which they are advertising?

Reading Math

Determinants The determinant is used to *determine* whether a system has a unique solution.

2 Cramer's Rule You can use determinants to solve systems of equations. If a determinant is nonzero, then the system has a unique solution. If a determinant is 0, then the system either has no solution or infinite solutions. A method called **Cramer's Rule** uses the coefficient matrix. The **coefficient matrix** is a matrix that contains only the coefficients of the system.

Key Concept Cramer's Rule

Let C be the coefficient matrix of the system $ax + by = m$ \rightarrow $\begin{bmatrix} a & b \\ f & g \end{bmatrix}$.

The solution of this system is $x = \frac{\begin{vmatrix} m & b \\ n & g \end{vmatrix}}{|C|}$ and $y = \frac{\begin{vmatrix} a & m \\ f & n \end{vmatrix}}{|C|}$, if $|C| \neq 0$.

Test-Taking Tip

Cramer's Rule When the determinant of the coefficient matrix C is 0, the system does not have a unique solution.

Example 4 Solve a System of Two Equations

Solve the system by using Cramer's Rule.

$$5x - 6y = 15$$

$$3x + 4y = -29$$

$$x = \frac{\begin{vmatrix} m & b \\ n & g \end{vmatrix}}{|C|}$$

Cramer's Rule

$$y = \frac{\begin{vmatrix} a & m \\ f & n \end{vmatrix}}{|C|}$$

$$= \frac{\begin{vmatrix} 15 & -6 \\ -29 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & -6 \\ 3 & 4 \end{vmatrix}}$$

Substitute values.

$$= \frac{\begin{vmatrix} 5 & 15 \\ 3 & -29 \end{vmatrix}}{\begin{vmatrix} 5 & -6 \\ 3 & 4 \end{vmatrix}}$$

$$= \frac{15(4) - (-29)(-6)}{5(4) - (3)(-6)}$$

Evaluate.

$$= \frac{5(-29) - 3(15)}{5(4) - (3)(-6)}$$

$$= \frac{60 - 174}{20 + 18}$$

Multiply.

$$= \frac{-145 - 45}{20 + 18}$$

$$= -\frac{114}{38}$$

Add and subtract.

$$= -\frac{190}{38}$$

$$= -3$$

Simplify.

$$= -5$$

The solution of the system is $(-3, -5)$.

CHECK $5(-3) - 6(-5) \stackrel{?}{=} 15$ $x = -3, y = -5$

$$-15 + 30 \stackrel{?}{=} 15$$
 Simplify.

$$15 = 15 \quad \checkmark$$

$$3(-3) + 4(-5) \stackrel{?}{=} -29$$
 $x = -3, y = -5$

$$-9 - 20 \stackrel{?}{=} -29$$
 Simplify.

$$-29 = -29 \quad \checkmark$$

Guided Practice

Solve each system using Cramer's Rule.

4A. $7x + 3y = 37$

$$-5x - 7y = -41$$

4B. $8x - 5y = 70$

$$9x + 7y = 3$$



Cramer's Rule can also be used for systems of three equations.

KeyConcept Cramer's Rule for a System of Three Equations

Let C be the coefficient matrix of the system

$$\begin{aligned} ax + by + cz &= m \\ fx + gy + hz &= n \\ jx + ky + \ell z &= p \end{aligned} \rightarrow \begin{bmatrix} a & b & c \\ f & g & h \\ j & k & \ell \end{bmatrix}$$

The solution of this system is $x = \frac{\begin{vmatrix} m & b & c \\ n & g & h \\ p & k & \ell \end{vmatrix}}{|C|}$, $y = \frac{\begin{vmatrix} a & m & c \\ f & n & h \\ j & p & \ell \end{vmatrix}}{|C|}$, and $z = \frac{\begin{vmatrix} a & b & m \\ f & g & n \\ j & k & p \end{vmatrix}}{|C|}$, if $|C| \neq 0$.



Example 5 Solve a System of Three Equations

Solve the system by using Cramer's Rule.

$$\begin{aligned} 4x + 5y - 6z &= -14 \\ 3x - 2y + 7z &= 47 \\ 7x - 6y - 8z &= 15 \end{aligned}$$

$$x = \frac{\begin{vmatrix} m & b & c \\ n & g & h \\ p & k & \ell \end{vmatrix}}{|C|}$$

$$y = \frac{\begin{vmatrix} a & m & c \\ f & n & h \\ j & p & \ell \end{vmatrix}}{|C|}$$

$$z = \frac{\begin{vmatrix} a & b & m \\ f & g & n \\ j & k & p \end{vmatrix}}{|C|}$$

$$= \frac{\begin{vmatrix} -14 & 5 & -6 \\ 47 & -2 & 7 \\ 15 & -6 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & -6 \\ 3 & -2 & 7 \\ 7 & -6 & -8 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 4 & -14 & -6 \\ 3 & 47 & 7 \\ 7 & 15 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & -6 \\ 3 & -2 & 7 \\ 7 & -6 & -8 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 4 & 5 & -14 \\ 3 & -2 & 47 \\ 7 & -6 & 15 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & -6 \\ 3 & -2 & 7 \\ 7 & -6 & -8 \end{vmatrix}}$$

$$= \frac{3105}{621} \text{ or } 5$$

$$= -\frac{1242}{621} \text{ or } -2$$

$$= \frac{2484}{621} \text{ or } 4$$

The solution of the system is $(5, -2, 4)$.

StudyTip

Check for Accuracy
Always substitute your answers into the initial equations to confirm accuracy.

CHECK $4(5) + 5(-2) - 6(4) \stackrel{?}{=} -14$ $3(5) - 2(-2) + 7(4) \stackrel{?}{=} 47$
 $20 - 10 - 24 \stackrel{?}{=} -14$ $15 + 4 + 28 \stackrel{?}{=} 47$
 $-14 = -14$ ✓ $47 = 47$ ✓

$7(5) - 6(-2) - 8(4) \stackrel{?}{=} 15$
 $35 + 12 - 32 \stackrel{?}{=} 15$
 $15 = 15$ ✓

GuidedPractice

Solve each system using Cramer's Rule.

5A. $3x + 5y + 2z = -7$
 $-4x + 3y - 5z = -19$
 $5x + 4y - 7z = -15$

5B. $6x + 5y + 2z = -1$
 $-x + 3y + 7z = 12$
 $5x - 7y - 3z = -52$



Example 1 Evaluate each determinant.

1. $\begin{vmatrix} 8 & 6 \\ 5 & 7 \end{vmatrix}$

2. $\begin{vmatrix} -6 & -6 \\ 8 & 10 \end{vmatrix}$

3. $\begin{vmatrix} -4 & 12 \\ 9 & 5 \end{vmatrix}$

4. $\begin{vmatrix} 16 & -10 \\ -8 & 5 \end{vmatrix}$

Example 2 Evaluate each determinant using diagonals.

5. $\begin{vmatrix} 3 & -2 & 2 \\ -4 & 2 & -5 \\ -3 & 1 & 4 \end{vmatrix}$

6. $\begin{vmatrix} 2 & -3 & 5 \\ -4 & 6 & -2 \\ 4 & -1 & -6 \end{vmatrix}$

7. $\begin{vmatrix} 8 & 4 & 0 \\ -2 & -6 & -1 \\ 5 & -3 & 6 \end{vmatrix}$

8. $\begin{vmatrix} -5 & -3 & 4 \\ -2 & -4 & -3 \\ 8 & -2 & 4 \end{vmatrix}$

9. $\begin{vmatrix} 8 & 3 & 4 \\ 2 & 4 & 2 \\ 1 & 6 & 5 \end{vmatrix}$

10. $\begin{vmatrix} -4 & 3 & 0 \\ 1 & 5 & -2 \\ -1 & -8 & -3 \end{vmatrix}$

11. $\begin{vmatrix} 2 & -6 & -3 \\ 7 & 9 & -4 \\ -6 & 4 & 9 \end{vmatrix}$

12. $\begin{vmatrix} -5 & -6 & 7 \\ 4 & 0 & 5 \\ -3 & 8 & 2 \end{vmatrix}$

Example 4 Use Cramer's Rule to solve each system of equations.

13. $\begin{cases} 4x - 5y = 39 \\ 3x + 8y = -6 \end{cases}$

14. $\begin{cases} 5x + 6y = 20 \\ -3x - 7y = -29 \end{cases}$

15. $\begin{cases} -8a - 5b = -27 \\ 7a + 6b = 22 \end{cases}$

16. $\begin{cases} 10c - 7d = -59 \\ 6c + 5d = -63 \end{cases}$

Examples 3–5 17. **CCSS PERSEVERANCE** The “Bermuda Triangle” is an area located off the southeastern Atlantic coast of the United States, and is noted for reports of unexplained losses of ships, small boats, and aircraft.

- Find the area of the triangle on the map.
- Suppose each grid represents 175 miles. What is the area of the Bermuda Triangle?



Use Cramer's Rule to solve each system of equations.

18. $\begin{cases} 4x - 2y + 7z = 26 \\ 5x + 3y - 5z = -50 \\ -7x - 8y - 3z = 49 \end{cases}$

19. $\begin{cases} -3x - 5y + 10z = -4 \\ -8x + 2y - 3z = -91 \\ 6x + 8y - 7z = -35 \end{cases}$

20. $\begin{cases} 6x - 5y + 2z = -49 \\ -5x - 3y - 8z = -22 \\ -3x + 8y - 5z = 55 \end{cases}$

21. $\begin{cases} -9x + 5y + 3z = 50 \\ 7x + 8y - 2z = -60 \\ -5x + 7y + 5z = 46 \end{cases}$

22. $\begin{cases} x + 2y = 12 \\ 3y - 4z = 25 \\ x + 6y + z = 20 \end{cases}$

23. $\begin{cases} 9a + 7b = -30 \\ 8b + 5c = 11 \\ -3a + 10c = 73 \end{cases}$

24. $\begin{cases} 2n + 3p - 4w = 20 \\ 4n - p + 5w = 13 \\ 3n + 2p + 4w = 15 \end{cases}$

25. $\begin{cases} x + y + z = 12 \\ 6x - 2y - z = 16 \\ 3x + 4y + 2z = 28 \end{cases}$



Examples 1–2 Evaluate each determinant.

$$26. \begin{vmatrix} -7 & 12 \\ 5 & 6 \end{vmatrix}$$

$$27. \begin{vmatrix} -8 & -9 \\ 11 & 12 \end{vmatrix}$$

$$28. \begin{vmatrix} -5 & 8 \\ -6 & -7 \end{vmatrix}$$

$$29. \begin{vmatrix} 3 & 5 & -2 \\ -1 & -4 & 6 \\ -6 & -2 & 5 \end{vmatrix}$$

$$30. \begin{vmatrix} 2 & 0 & -6 \\ -3 & -4 & -5 \\ -2 & 5 & 8 \end{vmatrix}$$

$$31. \begin{vmatrix} -5 & -1 & -2 \\ 1 & 8 & 4 \\ 0 & -6 & 9 \end{vmatrix}$$

$$32. \begin{vmatrix} 6 & -3 & -5 \\ 0 & -7 & 0 \\ 3 & -6 & -4 \end{vmatrix}$$

$$33. \begin{vmatrix} -8 & -3 & -9 \\ 0 & 0 & 0 \\ 8 & -2 & -4 \end{vmatrix}$$

$$34. \begin{vmatrix} 1 & 6 & 7 \\ -2 & -5 & -8 \\ 4 & 4 & 9 \end{vmatrix}$$

$$35. \begin{vmatrix} 1 & -8 & -9 \\ 6 & 5 & -6 \\ -2 & -8 & 10 \end{vmatrix}$$

$$36. \begin{vmatrix} 5 & -5 & -5 \\ -8 & -3 & -2 \\ -2 & 4 & 6 \end{vmatrix}$$

$$37. \begin{vmatrix} -4 & 1 & -2 \\ 10 & 12 & 9 \\ -6 & 0 & 13 \end{vmatrix}$$

38. **TRAVEL** Mr. Smith's art class took a bus trip to an art museum. The bus averaged 65 miles per hour on the highway and 25 miles per hour in the city. The art museum is 375 miles away from the school, and it took the class 7 hours to get there. Use Cramer's Rule to find how many hours the bus was on the highway and how many hours it was driving in the city.

Examples 4–5 Use Cramer's Rule to solve each system of equations.

$$39. \begin{cases} 6x - 5y = 73 \\ -7x + 3y = -71 \end{cases}$$

$$40. \begin{cases} 10a - 3b = -34 \\ 3a + 8b = -28 \end{cases}$$

$$41. \begin{cases} -4c - 5d = -39 \\ 5c + 8d = 54 \end{cases}$$

$$42. \begin{cases} -6f - 8g = -22 \\ -11f + 5g = -60 \end{cases}$$

$$43. \begin{cases} 9r + 4s = -55 \\ -5r - 3s = 36 \end{cases}$$

$$44. \begin{cases} -11u - 7v = 4 \\ 9u + 4v = -24 \end{cases}$$

$$45. \begin{cases} 5x - 4y + 6z = 58 \\ -4x + 6y + 3z = -13 \\ 6x + 3y + 7z = 53 \end{cases}$$

$$46. \begin{cases} 8x - 4y + 7z = 34 \\ 5x + 6y + 3z = -21 \\ 3x + 7y - 8z = -85 \end{cases}$$

47. **DOUGHNUTS** Mi-Ling is ordering doughnuts for a class party. The box contains 2 dozen doughnuts, some of which are plain and some of which are jelly-filled. The plain doughnuts each cost \$0.50, and the jelly-filled cost \$0.60. If the total cost is \$12.60, use Cramer's Rule to find how many jelly-filled doughnuts Mi-Ling ordered.



- Examples 3–4 48. **CCSS PERSEVERANCE** The salary for each of the stars of a new movie is \$5 million, and the supporting actors each receive \$1 million. The total amount spent for the salaries of the actors and actresses is \$19 million. If the cast has 7 members, use Cramer's Rule to find the number of stars in the movie.

49. **ARCHAEOLOGY** Archaeologists found whale bones at coordinates $(0, 3)$, $(4, 7)$, and $(5, 9)$. If the units of the coordinates are meters, find the area of the triangle formed by these finds.

Use Cramer's Rule to solve each system of equations.

50. $6a - 7b = -55$
 $2a + 4b - 3c = 35$
 $-5a - 3b + 7c = -37$

51. $3a - 5b - 9c = 17$
 $4a - 3c = 31$
 $-5a - 4b - 2c = -42$

52. $4x - 5y = -2$
 $7x + 3z = -47$
 $8y - 5z = -63$

53. $7x + 8y + 9z = -149$
 $-6x + 7y - 5z = 54$
 $4x + 5y - 2z = -44$

54. **GARDENING** Rob wants to build a triangular flower garden. To plan out his garden he uses a coordinate grid where each of the squares represents one square foot. The coordinates for the vertices of his garden are $(-1, 7)$, $(2, 6)$, and $(4, -3)$. Find the area of the garden.

55. **FINANCIAL LITERACY** A vendor sells small drinks for \$1.15, medium drinks for \$1.75, and large drinks for \$2.25. During a week in which he sold twice as many small drinks as medium drinks, his total sales were \$2,238.75 for 1385 drinks.

- Use Cramer's Rule to determine how many of each drink were sold.
- The vendor decided to increase the price for small drinks to \$1.25 the next week. The next week, he sold 140 fewer small drinks, 125 more medium drinks, and 35 more large drinks. Calculate his sales for that week.
- Was raising the price of the small drink a good business move? Explain.

H.O.T. Problems Use Higher-Order Thinking Skills

56. **REASONING** Some systems of equations cannot be solved by using Cramer's Rule.

a. Find the value of $\begin{vmatrix} a & b \\ f & g \end{vmatrix}$. When is the value 0?

b. Choose values for $a, b, f,$ and g to make the determinant of the coefficient matrix 0. What type of system is formed?

57. **REASONING** What can you determine about the solution of a system of linear equations if the determinant of the coefficients is 0?

58. **CCSS CRITIQUE** James and Amber are finding the value of $\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix}$.

$$\begin{array}{c} \text{James} \\ \begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - (-15) \\ = 31 \end{array}$$

$$\begin{array}{c} \text{Amber} \\ \begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - 15 \\ = 1 \end{array}$$

Is either of them correct? Explain your reasoning.

59. **CHALLENGE** Find the determinant of a 3×3 matrix defined by

$$a_{mn} = \begin{cases} 0 & \text{if } m + n \text{ is even} \\ m + n & \text{if } m + n \text{ is odd} \end{cases}$$

60. **OPEN ENDED** Write a 2×2 matrix with each of the following characteristics.

- The determinant equals 0.
- The determinant equals 25.
- The elements are all negative numbers and the determinant equals -32 .

61. **WRITING IN MATH** Describe the possible graphical representations of a 2×2 system of linear equations if the determinant of the matrix of coefficients is 0.



Standardized Test Practice

62. Tyler paid \$25.25 to play three games of miniature golf and two rides on go-karts. Brent paid \$25.75 for four games of miniature golf and one ride on the go-karts. How much does one game of miniature golf cost?

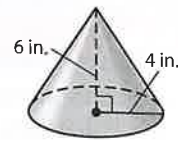
A \$4.25 C \$5.25
B \$4.75 D \$5.75

63. Use the table to determine the expression that best represents the number of faces of any prism having a base with n sides.

Base	Sides of Base	Faces of Prisms
triangle	3	5
quadrilateral	4	6
pentagon	5	7
hexagon	6	8
heptagon	7	9
octagon	8	10

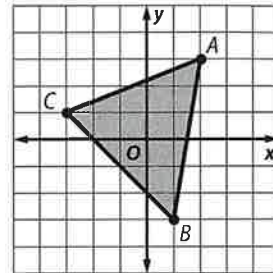
F $2(n - 1)$ H $n + 2$
G $2(n + 1)$ J $2n$

64. **SHORT RESPONSE** A right circular cone has radius 4 inches and height 6 inches.



What is the lateral area of the cone? (lateral area of cone = $\pi r \ell$, where ℓ = slant height)

65. **SAT/ACT** Find the area of $\triangle ABC$.



A 10 units² D 14 units²
B 12 units² E 16 units²
C 13 units²

Spiral Review

Determine whether each matrix product is defined. If so, state the dimensions of the product. (Lesson 3-6)

66. $A_{4 \times 2} \cdot B_{2 \times 6}$

67. $C_{5 \times 4} \cdot D_{5 \times 3}$

68. $E_{2 \times 7} \cdot F_{7 \times 1}$

69. **BUSINESS** The table lists the prices at the Sandwich Shoppe. (Lesson 3-5)

- List the prices in a 4×3 matrix.
- The manager decides to cut the prices of every item by 20%. List this new set of data in a 4×3 matrix.
- Subtract the second matrix from the first and determine the savings to the customer for each sandwich.

Sandwich	Small	Medium	Large
ham	\$4.50	\$6.75	\$9.50
salami	\$4.50	\$6.75	\$9.50
veggie	\$4.00	\$6.25	\$8.75
meatball	\$4.75	\$7.50	\$10.25

Graph each function. (Lesson 2-6)

70. $f(x) = 2|x - 3| - 4$

71. $f(x) = -3|2x| + 4$

72. $f(x) = |3x - 1| + 2$

Skills Review

Solve each system of equations.

73. $2x - 5y = -26$

74. $4y + 6x = 10$

75. $-3x - 2y = 17$

$5x + 3y = -34$

$2x - 7y = 22$

$-4x + 5y = -8$

Solving Systems of Equations Using Inverse Matrices

Then

- You solved systems of linear equations algebraically.

Now

- Find the inverse of a 2×2 matrix.
- Write and solve matrix equations for a system of equations.

Why?

- Maria's Sandwich Shop offers three lunch options as shown at the right.

To determine how much each individual item costs, you can solve the following matrix equation in which w represents the cost of a sandwich, s the cost of a side, and d the cost of a drink.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 4 & 3 & 4 \end{bmatrix} \begin{bmatrix} w \\ s \\ d \end{bmatrix} = \begin{bmatrix} 9 \\ 16.50 \\ 30.75 \end{bmatrix}$$



New Vocabulary

- identity matrix
- square matrix
- inverse matrix
- matrix equation
- variable matrix
- constant matrix



Common Core State Standards

Content Standards

A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

Mathematical Practices

5 Use appropriate tools strategically.

1 Identity and Inverse Matrices Recall that in real numbers, two numbers are multiplicative inverses if their product is the multiplicative identity, 1. Similarly, for matrices, the **identity matrix** is a *square matrix* that, when multiplied by another matrix, equals that same matrix. A **square matrix** is a matrix with the same number of rows and columns.

2×2 Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3×3 Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Key Concept Identity Matrix for Multiplication

Words

The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix A of the same dimension as I , $A \cdot I = I \cdot A = A$.

Symbols

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Two $n \times n$ matrices are **inverses** of each other if their product is the identity matrix. If matrix A has an inverse symbolized by A^{-1} , then $A \cdot A^{-1} = A^{-1} \cdot A = I$.



Example 1 Verify Inverse Matrices

Determine whether the matrices in each pair are inverses.

a. $A = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$

If A and B are inverses, then $A \cdot B = B \cdot A = I$.

$$A \cdot B = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$$

Write an equation.

$$= \begin{bmatrix} -1 + 1 & 2 - 2 \\ -\frac{1}{2} + \frac{1}{2} & 1 - 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix multiplication

Since $A \cdot B \neq I$, they are *not* inverses.

b. $F = \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix}$ and $G = \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$

If F and G are inverses, then $F \cdot G = G \cdot F = I$.

$$F \cdot G = \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$$

Write an equation.

$$= \begin{bmatrix} \frac{9}{4} - \frac{5}{4} & \frac{15}{8} - \frac{15}{8} \\ -\frac{6}{4} + \frac{6}{4} & -\frac{10}{8} + \frac{18}{8} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix multiplication

$$G \cdot F = \begin{bmatrix} \frac{3}{4} & \frac{5}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix}$$

Write an equation.

$$= \begin{bmatrix} \frac{9}{4} - \frac{10}{8} & -\frac{15}{4} + \frac{30}{8} \\ \frac{3}{4} - \frac{6}{8} & -\frac{5}{4} + \frac{18}{8} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix multiplication

Since $F \cdot G = G \cdot F = I$, F and G are inverses.

Guided Practice

1. Determine whether $X = \begin{bmatrix} 4 & -1 \\ 2 & -2 \end{bmatrix}$ and $Y = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$ are inverses of each other.

Some matrices do not have inverses. You can determine whether a matrix has an inverse by using the determinant.

Key Concept Inverse of a 2×2 Matrix

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Notice that $ad - bc$ is the value of $\det A$. Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.

Study Tip

CCSS Structure

Since multiplication of matrices is not commutative, it is necessary to check the product in both orders.





Math HistoryLink

Seki Kowa (1642–1708)
Known as The Arithmetical Sage, Seki Kowa was the first to develop the theory of determinants.

Example 2 Find the Inverse of a Matrix

Find the inverse of each matrix, if it exists.

a. $P = \begin{bmatrix} 7 & -5 \\ 2 & -1 \end{bmatrix}$

$$\begin{vmatrix} 7 & -5 \\ 2 & -1 \end{vmatrix} = -7 - (-10) \text{ or } 3 \quad \text{Find the determinant.}$$

Since the determinant does not equal 0, P^{-1} exists.

$$P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{Definition of inverse}$$

$$= \frac{1}{7(-1) - (-5)(2)} \begin{bmatrix} -1 & 5 \\ -2 & 7 \end{bmatrix} \quad a = 7, b = -5, c = 2, d = -1$$

$$= \frac{1}{3} \begin{bmatrix} -1 & 5 \\ -2 & 7 \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{1}{3} & \frac{5}{3} \\ -\frac{2}{3} & \frac{7}{3} \end{bmatrix} \quad \text{Simplify.}$$

CHECK Find the product of the matrices. If the product is I , then they are inverses.

$$\begin{bmatrix} 7 & -5 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{3} & \frac{5}{3} \\ -\frac{2}{3} & \frac{7}{3} \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} + \frac{10}{3} & \frac{35}{3} - \frac{35}{3} \\ -\frac{2}{3} + \frac{2}{3} & \frac{10}{3} - \frac{7}{3} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

b. $Q = \begin{bmatrix} -8 & -6 \\ 12 & 9 \end{bmatrix}$

$$\begin{vmatrix} -8 & -6 \\ 12 & 9 \end{vmatrix} = -72 - (-72) = 0 \quad \text{Find the determinant.}$$

Since the determinant equals 0, Q^{-1} does not exist.

Guided Practice

2A. $\begin{bmatrix} 3 & 7 \\ 1 & -4 \end{bmatrix}$

2B. $\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$

2 Matrix Equations Matrices can be used to represent and solve systems of equations. You can write a **matrix equation** to solve the system of equations below.

$$\begin{aligned} x + 2y &= 9 \\ 3x - 6y &= 3 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} x + 2y \\ 3x - 6y \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

Write the left side of the matrix equation as the product of the coefficient matrix and the variable matrix. Write the right side as a constant matrix.

$$\begin{matrix} A \\ \begin{bmatrix} 1 & 2 \\ 3 & -6 \end{bmatrix} \end{matrix} \cdot \begin{matrix} X \\ \begin{bmatrix} x \\ y \end{bmatrix} \end{matrix} = \begin{matrix} B \\ \begin{bmatrix} 9 \\ 3 \end{bmatrix} \end{matrix}$$

coefficient matrix

variable matrix
only the variables
of a system

constant matrix
only the constants
of a system



StudyTip

Inverse You can use this method to solve systems of equations only if A has an inverse. If A does not have an inverse, then the system either has no solution or infinitely many solutions.

Then solve the matrix equation in the same way that you would solve any other equation.

$$\begin{array}{lll} ax = b & \text{Write the equation.} & AX = B \\ \left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b & \text{Multiply each side by the inverse of the coefficient, if it exists.} & A^{-1}AX = A^{-1}B \\ 1x = \frac{b}{a} & \left(\frac{1}{a}\right)a = 1, A^{-1}A = I & IX = A^{-1}B \\ x = \frac{b}{a} & 1x = x, IX = X & X = A^{-1}B \end{array}$$

Notice that the solution of the matrix equation is the product of the inverse of the coefficient matrix and the constant matrix.



Real-World Example 3 Solve a System of Equations

TRAVEL Helena stopped for gasoline twice during a car trip. The price of gasoline at the first station where she stopped was \$3.75 per gallon. At the second station, the price was \$3.50 per gallon. Helena bought a total of 24.2 gallons of gasoline and spent \$88.05. How much gasoline did Helena buy at each gas station?

A system of equations to represent the situation is as follows.

$$\begin{aligned} x + y &= 24.2 \\ 3.75x + 3.50y &= 88.05 \end{aligned}$$

The matrix equation is $\begin{bmatrix} 1 & 1 \\ 3.75 & 3.50 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24.2 \\ 88.05 \end{bmatrix}$.

Step 1 Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{3.50 - 3.75} \begin{bmatrix} 3.50 & -1 \\ -3.75 & 1 \end{bmatrix} \text{ or } -\frac{1}{0.25} \begin{bmatrix} 3.50 & -1 \\ -3.75 & 1 \end{bmatrix}$$

Step 2 Multiply each side of the matrix equation by the inverse matrix.

$$\begin{aligned} -\frac{1}{0.25} \begin{bmatrix} 3.50 & -1 \\ -3.75 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 3.75 & 3.50 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} &= -\frac{1}{0.25} \begin{bmatrix} 3.50 & -1 \\ -3.75 & 1 \end{bmatrix} \cdot \begin{bmatrix} 24.2 \\ 88.05 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} &= -\frac{1}{0.25} \begin{bmatrix} -3.35 \\ -2.70 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 13.4 \\ 10.8 \end{bmatrix} \end{aligned}$$

The solution is (13.4, 10.8), where x represents the amount of gas Helena purchased at the first gas station, and y represents the amount purchased at the second gas station.

CHECK You can check your answer by using inverses.

Enter $\begin{bmatrix} 1 & 1 \\ 3.75 & 3.50 \end{bmatrix}$ as matrix A .

Enter $\begin{bmatrix} 24.2 \\ 88.05 \end{bmatrix}$ as matrix B .

Multiply the inverse of A by B .



GuidedPractice

3. COMIC BOOKS Dante and Erica just returned from a comic book store that sells new and used comic books. Dante spent \$11.25 on 3 new and 4 old books, and Erica spent \$15.75 on 10 used and 3 new ones. If comic books of one type are sold at the same price, what is the price in dollars of a new comic book?



Real-WorldLink

Average gas prices increased fivefold from \$0.70 per gallon in 1977 to \$3.50 per gallon in 2007.

Source: U.S. Department of Energy

Example 1 Determine whether the matrices in each pair are inverses.

1. $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ 2. $C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}$
 3. $F = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, G = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$ 4. $H = \begin{bmatrix} -3 & -1 \\ -4 & -2 \end{bmatrix}, J = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$

Example 2 Find the inverse of each matrix, if it exists.

5. $\begin{bmatrix} 6 & -3 \\ -1 & 0 \end{bmatrix}$ 6. $\begin{bmatrix} 2 & -4 \\ -3 & 0 \end{bmatrix}$
 7. $\begin{bmatrix} -3 & 0 \\ 5 & 2 \end{bmatrix}$ 8. $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

Example 3 Use a matrix equation to solve each system of equations.

9. $-2x + y = 9$ 10. $4x - 2y = 22$ 11. $-2x + y = -4$
 $x + y = 3$ $6x + 9y = -3$ $3x + y = 1$

12. **MONEY** Kevin had 25 quarters and dimes. The total value of all the coins was \$4. How many quarters and dimes did Kevin have?

Practice and Problem Solving

Extra Practice is on page F

Example 1 Determine whether each pair of matrices are inverses of each other.

13. $K = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ 14. $M = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix}, N = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
 15. $P = \begin{bmatrix} 4 & 0 \\ 3 & 0 \end{bmatrix}, Q = \begin{bmatrix} -1 & -1 \\ \frac{2}{3} & 5 \end{bmatrix}$ 16. $R = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix}, S = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

Example 2 Find the inverse of each matrix, if it exists.

17. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ 18. $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ 19. $\begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}$
 20. $\begin{bmatrix} 1 & -1 \\ -6 & -1 \end{bmatrix}$ **21.** $\begin{bmatrix} -5 & -4 \\ 4 & 2 \end{bmatrix}$ 22. $\begin{bmatrix} -5 & 9 \\ 4 & -8 \end{bmatrix}$
 23. $\begin{bmatrix} 6 & -5 \\ 4 & 9 \end{bmatrix}$ 24. $\begin{bmatrix} -4 & -2 \\ 7 & 8 \end{bmatrix}$ 25. $\begin{bmatrix} -6 & 8 \\ 8 & -7 \end{bmatrix}$

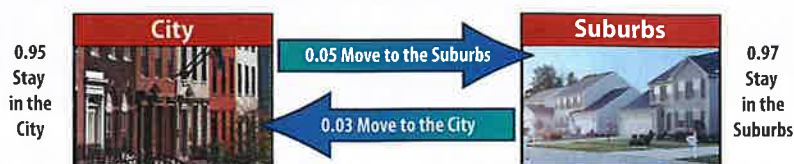
Example 3 26. **BAKING** Peggy is preparing a colored frosting for a cake. For the right shade of purple, she needs 25 milliliters of a 44% concentration food coloring. The store has a 25% red and a 50% blue concentration of food coloring. How many milliliters each of blue food coloring and red food coloring should be mixed to make the necessary amount of purple food coloring?

CCSS PERSEVERANCE Use a matrix equation to solve each system of equations.

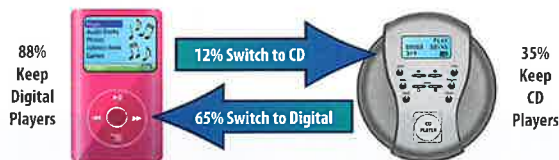
27. $-x + y = 4$ 28. $-x + y = 3$ 29. $x + y = 4$
 $-x + y = -4$ $-2x + y = 6$ $-4x + y = 9$
 30. $3x + y = 3$ 31. $y - x = 5$ 32. $4x + 2y = 6$
 $5x + 3y = 6$ $2y - 2x = 8$ $6x - 3y = 9$
 33. $1.6y - 0.2x = 1$ 34. $4y - x = -2$ 35. $2y - 4x = 3$
 $0.4y - 0.1x = 0.5$ $3y - x = 6$ $4x - 3y = -6$



36. **POPULATIONS** The diagram shows the annual percent migration between a city and its suburbs.



- Write a matrix to represent the transitions in city population and suburb population.
 - There are currently 16,275 people living in the city and 17,552 people living in the suburbs. Assuming that the trends continue, predict the number of people who will live in the suburbs next year.
 - Use the inverse of the matrix from part b to find the number of people who lived in the city last year.
37. **MUSIC** The diagram shows the trends in digital audio player and portable CD player ownership over the past five years for Central City. Every person in Central City has either a digital audio player or a portable CD player. Central City has a stable population of 25,000 people, of whom 17,252 own digital audio players and 7748 own portable CD players.



- Write a matrix to represent the transitions in player ownership.
- Assume that the trends continue. Predict the number of people who will own digital audio players next year.
- Use the inverse of the matrix from part b to find the number of people who owned digital audio players last year.

H.O.T. Problems Use Higher-Order Thinking Skills

38. **CCSS CRITIQUE** Cody and Megan are setting up matrix equations for the system $5x + 7y = 19$ and $3y + 4x = 10$. Is either of them correct? Explain your reasoning.

$$\begin{array}{c} \text{Cody} \\ \left[\begin{array}{cc} 5 & 7 \\ 3 & 4 \end{array} \right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \text{Megan} \\ \left[\begin{array}{cc} 5 & 7 \\ 4 & 3 \end{array} \right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \end{bmatrix} \end{array}$$

39. **CHALLENGE** Describe what a matrix equation with infinite solutions looks like.
40. **REASONING** Determine whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning.
A square matrix has a multiplicative inverse.
41. **OPEN ENDED** Write a matrix equation that does not have a solution.
42. **WRITING IN MATH** When would you prefer to solve a system of equations using algebraic methods, and when would you prefer to use matrices? Explain.



Standardized Test Practice

43. The Yogurt Shoppe sells cones in three sizes: small, \$0.89; medium, \$1.19; and large, \$1.39. One day Santos sold 52 cones. He sold seven more medium cones than small cones. If he sold \$58.98 in cones, how many medium cones did he sell?

A 11 B 17 C 24 D 36

44. The chart shows an expression evaluated for different values of x .

x	$x^2 + x + 1$
1	3
2	7
3	13
4	21

A student concludes that for all values of x , $x^2 + x + 1$ produces a prime number. Which value of x serves as a counterexample to prove this conclusion false?

F -4 G -3 H 2 J 4

45. **SHORT RESPONSE** What is the solution of the system of equations $6a + 8b = 5$ and $10a - 12b = 2$?

46. **SAT/ACT** Each year at Capital High School the students vote to choose the theme of the homecoming dance. The theme "A Night Under the Stars" received 225 votes, and "The Time of My Life" received 480 votes. If 40% of girls voted for "A Night Under the Stars" and 75% of boys voted for "The Time of My Life," how many girls and boys voted?

A 176 boys and 351 girls
 B 395 boys and 310 girls
 C 380 boys and 325 girls
 D 705 boys and 325 girls
 E 854 boys and 176 girls

Spiral Review

Evaluate each determinant. (Lesson 3-7)

47. $\begin{vmatrix} 8 & -3 \\ 6 & -9 \end{vmatrix}$

48. $\begin{vmatrix} 9 & -7 \\ -5 & -3 \end{vmatrix}$

49. $\begin{vmatrix} 8 & 6 & -1 \\ -4 & 5 & 1 \\ -3 & -2 & 9 \end{vmatrix}$

Find each product, if possible. (Lesson 3-6)

50. $\begin{bmatrix} 4 & 2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 5 & 1 \end{bmatrix}$

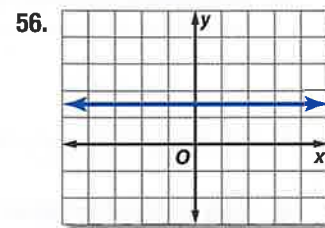
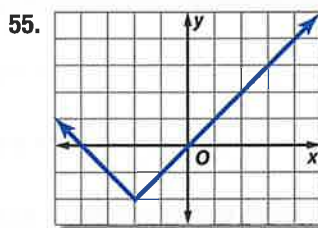
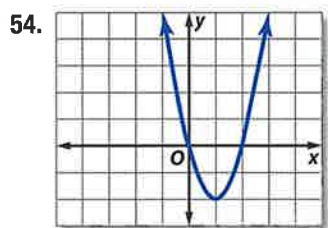
51. $\begin{bmatrix} 8 & -2 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

52. $\begin{bmatrix} -3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -6 & -8 \\ -4 & 5 \end{bmatrix}$

53. **MILK** The Yoder Family Dairy produces at most 200 gallons of skim and whole milk each day for delivery to large bakeries and restaurants. Regular customers require at least 15 gallons of skim and 21 gallons of whole milk each day. If the profit on a gallon of skim milk is \$0.82 and the profit on a gallon of whole milk is \$0.75, how many gallons of each type of milk should the dairy produce each day to maximize profits? (Lesson 3-4)

Skills Review

Identify the type of the function represented by each graph.





Using a TI-83/84 Plus graphing calculator, you can solve a system of linear equations using the MATRIX function. An **augmented matrix** contains the coefficient matrix with an extra column containing the constant terms. You can use a graphing calculator to reduce the augmented matrix so that the solution of the system of equations can be easily determined.



Example

Write an augmented matrix for the following system of equations. Then solve the system by using a graphing calculator.

$$\begin{aligned} 2x + y + z &= 1 \\ 3x + 2y + 3z &= 12 \\ 4x + y + 2z &= -1 \end{aligned}$$

Step 1 Write the augmented matrix and enter it into a calculator.

The augmented matrix $B = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 3 & 2 & 3 & 12 \\ 4 & 1 & 2 & -1 \end{array} \right]$.

Begin by entering the matrix.

KEYSTROKES: **2nd** **[MATRIX]** **▶** **▶** **ENTER** 3 **ENTER** 4 **ENTER** 2 **ENTER** 1
ENTER 1 **ENTER** 1 **ENTER** 3 **ENTER** 2 **ENTER** 3 **ENTER** 12
ENTER 4 **ENTER** 1 **ENTER** 2 **ENTER** (←) 1 **ENTER**

Step 2 Find the reduced row echelon form (rref) using the graphing calculator.

KEYSTROKES: **2nd** **[QUIT]** **2nd** **[MATRIX]** **▶** **ALPHA** **[B]** **2nd** **[MATRIX]**
ENTER **)** **ENTER**

Study the reduced echelon matrix. The first three columns are the same as a 3×3 identity matrix. The first row represents $x = -4$, the second row represents $y = 3$, and the third row represents $z = 6$. The solution is $(-4, 3, 6)$.



Exercises

Write an augmented matrix for each system of equations. Then solve with a graphing calculator.

- | | | |
|-------------------------------------|--|---|
| 1. $3x + 2y = -4$
$4x + 7y = 13$ | 2. $2x + y = 6$
$6x - 2y = 0$ | 3. $2x + 2y = -4$
$7x + 3y = 10$ |
| 4. $4x + 6y = 0$
$8x - 2y = 7$ | 5. $6x - 4y + 2z = -4$
$2x - 2y + 6z = 10$
$2x + 2y + 2z = -2$ | 6. $5x - 5y + 5z = 10$
$5x - 5z = 5$
$5y + 10z = 0$ |

3 Study Guide and Review

Study Guide

Key Concepts

Systems of Equations and Inequalities (Lessons 3-1 and 3-2)

- In the substitution method, one equation is solved for a variable and substituted to find the value of another variable. In the elimination method, one variable is eliminated by adding or subtracting the equations.
- The solution of a system of inequalities is found by graphing the inequalities and determining the intersection of the graphs.

Linear Programming (Lesson 3-3)

- Linear programming is a method for finding maximum or minimum values of a function over a given system of inequalities with each inequality representing a constraint.

Systems of Equations in Three Variables (Lesson 3-4)

- A system of equations in three variables can be solved algebraically by using the substitution method or the elimination method.

Operations with Matrices (Lessons 3-5 and 3-6)

- Matrices can be added or subtracted if they have the same dimensions. Add or subtract corresponding elements.
- Two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Solving Systems Using Cramer's Rule (Lesson 3-7)

- If a determinant is nonzero, then the system has a unique solution. If the determinant is 0, then the system either has no solution or infinite solutions.

Solving Systems with Inverse Matrices (Lesson 3-8)

- An identity matrix is a square matrix with ones on the main diagonal and zeros in the other positions.
- Two matrices are inverses of each other if their product is the identity matrix.
- To solve a matrix equation, find the inverse of the coefficient matrix. Then multiply each side of the equation by the inverse of the coefficient matrix.

FOLDABLES Study Organizer

Be sure the Key Concepts are noted in your Foldable.



Key Vocabulary

- | | |
|-----------------------------|--------------------------------|
| bounded (p. 154) | inconsistent (p. 137) |
| break-even point (p. 136) | independent (p. 137) |
| coefficient matrix (p. 192) | inverse matrices (p. 198) |
| consistent (p. 137) | matrix (p. 169) |
| constant matrix (p. 200) | matrix equation (p. 200) |
| Cramer's Rule (p. 192) | optimize (p. 156) |
| dependent (p. 137) | ordered triple (p. 161) |
| determinant (p. 189) | scalar (p. 173) |
| diagonal rule (p. 190) | scalar multiplication (p. 173) |
| dimensions (p. 169) | substitution method (p. 138) |
| elimination method (p. 139) | unbounded (p. 154) |
| feasible region (p. 154) | variable matrix (p. 200) |
| identity matrix (p. 198) | |

Vocabulary Check

Choose the term from above to complete each sentence.

1. A feasible region that is open and can go on forever is called _____.
2. To _____ means to seek the best price or profit using linear programming.
3. A matrix that contains the constants in a system of equations is called a(n) _____.
4. A matrix can be multiplied by a constant called a(n) _____.
5. The _____ of a matrix with 4 rows and 3 columns are 4×3 .
6. A system of equations is _____ if it has at least one solution.
7. The _____ matrix is a square matrix that, when multiplied by another matrix, equals that same matrix.
8. The _____ is the point at which income equals cost.
9. A system of equations is _____ if it has no solutions.
10. If the product of two matrices is the identity matrix, they are _____.



Lesson-by-Lesson Review

3-1 Solving Systems of Equations

Solve each system of equations by graphing.

- | | |
|------------------------|-----------------------------------|
| 11. $3x + 4y = 8$ | 12. $x + \frac{8}{3}y = 12$ |
| $x - 3y = -6$ | $\frac{1}{2}x + \frac{4}{3}y = 6$ |
| 13. $y - 3x = 13$ | 14. $6x - 14y = 5$ |
| $y = \frac{1}{3}x + 5$ | $3x - 7y = 5$ |

15. **LAWN CARE** André and Paul each mow lawns. André charges a \$30 service fee and \$10 per hour. Paul charges a \$10 service fee and \$15 per hour. After how many hours will André and Paul charge the same amount?

Solve each system of equations by using either substitution or elimination.

- | | |
|-------------------|-------------------|
| 16. $x + y = 6$ | 17. $5x - 2y = 4$ |
| $3x - 2y = -2$ | $-2y + x = 12$ |
| 18. $x + y = 3.5$ | 19. $3y - 5x = 0$ |
| $x - y = 7$ | $2y - 4x = -2$ |

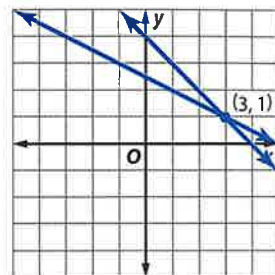
20. **SCHOOL SUPPLIES** At an office supply store, Emilio bought 3 notebooks and 5 pens for \$13.75. If a notebook costs \$1.25 more than a pen, how much does a notebook cost? How much does a pen cost?

Example 1

Solve the system of equations by graphing.

$$\begin{aligned} x + y &= 4 \\ x + 2y &= 5 \end{aligned}$$

Graph both equations on the coordinate plane.



The solution of the system is (3, 1).

Example 2

Solve the system of equations by using either substitution or elimination.

$$\begin{aligned} 3x + 2y &= 1 \\ y &= -x + 1 \end{aligned}$$

Substitute $-x + 1$ for y in the first equation. Then solve for y .

$3x + 2y = 1$	$y = -x + 1$
$3x + 2(-x + 1) = 1$	$= -(-1) + 1$
$3x - 2x + 2 = 1$	$= 2$
$x + 2 = 1$	
$x = -1$	

The solution is $(-1, 2)$.

3-2 Solving Systems of Inequalities by Graphing

Solve each system of inequalities by graphing.

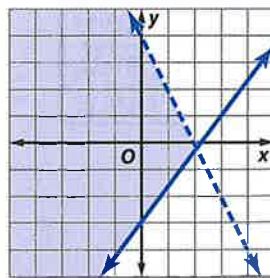
- | | |
|--------------------|-----------------|
| 21. $y < 2x - 3$ | 22. $ y > 2$ |
| $y \geq 4$ | $x > 3$ |
| 23. $y \geq x + 3$ | 24. $y > x + 1$ |
| $2y \leq x - 5$ | $x < -2$ |

25. **JEWELRY** Payton makes jewelry to sell at her mother's clothing store. She spends no more than 3 hours making jewelry on Saturdays. It takes her 15 minutes to set up her supplies and 25 minutes to make each bracelet. Draw a graph that represents this.

Example 3

Solve the system of inequalities by graphing.

$$\begin{aligned} y &\geq \frac{3}{2}x - 3 \\ y &< 4 - 2x \end{aligned}$$



The solution of the system is the region that satisfies both inequalities. The solution of this system is the shaded region.

3-3 Optimization with Linear Programming

26. **FLOWERS** A florist can make a grand arrangement in 18 minutes or a simple arrangement in 10 minutes. The florist makes at least twice as many of the simple arrangements as the grand arrangements. The florist can work only 40 hours per week. The profit on the simple arrangements is \$10 and the profit on the grand arrangements is \$25. Find the number and type of arrangements that the florist should produce to maximize profit.

27. **MANUFACTURING** A shoe manufacturer makes outdoor and indoor soccer shoes. There is a two-step process for both kinds of shoes. Each pair of outdoor shoes requires 2 hours in step one and 1 hour in step two, and produces a profit of \$20. Each pair of indoor shoes requires 1 hour in step one and 3 hours in step two, and produces a profit of \$15. The company has 40 hours of labor available per day for step one and 60 hours available for step two. What is the manufacturer's maximum profit? What is the combination of shoes for this profit?

Example 4

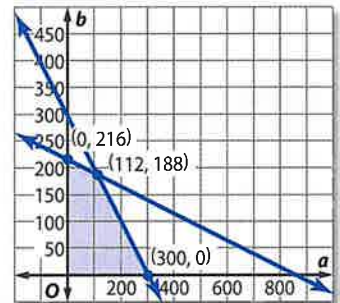
A gardener is planting two types of herbs in a 5184-square inch garden. Herb A requires 6 square inches of space, and herb B requires 24 square inches of space. The gardener will plant no more than 300 plants. If herb A can be sold for \$8 and herb B can be sold for \$12, how many of each herb should be sold to maximize income?

Let a = the number of herb A and
 b = the number of herb B.

$$a \geq 0, b \geq 0,$$

$$6a + 24b \leq 5184,$$

$$\text{and } a + b \leq 300$$



Graph the inequalities. The vertices of the feasible region are $(0, 0)$, $(300, 0)$, $(0, 216)$, and $(112, 188)$.

The profit function is $f(a, b) = 8a + 12b$.

The maximum value of \$3152 occurs at $(112, 188)$. So the gardener should plant 112 of herb A and 188 of herb B.

3-4 Systems of Equations in Three Variables

Solve each system of equations.

28. $a - 4b + c = 3$ 29. $2x - z = 14$
 $b - 3c = 10$ $3x - y + 5z = 0$
 $3b - 8c = 24$ $4x + 2y + 3z = -2$

30. **AMUSEMENT PARKS** Dustin, Luis, and Marci went to an amusement park. They purchased snacks from the same vendor. Their snacks and how much they paid are listed in the table. How much did each snack cost?

Name	Hot Dogs	Popcorn	Soda	Price
Dustin	1	2	3	\$15.25
Luis	2	0	3	\$14.00
Marci	1	2	1	\$10.25

Example 5

Solve the system of equations.

$$x + y + 2z = 6$$

$$2x + 5z = 12$$

$$x + 2y + 3z = 9$$

$$2x + 2y + 4z = 12 \quad \text{Equation 1} \times 2$$

$$\begin{array}{r} (-) x + 2y + 3z = 9 \quad \text{Equation 3} \\ \hline x + z = 3 \quad \text{Subtract.} \end{array}$$

Solve the system of two equations.

$$2x + 5z = 12 \quad \text{Equation 2}$$

$$\begin{array}{r} (-) 2x + 2z = 6 \quad 2 \times (x + z = 3) \\ \hline 3z = 6 \quad \text{Subtract.} \\ z = 2 \quad \text{Divide each side by 3.} \end{array}$$

Substitute 2 for z in one of the equations with two variables, and solve for y . Then, substitute 2 for z and the value you got for y into an equation from the original system to solve for x .

The solution is $(1, 1, 2)$.

3-5 Operations with Matrices

Perform the indicated operations. If the matrix does not exist, write *impossible*.

$$31. 3 \left(\begin{bmatrix} -2 & 0 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 9 \\ -3 & -4 \end{bmatrix} \right)$$

$$32. \begin{bmatrix} 2 \\ -6 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

33. **RETAIL** Current Fashions buys shirts, jeans, and shoes from a manufacturer, marks them up, and then sells them. The table shows the purchase price and the selling price.

Item	Purchase Price	Selling Price
shirts	\$15	\$35
jeans	\$25	\$55
shoes	\$30	\$85

- Write a matrix for the purchase price.
- Write a matrix for the selling price.
- Use matrix operations to find the profit on 1 shirt, 1 pair of jeans, and 1 pair of shoes.

Example 6

Find $2A + 3B$ if $A = \begin{bmatrix} 9 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$.

$$2B = 2 \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 8 \\ 6 & 14 \end{bmatrix}$$

$$3A = 3 \begin{bmatrix} 9 & 1 \\ 1 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 27 & 3 \\ 3 & 6 \end{bmatrix}$$

$$2B + 3A = \begin{bmatrix} 2 & 8 \\ 6 & 14 \end{bmatrix} + \begin{bmatrix} 27 & 3 \\ 3 & 6 \end{bmatrix} \text{ or } \begin{bmatrix} 29 & 11 \\ 9 & 20 \end{bmatrix}$$

Example 7

Find $3C - 5D$ if $C = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$ and $D = [9 \ 8]$.

$$3C - 5D = 3 \begin{bmatrix} 3 \\ -7 \end{bmatrix} - 5[9 \ 8].$$

Because the dimensions are different, you cannot subtract the matrices.

3-6 Multiplying Matrices

Find each product, if possible.

$$34. [3 \ -7] \cdot \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

$$35. \begin{bmatrix} -3 & 0 & 2 \\ 6 & -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 8 & -1 \\ -4 & 3 \\ 6 & 7 \end{bmatrix}$$

$$36. \begin{bmatrix} 2 & 11 \\ 0 & -3 \\ -6 & 7 \end{bmatrix} \cdot \begin{bmatrix} 0 & 8 & -5 \\ 12 & 0 & 9 \\ 4 & -6 & 7 \end{bmatrix}$$

37. **GROCERIES** Martin bought 1 gallon of milk, 2 apples, 4 frozen dinners, and 1 box of cereal. The following matrix shows the prices for each item, respectively.

$$[\$2.59 \ \$0.49 \ \$5.25 \ \$3.99]$$

Use matrix multiplication to find the total amount of money Martin spent at the grocery store.

Example 8

Find XY if $X = \begin{bmatrix} 0 & -6 \\ 3 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$.

$$XY = \begin{bmatrix} 0 & -6 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -1 \end{bmatrix} \quad \text{Write an equation.}$$

$$= \begin{bmatrix} 0(8) + (-6)(-1) \\ 3(8) + 5(-1) \end{bmatrix} \quad \text{Multiply columns by rows.}$$

$$= \begin{bmatrix} 6 \\ 19 \end{bmatrix} \quad \text{Simplify.}$$

3-7 Solving Systems of Equations Using Cramer's Rule

Evaluate each determinant.

38. $\begin{vmatrix} 2 & 4 \\ 7 & -3 \end{vmatrix}$

39. $\begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

Use Cramer's Rule to solve each system of equations.

40. $3x - y = 0$
 $5x + 2y = 22$

41. $5x + 2y = 4$
 $3x + 4y + 2z = 6$
 $7x + 3y + 4z = 29$

42. **JEWELRY** Alana paid \$98.25 for 3 necklaces and 2 pairs of earrings. Petra paid \$133.50 for 2 necklaces and 4 pairs of earrings. Use Cramer's Rule to find out how much 1 necklace costs and how much 1 pair of earrings costs.

Example 9

Evaluate $\begin{vmatrix} 4 & -6 \\ 2 & 5 \end{vmatrix}$.

$\begin{vmatrix} 4 & -6 \\ 2 & 5 \end{vmatrix} = 4(5) - (-6)(2)$
 $= 20 + 12$ or 32

Definition of determinant

Simplify.

Example 10

Use Cramer's Rule to solve $2a + 6b = -1$ and $a + 8b = 2$

$a = \frac{\begin{vmatrix} -1 & 6 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix}}$

Cramer's Rule

$b = \frac{\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix}}$

$= \frac{-8 - 12}{16 - 6}$

Evaluate each determinant.

$= \frac{4 + 1}{16 - 6}$

$= \frac{-20}{10}$ or -2

Simplify.

$= \frac{5}{10}$ or $\frac{1}{2}$

The solution is $(-2, \frac{1}{2})$.

3-8 Solving Systems of Equations Using Inverse Matrices

Find the inverse of each matrix, if it exists.

43. $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$

44. $\begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$

45. $\begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$

Use a matrix equation to solve each system of equations.

46. $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

47. $\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

48. **HEALTH FOOD** Heath sells nuts and raisins by the pound. Sonia bought 2 pounds of nuts and 2 pounds of raisins for \$23.50. Drew bought 3 pounds of nuts and 1 pound of raisins for \$22.25. What is the cost of 1 pound of nuts and 1 pound of raisins?

Example 11

Solve $\begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 36 \end{bmatrix}$.

Step 1 Find the inverse of the coefficient matrix.

$A^{-1} = \frac{1}{-12 - (-15)} \begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix}$ or $\frac{1}{3} \begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix}$

Step 2 Multiply each side by the inverse matrix.

$\frac{1}{3} \begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 15 \\ 36 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 90 \\ 27 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \end{bmatrix}$

3 Practice Test

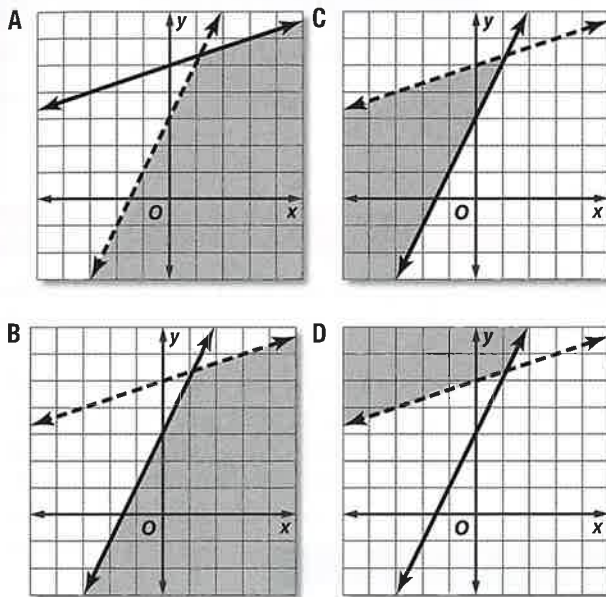
Solve each system of equations by using either substitution or elimination.

- $y = x + 4$
 $x + y = -12$
- $3x + 5y = -7$
 $6x - 4y = 0$
- $5x + 2y = 4$
 $3y - 4x = -40$
- $8x - 3y = -13$
 $-3x + 5y = 1$

5. **MULTIPLE CHOICE** Which graph shows the solution of the system of inequalities?

$$y \leq 2x + 3$$

$$y < \frac{1}{3}x + 5$$



Solve each system of inequalities by graphing.

- $x + y > 6$
 $x - y < 0$
- $y \geq 2x - 5$
 $y \leq x + 4$
- $3x + 4y \leq 12$
 $6x - 3y \geq 18$
- $5y + 2x \leq 20$
 $4x + 3y > 12$

10. **SALONS** Sierra King is a nail technician. She allots 20 minutes for a manicure and 45 minutes for a pedicure in her 7-hour work day. No more than 5 pedicures can be scheduled each day. The prices are \$18 for a manicure and \$45 for a pedicure. How many manicures and pedicures should Ms. King schedule to maximize her daily income? What is her maximum daily income?

11. **COLLEGE FOOTBALL** Darren McFadden of Arkansas placed second overall in the Heisman Trophy voting. Players are given 3 points for every first-place vote, 2 points for every second-place vote, and 1 point for every third-place vote. McFadden received 490 total votes for first, second, and third place, for a total of 878 points. If he had 4 more than twice as many second-place votes as third-place votes, how many votes did he receive for each place?

Perform the indicated operations. If the matrix does not exist, write *impossible*.

- $-3 \begin{bmatrix} 4a \\ 0 \\ -3 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$
- $\begin{bmatrix} -3 & 0 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ -6 & 0 \end{bmatrix}$
- $\begin{bmatrix} 2 & 0 \\ -3 & 5 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix}$
- $\begin{bmatrix} -5 & 7 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -2 \\ 9 & 0 & 1 \end{bmatrix}$

16. **MULTIPLE CHOICE** What is the value of

$$\begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} ?$$

F -44

H $\frac{1}{44}$

G $-\frac{1}{44}$

J 44

Find the inverse of each matrix, if it exists.

- $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
- $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$
- $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$

Use Cramer's Rule to solve each system of equations.

- $2x - y = -9$
 $x + 2y = 8$
- $x - y + 2z = 0$
 $3x + z = 11$
 $-x + 2y = 0$



Short Answer Questions

Short answer questions require you to provide a solution to the problem, along with a method, explanation, and/or justification used to arrive at the solution.

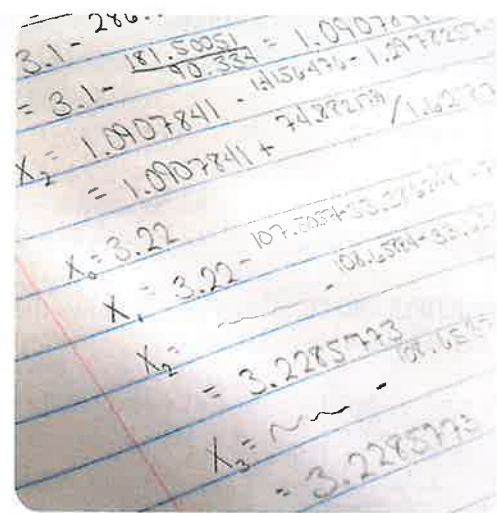
Strategies for Solving Short Answer Questions

Short answer questions are typically graded using a **rubric**, or a scoring guide. The following is an example of a short answer question scoring rubric.

Scoring Rubric	
Criteria	Score
Full Credit: The answer is correct and a full explanation is provided that shows each step.	2
Partial Credit: <ul style="list-style-type: none"> The answer is correct but the explanation is incomplete. The answer is incorrect but the explanation is correct. 	1
No Credit: Either an answer is not provided or the answer does not make sense.	0

In solving short answer questions, remember to...

- explain your reasoning or state your approach to solving the problem.
- show all of your work or steps.
- check your answer if time permits.



Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Company A charges a monthly fee of \$14.50 plus \$0.05 per minute for cell phone service. Company B charges \$20.00 per month plus \$0.04 per minute. For what number of minutes would the total monthly charge be the same with each company?

Read the problem carefully. You are given information about two different cell phone companies and their monthly charges. Since the situation involves a fixed amount and a variable rate, you can set up and solve a system of equations.

Example of a 2-point response:

Set up and solve a system of equations.

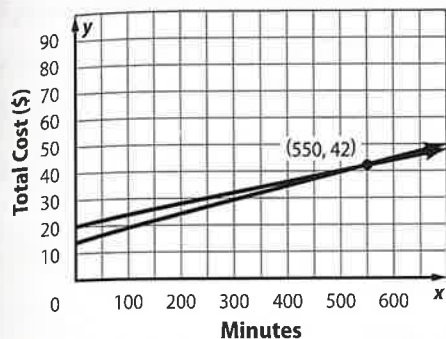
flat fee + rate \times minutes = total charges

y = total charges, x = minutes used

$y = 14.5 + 0.05x$ (Company A)

$y = 20 + 0.04x$ (Company B)

Solve the system by graphing.



The solution is $(550, 42)$. So, with each company, if the customer uses 550 minutes the total monthly charge is \$42.

The steps, calculations, and reasoning are clearly stated. The student also arrives at the correct answer. So, this response is worth the full 2 points.

Exercises

Read each problem. Then use the information in the problem to solve.

- Shawn and Jerome borrowed \$1400 to start a lawn mowing business. They charge their customers \$45 per lawn, and with each lawn that they mow, they incur \$10.50 in operating expenses. How many lawns must they mow in order to start earning a profit?
- A circle of radius r is circumscribed about a square. What is the exact ratio of the area of the circle to the area of the square?
- Mr. Williams can spend no more than \$50 on art supplies. Packages of paint brushes cost \$4.75 each, and boxes of colored pencils cost \$6.50 each. He wants to buy at least 2 packages of each supply. Write a system of inequalities and plot the feasible region on a coordinate grid. Give three different solutions to the system.
- Marla sells engraved necklaces over the Internet. She purchases 50 necklaces for \$400, and it costs her an additional \$3 for each personalized engraving. If she charges \$20 for each necklace, how many will she need to sell in order to make a profit of at least \$225?
- An auto dealership sold 7378 cars during 2011. This was an 8.5% increase in the number of cars sold during 2010. What was the increase in the number of cars sold in 2011?
- The sides of two similar triangles are in a ratio of 3:5. If the area of the larger triangle is 600 square centimeters, what is the area of the smaller triangle?
- Raul had \$35 in a savings account and started adding \$25 a week. At the same time, his sister Tina had \$365 in her account and began spending \$30 a week. After how many weeks will Raul and Tina have the same amount in their savings accounts?
- A city planner wishes to build a sidewalk diagonally across a rectangular-shaped park. The park measures 140 feet by 225 feet. It will cost \$30 per foot to construct the sidewalk. What will be the total cost of the sidewalk?

Standardized Test Practice

Cumulative, Chapters 1 through 3

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Matrix L shows the average low temperature, in degrees Fahrenheit, each month where Terrance lives. Matrix H shows the monthly average high temperature.

$$L = \begin{bmatrix} 24.1 & 27.7 & 35.9 \\ 44.1 & 53.6 & 62.2 \\ 66.4 & 64.9 & 57.9 \\ 46.4 & 37.3 & 28.4 \end{bmatrix}$$

$$H = \begin{bmatrix} 39.9 & 45.2 & 55.3 \\ 65.1 & 74.0 & 82.3 \\ 85.9 & 84.6 & 78.1 \\ 66.9 & 54.5 & 44.3 \end{bmatrix}$$

Which operation would you use to find the difference between the average high temperature and the average low temperature each month?

- A $L + H$ C $H \times L$
 B $H - L$ D $L - H$
2. Find $[3 \ 1] \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, if possible.
 F $[-3]$ H $\begin{bmatrix} 8 & -4 \\ 12 & 6 \end{bmatrix}$
 G $[11]$ J undefined
3. Which equation is equivalent to $4x - 3(2x + 7) = 5x$?
 A $-2x - 21 = 5x$ C $-2x + 21 = 5x$
 B $-2x + 7 = 5x$ D $6x - 7 = 5x$

Test-Taking Tip

Question 2 The product of a 1-by-2 matrix and a 2-by-1 matrix is a 1-by-1 matrix. So, answer choices H and J can be eliminated.

4. Triangle DEF has vertices $D(-6, 2)$, $E(3, 5)$, and $F(8, -7)$. Evaluate the determinant below to find the area of the triangle.

$$A = \frac{1}{2} \begin{vmatrix} -6 & 2 & 1 \\ 3 & 5 & 1 \\ 8 & -7 & 1 \end{vmatrix}$$

- F 54.5 square units
 G 58 square units
 H 60 square units
 J 61.5 square units
5. Suppose Kendall sells apples and tomatoes at a farmer's market. If he sold 280 items one morning and earned \$65.20, how many apples did he sell?

Item	Cost
apple	\$0.25
tomato	\$0.20

- A 96 C 168
 B 126 D 184

6. What are the dimensions of $D = \begin{bmatrix} 4 & -6 \\ 9 & 2 \\ 1 & 0 \\ -3 & -5 \end{bmatrix}$?
- F 4×2
 G 2×4
 H 4×8
 J 8×4

7. What is the solution set of $6 - |x + 7| \leq -2$?
 A $\{x \mid -15 \leq x \leq 1\}$
 B $\{x \mid -1 \leq x \leq 3\}$
 C $\{x \mid x \leq -1 \text{ or } x \geq 3\}$
 D $\{x \mid x \leq -15 \text{ or } x \geq 1\}$

Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

8. Does matrix B have an inverse? Explain why or why not.

$$B = \begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$$

9. **GRIDDED RESPONSE** Evaluate the determinant of

$$W = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & -4 \\ 0 & -1 & 1 \end{bmatrix}$$

10. Valeria has 14 quarters and dimes. The total value of all the coins is \$2.75. Use this information to answer each question.
- Let d represent the number of dimes that Valeria has, and let q represent the number of quarters. Write a system of equations to model the situation.
 - Write a matrix equation that can be used to solve for d and q .
 - Solve your matrix equation using inverses. How many dimes and quarters does Valeria have?
11. **GRIDDED RESPONSE** Andrea is using a coordinate grid to design a new deck for her backyard. The deck is represented by the intersection of $y \leq 20$, $x \leq 16$, $y \geq 0$, $x \geq 0$ and $y \leq -x + 32$. If each unit of the coordinate grid represents 1 foot, what is the area of the deck? Express your answer in square feet.

12. What are the coordinates of the x - and y -intercepts of the graph of $2y = 4x + 3$?

Extended Response

Record your answers on a sheet of paper. Show your work.

13. Suppose Tonya is baking cookies and muffins for a bake sale. Each tray of cookies uses 5 cups of flour and 2 cups of sugar. Each tray of muffins uses 5 cups of flour and 1 cup of sugar. She has 40 cups of flour and 15 cups of sugar available for baking. Tonya will make \$12 profit for each tray of cookies that is sold and \$8 profit for each tray of muffins sold.
- Let x represent the number of trays of cookies baked, and let y represent the number of trays of muffins baked. Write a system of inequalities to model the different number of trays Tonya can bake.
 - Graph the system of inequalities to show the feasible region. List the coordinates of the vertices of the feasible region.
 - Write a profit function for selling x trays of cookies and y trays of muffins.
 - How many trays of cookies and muffins should Tonya bake to maximize the profit? What will the total profit be?

Need ExtraHelp?

If you missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13
Go to Lesson...	3-5	3-6	1-2	3-7	3-1	3-5A	2-8	3-8	3-7	3-8	3-2	2-2	3-3

