CHAPTER 4

# Quadratic Functions and Relations

## Then

- You graphed linear equations and inequalities.

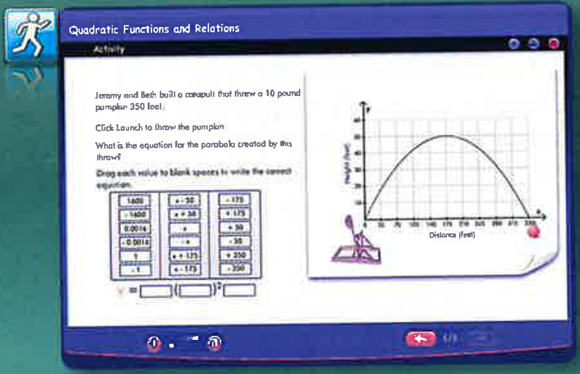
## Now

You will:

- Graph quadratic functions.
- Solve quadratic equations.
- Perform operations with complex numbers.
- Graph and solve quadratic inequalities.

## Why? ▲

**MOTION** The path that a soccer ball or a firework takes can be modeled by a quadratic function. Quadratic functions can map an object in motion. In this chapter you will look at a pumpkin catapult, an amusement park ride, and a diver in motion.



Jeremy and Beth built a catapult that threw a 10 pound pumpkin 350 feet.

Click Launch to throw the pumpkin.

What is the equation for the parabola created by this throw?

Drag each value to blank spaces to write the correct equation.

**connectED.mcgraw-hill.com** **Your Digital Math Portal**

Animation | Vocabulary | eGlossary | Personal Tutor | Virtual Manipulatives | Graphing Calculator | Audio | Foldables | Self-Check Practice | Worksheets

Blick 100/Allsport Concepts/Getty Images

LESSON

# 4-1 Graphing Quadratic Functions

## Then

- You identified and manipulated graphs of functions.

## Now

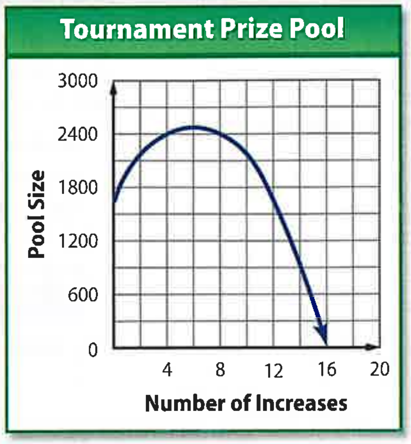
- **1** Graph quadratic functions.

**2** Find and interpret the maximum and minimum values of a quadratic function.

## Why?

- Eddie is organizing a charity tournament. He plans to charge a $20 entry fee for each of the 80 players. He recently decided to raise the entry fee by $5, and 5 fewer players entered with the increase. He used this information to determine how many fee increases will maximize the money raised.

The quadratic function at the right represents this situation. The tournament prize pool increases when he first increases the fee, but eventually the pool starts to decrease as the fee gets even higher.



### NewVocabulary

quadratic function
quadratic term
linear term
constant term
parabola
axis of symmetry
vertex
maximum value
minimum value

### Common Core State Standards

**Content Standards**

**A.SSE.1.a** Interpret parts of an expression, such as terms, factors, and coefficients.

**F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

**Mathematical Practices**

1 Make sense of problems and perseverance in solving them.

**1 Graph Quadratic Functions** In a **quadratic function**, the greatest exponent is 2. These functions can have a **quadratic term**, a **linear term**, and a **constant term**. The general quadratic function is shown below.

$$f(x) = \underbrace{ax^2}_{\text{quadratic term}} + \underbrace{bx}_{\text{linear term}} + \underbrace{c}_{\text{constant term}}, \text{ where } a \neq 0$$

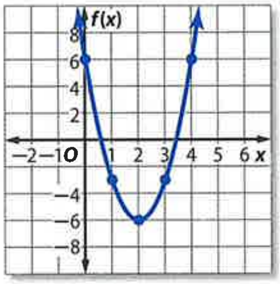
The graph of a quadratic function is called a **parabola**. To graph a quadratic function, graph ordered pairs that satisfy the function.

### Example 1 Graph a Quadratic Function by Using a Table

**Graph $f(x) = 3x^2 - 12x + 6$ by making a table of values.**

Choose integer values for $x$, and evaluate the function for each value. Graph the resulting coordinate pairs, and connect the points with a smooth curve.

| $x$ | $3x^2 - 12x + 6$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|---|
| 0 | $3(0)^2 - 12(0) + 6$ | 6 | (0, 6) |
| 1 | $3(1)^2 - 12(1) + 6$ | −3 | (1, −3) |
| 2 | $3(2)^2 - 12(2) + 6$ | −6 | (2, −6) |
| 3 | $3(3)^2 - 12(3) + 6$ | −3 | (3, −3) |
| 4 | $3(4)^2 - 12(4) + 6$ | 6 | (4, 6) |



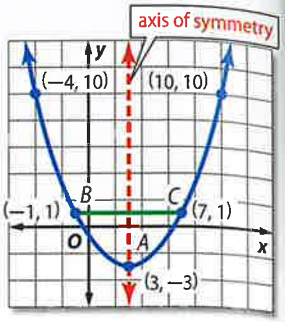
**GuidedPractice**

**Graph each function by making a table of values.**

**1A.** $g(x) = -2x^2 + 8x - 3$

**1B.** $h(x) = 4x^2 - 8x + 1$

Notice in Example 1 that there seemed to be a pattern in the values for $f(x)$. This is due to the axis of symmetry of parabolas. The **axis of symmetry** is a line through the graph of a parabola that divides the graph into two congruent halves. Each side of the parabola is a reflection of the other side.



The axis of symmetry will intersect a parabola at only one point, called the **vertex**. The vertex of the graph at the right is $A(3, -3)$.

**ReviewVocabulary**

**Symmetry** When something is symmetrical, its opposite sides are mirror images of each other.

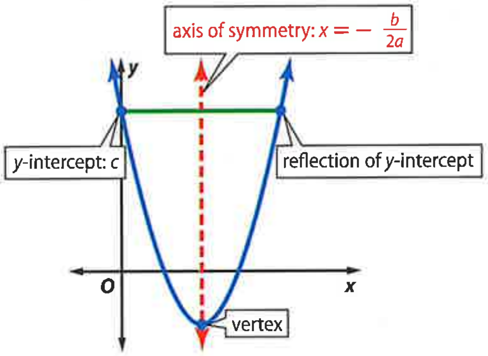
Notice that the $x$-coordinates of points $B$ and $C$ are both 4 units away from the $x$-coordinate of the vertex, and they have the same $y$-coordinate. This is due to the symmetrical nature of the graph.

## KeyConcept Graph of a Quadratic Function—Parabola

**Words** Consider the graph of $y = ax^2 + bx + c$, where $a \neq 0$.

- The $y$-intercept is $a(0)^2 + b(0) + c$ or $c$.
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$.
- The $x$-coordinate of the vertex is $-\frac{b}{2a}$.

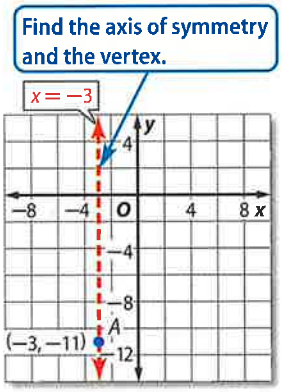
**Model**

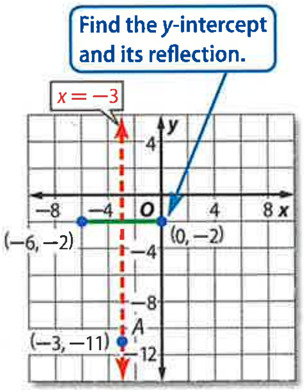


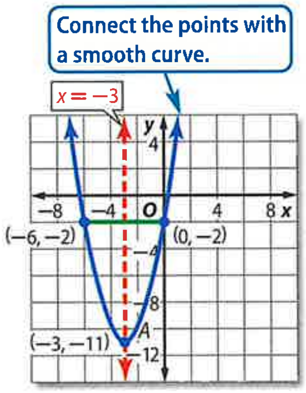
Now you can use the axis of symmetry to help plot points and graph a parabola. For $y = x^2 + 6x - 2$ below, the axis of symmetry is $x = -\frac{b}{2a} = -\frac{6}{2(1)}$ or $x = -3$.

**StudyTip**

**Plotting Reflections** The reflection of a point is its mirror image on the other side of the axis of symmetry.


Find the axis of symmetry and the vertex.


Find the $y$-intercept and its reflection.


Connect the points with a smooth curve.

**Study Tip**

**Quadratic Form** Make sure the function is in standard quadratic form, $y = ax^2 + bx + c$, before graphing.

## Example 2 Axis of Symmetry, y-intercept, and Vertex

**Consider $f(x) = x^2 + 4x - 3$.**

**a. Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex.**

The function is of the form $f(x) = ax^2 + bx + c$, so we can identify $a$, $b$, and $c$.

$$\begin{array}{l} f(x) = ax^2 + bx + c \\ \quad\quad\;\;\downarrow \quad\;\; \downarrow \quad \downarrow \\ f(x) = 1x^2 + 4x - 3 \quad \rightarrow \quad a = 1, b = 4, \text{ and } c = -3 \end{array}$$

The $y$-intercept is $c = -3$.

Use $a$ and $b$ to find the equation of the axis of symmetry.

$$\begin{aligned} x &= -\frac{b}{2a} && \text{Equation of the axis of symmetry} \\ &= -\frac{4}{2(1)} && a = 1 \text{ and } b = 4 \\ &= -2 && \text{Simplify.} \end{aligned}$$

The equation of the axis of symmetry is $x = -2$. Therefore, the $x$-coordinate of the vertex is $-2$.

**Study Tip**

**Fractions** When the $x$-coordinate of the vertex is a fraction, select the nearest integer for the next point to avoid using fractions and simplify the calculations.

**b. Make a table of values that includes the vertex.**

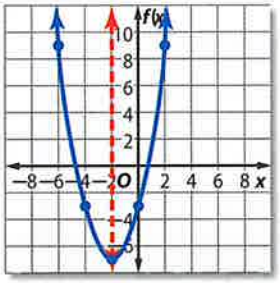
Select five specific points, with the vertex in the middle and two points on either side of the vertex, including the $y$-intercept and its reflection. Use symmetry to determine the $y$-values of the reflections.

| $x$ | $x^2 + 4x - 3$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|---|
| −6 | $(-6)^2 + 4(-6) - 3$ | 9 | (−6, 9) |
| −4 | $(-4)^2 + 4(-4) - 3$ | −3 | (−4, −3) |
| −2 | $(-2)^2 + 4(-2) - 3$ | −7 | (−2, −7) |
| 0 | $(0)^2 + 4(0) - 3$ | −3 | (0, −3) |
| 2 | $(2)^2 + 4(2) - 3$ | 9 | (2, 9) |

**c. Use this information to graph the function.**

Graph the points from the table and connect them with a smooth curve.

Draw the axis of symmetry, $x = -2$, as a dashed line. The graph should be symmetrical about this line.



### Guided Practice

**2.** Consider $f(x) = -5x^2 - 10x + 6$.

**A.** Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex.

**B.** Make a table of values that includes the vertex.

**C.** Use this information to graph the function.

**WatchOut!**

**Maxima and Minima** The terms *minimum point* and *minimum value* are not interchangeable. The minimum point on the graph of a quadratic function is the ordered pair that describes the location of the vertex. The minimum value of a function is the $y$-coordinate of the minimum point. It is the smallest value obtained when $f(x)$ is evaluated for all values of $x$.

## 2 Maximum and Minimum Values

The $y$-coordinate of the vertex of a quadratic function is the **maximum value** or the **minimum value** of the function. These values represent the greatest or lowest possible value the function can reach.

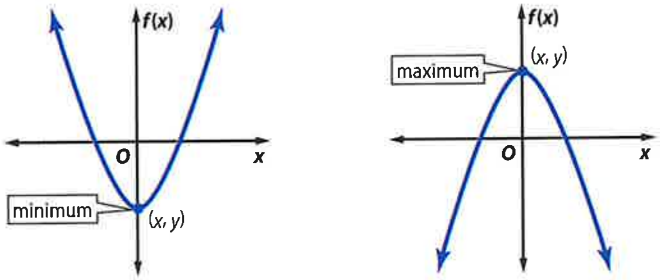
### KeyConcept Maximum and Minimum Value

**Words** The graph of $f(x) = ax^2 + bx + c$, where $a \neq 0$,

- opens up and has a minimum value when $a > 0$, and
- opens down and has a maximum value when $a < 0$.

**Model**

$a$ is positive. $a$ is negative.



The $y$-coordinate is the minimum value.

The $y$-coordinate is the maximum value.

### Example 3 Maximum or Minimum Values

**Consider $f(x) = -4x^2 + 12x + 18$.**

**a. Determine whether the function has a *maximum* or *minimum* value.**

For this function, $a = -4$, so the graph opens down and the function has a maximum value.

**b. State the maximum or minimum value of the function.**

The maximum value of the function is the $y$-coordinate of the vertex.

The $x$-coordinate of the vertex is $-\frac{12}{2(-4)}$ or 1.5.

Find the $y$-coordinate of the vertex by evaluating the function for $x = 1.5$.

$$
\begin{aligned}
f(x) &= -4x^2 + 12x + 18 && \text{Original function} \\
&= -4(1.5)^2 + 12(1.5) + 18 && x = 1.5 \\
&= -9 + 18 + 18 \text{ or } 27 && \text{The maximum value of the function is 27.}
\end{aligned}
$$

**c. State the domain and range of the function.**

The domain is all real numbers. The range is all real numbers less than or equal to the maximum value, or $\{f(x) \mid f(x) \leq 27\}$.

**StudyTip**

**Domain and Range** The domain of a quadratic function will always be all real numbers. The range will either be all real numbers less than or equal to the maximum or all real numbers greater than or equal to the minimum.

### GuidedPractice

**3.** Consider $f(x) = 4x^2 - 24x + 11$.

**A.** Determine whether the function has a maximum or minimum value.

**B.** State the maximum or minimum value of the function.

**C.** State the domain and range of the function.

**Real-WorldLink**

As of 2006, there were approximately 900,000 public charities in the United States.

**Source:** National Center for Charitable Statistics

## Real-World Example 4 Quadratic Equations in the Real World

**CHARITY** **Refer to the beginning of the lesson.**

**a. How much should Eddie charge in order to maximize charity income?**

| | | | | | |
|---|---|---|---|---|---|
| **Words** | Total | equals | fee | times | number of entrants. |
| **Variable** | Let $x =$ the number of price increases.<br>Let $P(x) =$ the total pool as a function of $x$. | | | | |
| **Equation** | $P(x)$ | $=$ | $20 + 5x$ | $\cdot$ | $(80 - 5x)$ |

Solve for the $x$-value of the vertex.

$P(x) = (20 + 5x) \cdot (80 - 5x)$

$= 20(80) + 20(-5x) + 5x(80) + 5x(-5x)$ Distribute.

$= 1600 - 100x + 400x - 25x^2$ Multiply.

$= 1600 + 300x - 25x^2$ Simplify.

$= -25x^2 + 300x + 1600$ $ax^2 + bx + c$ form

Use the formula for the axis of symmetry, $x = -\frac{b}{2a}$, to find the $x$-coordinate.

$x = -\frac{300}{2(-25)}$ or 6 $a = -25$ and $b = 300$

Eddie needs to have 6 price increases, so he should charge 20 + 6(5) or $50.

**b. What will be the maximum value of the pool?**

Find the maximum value of the quadratic function $P(x)$ by evaluating $P(6)$.

$P(x) = -25x^2 + 300x + 1600$ Total pool function

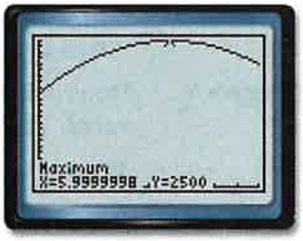
$P(6) = -25(6)^2 + 300(6) + 1600$ $x = 6$

$= -900 + 1800 + 1600$ or 2500 Simplify.

Thus, the maximum prize pool is $2500 after 6 price increases.

**CHECK** Graph the function on a graphing calculator and use the **CALC:maximum** function to confirm the solution.

Select a left bound of 0 and a right bound of 10. The calculator will display the coordinates of the maximum at the bottom of the screen.



[0, 10] scl: 1 by [0, 2500] scl: 100

The domain is $\{x \mid x \geq 0\}$ because there can be no negative increases in price. The range is $\{y \mid 0 \leq y \leq 2500\}$ because the prize pool cannot have a negative monetary value.

**StudyTip**

**CCSS Modeling** Use logic and the information from the problem to determine the domain and range that are reasonable in the situation.

### GuidedPractice

**4.** Suppose a different tournament that Eddie organizes has 120 players and the entry fee is $40. Each time he increases the fee by $5, he loses 10 players. Determine what the entry fee should be to maximize the value of the pool.

Peter Dazeley/Getty Images

## Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

**Examples 1–2** Complete parts a–c for each quadratic function.

**a.** Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex.

**b.** Make a table of values that includes the vertex.

**c.** Use this information to graph the function.

**1.** $f(x) = 3x^2$

**2.** $f(x) = -6x^2$

**3.** $f(x) = x^2 - 4x$

**4.** $f(x) = -x^2 - 3x + 4$

**5.** $f(x) = 4x^2 - 6x - 3$

**6.** $f(x) = 2x^2 - 8x + 5$

**Example 3** Determine whether each function has a *maximum* or *minimum* value, and find that value. Then state the domain and range of the function.

**7.** $f(x) = -x^2 + 6x - 1$

**8.** $f(x) = x^2 + 3x - 12$

**9.** $f(x) = 3x^2 + 8x + 5$

**10.** $f(x) = -4x^2 + 10x - 6$

**Example 4**

**11.** **BUSINESS** A store rents 1400 videos per week at \$2.25 per video. The owner estimates that they will rent 100 fewer videos for each \$0.25 increase in price. What price will maximize the income of the store?

## Practice and Problem Solving

Extra Practice is on page R4.

**Examples 1–2** Complete parts a–c for each quadratic function.

**a.** Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex.

**b.** Make a table of values that includes the vertex.

**c.** Use this information to graph the function.

**12.** $f(x) = 4x^2$

**13.** $f(x) = -2x^2$

**14.** $f(x) = x^2 - 5$

**15.** $f(x) = x^2 + 3$

**16.** $f(x) = 4x^2 - 3$

**17.** $f(x) = -3x^2 + 5$

**18.** $f(x) = x^2 - 6x + 8$

**⑲** $f(x) = x^2 - 3x - 10$

**20.** $f(x) = -x^2 + 4x - 6$

**21.** $f(x) = -2x^2 + 3x + 9$

**Example 3** Determine whether each function has a *maximum* or *minimum* value, and find that value. Then state the domain and range of the function.

**22.** $f(x) = 5x^2$

**23.** $f(x) = -x^2 - 12$

**24.** $f(x) = x^2 - 6x + 9$

**25.** $f(x) = -x^2 - 7x + 1$

**26.** $f(x) = 8x - 3x^2 + 2$

**27.** $f(x) = 5 - 4x - 2x^2$

**28.** $f(x) = 15 - 5x^2$

**29.** $f(x) = x^2 + 12x + 27$

**30.** $f(x) = -x^2 + 10x + 30$

**31.** $f(x) = 2x^2 - 16x - 42$

**Example 4**

**32.** CCSS **MODELING** A financial analyst determined that the cost, in thousands of dollars, of producing bicycle frames is $C = 0.000025f^2 - 0.04f + 40$, where $f$ is the number of frames produced.

**a.** Find the number of frames that minimizes cost.

**b.** What is the total cost for that number of frames?

**Complete parts a–c for each quadratic function.**

**a.** **Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex.**

**b.** **Make a table of values that includes the vertex.**

**c.** **Use this information to graph the function.**

**33.** $f(x) = 2x^2 - 6x - 9$

**34.** $f(x) = -3x^2 - 9x + 2$

**35.** $f(x) = -4x^2 + 5x$

**36.** $f(x) = 2x^2 + 11x$

**37.** $f(x) = 0.25x^2 + 3x + 4$

**38.** $f(x) = -0.75x^2 + 4x + 6$

**39.** $f(x) = \frac{3}{2}x^2 + 4x - \frac{5}{2}$

**40.** $f(x) = \frac{2}{3}x^2 - \frac{7}{3}x + 9$

**41.** **FINANCIAL LITERACY** A babysitting club sits for 50 different families. They would like to increase their current rate of $9.50 per hour. After surveying the families, the club finds that the number of families will decrease by about 2 for each $0.50 increase in the hourly rate.

**a.** Write a quadratic function that models this situation.

**b.** State the domain and range of this function as it applies to the situation.

**c.** What hourly rate will maximize the club's income? Is this reasonable?

**d.** What is the maximum income the club can expect to make?

**42.** **ACTIVITIES** Last year, 300 people attended the Franklin High School Drama Club's winter play. The ticket price was $8. The advisor estimates that 20 fewer people would attend for each $1 increase in ticket price.

**a.** What ticket price would give the greatest income for the Drama Club?

**b.** If the Drama Club raised its tickets to this price, how much income should it expect to bring in?

**CCSS TOOLS** **Use a calculator to find the maximum or minimum of each function. Round to the nearest hundredth if necessary.**

**43.** $f(x) = -9x^2 - 12x + 19$

**44.** $f(x) = 12x^2 - 21x + 8$

**45.** $f(x) = -8.3x^2 + 14x - 6$

**46.** $f(x) = 9.7x^2 - 13x - 9$

**47.** $f(x) = 28x - 15 - 18x^2$

**48.** $f(x) = -16 - 14x - 12x^2$

**Determine whether each function has a *maximum* or *minimum* value, and find that value. Then state the domain and range of the function.**

**49** $f(x) = -5x^2 + 4x - 8$

**50.** $f(x) = -4x^2 - 3x + 2$

**51.** $f(x) = -9 + 3x + 6x^2$

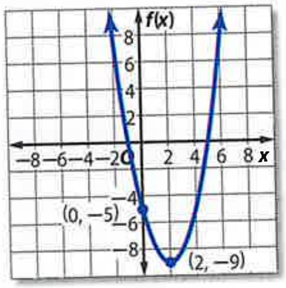
**52.** $f(x) = 2x - 5 - 4x^2$

**53.** $f(x) = \frac{2}{3}x^2 + 6x - 10$

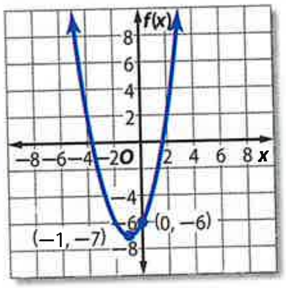
**54.** $f(x) = -\frac{3}{5}x^2 + 4x - 8$

**Determine the function represented by each graph.**

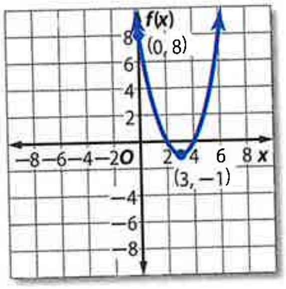
**55.**



**56.**



**57.**



**58.** **MULTIPLE REPRESENTATIONS** Consider $f(x) = x^2 - 4x + 8$ and $g(x) = 4x^2 - 4x + 8$.

**a.** **Tabular** Make a table of values for $f(x)$ and $g(x)$ if $-4 \le x \le 4$.

**b.** **Graphical** Graph $f(x)$ and $g(x)$.

**c.** **Verbal** Explain the difference in the shapes of the graphs of $f(x)$ and $g(x)$. What value was changed to cause this difference?

**d.** **Analytical** Predict the appearance of the graph of $h(x) = 0.25x^2 - 4x + 8$. Confirm your prediction by graphing all three functions if $-10 \le x \le 10$.

**59** **VENDING MACHINES** Omar owns a vending machine in a bowling alley. He currently sells 600 cans of soda per week at \$0.65 per can. He estimates that he will lose 100 customers for every \$0.05 increase in price and gain 100 customers for every \$0.05 decrease in price. (*Hint*: The charge *must* be a multiple of 5.)

**a.** Write and graph the related quadratic equation for a price increase.

**b.** If Omar lowers the price, what should it be to maximize his income?

**c.** What will be his income per week from the vending machine?

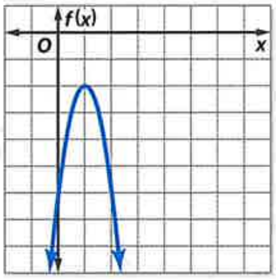
**60.** **BASEBALL** Lolita throws a baseball into the air and the height $h$ of the ball in feet at a given time $t$ in seconds after she releases the ball is given by the function $h(t) = -16t^2 + 30t + 5$.

**a.** State the domain and range for this situation.

**b.** Find the maximum height the ball will reach.

## H.O.T. Problems Use Higher-Order Thinking Skills

**61.** **CRITIQUE** Trent thinks that the function $f(x)$ graphed below, and the function $g(x)$ described next to it have the same maximum. Madison thinks that $g(x)$ has a greater maximum. Is either of them correct? Explain your reasoning.



$g(x)$ is a quadratic function with roots of 4 and 2 and a $y$-intercept of $-8$.

**62.** **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

*In a quadratic function, if two x-coordinates are equidistant from the axis of symmetry, then they will have the same y-coordinate.*

**63.** **CHALLENGE** The table at the right represents some points on the graph of a quadratic function.

**a.** Find the values of $a$, $b$, $c$, and $d$.

**b.** What is the $x$-coordinate of the vertex?

**c.** Does the function have a maximum or a minimum?

| $x$ | $y$ |
|---|---|
| $-20$ | $-377$ |
| $c$ | $-13$ |
| $-5$ | $-2$ |
| $-1$ | 22 |
| $d - 1$ | $a$ |
| 5 | $a - 24$ |
| 7 | $-b$ |
| 15 | $-202$ |
| $14 - c$ | $-377$ |

**64.** **OPEN ENDED** Give an example of a quadratic function with a

**a.** maximum of 8. **b.** minimum of $-4$. **c.** vertex of $(-2, 6)$.

**65.** **WRITING IN MATH** Describe how you determine whether a function is quadratic and if it has a maximum or minimum value.

## Standardized Test Practice

**66.** Which expression is equivalent to $\frac{8!}{5!}$?

A $\frac{8}{5}$

B $8 \cdot 7 \cdot 6$

C 3!

D $8 \cdot 7 \cdot 6 \cdot 5$

**67. SAT/ACT** The price of coffee beans is $d$ dollars for 6 ounces, and each ounce makes $c$ cups of coffee. In terms of $c$ and $d$, what is the cost of the coffee beans required to make 1 cup of coffee?

F $\frac{cd}{6}$

G $\frac{6c}{d}$

H $\frac{6}{cd}$

J $6cd$

K $\frac{d}{6c}$

**68. SHORT RESPONSE** Each side of the square base of a pyramid is 20 feet, and the pyramid's height is 90 feet. What is the volume of the pyramid?

**69.** Which ordered pair is the solution of the following system of equations?

$$3x - 5y = 11$$
$$3x - 8y = 5$$

A (2, 1)

B (7, −2)

C (7, 2)

D $\left(\frac{1}{3}, -2\right)$

## Spiral Review

**Find the inverse of each matrix, if it exists.** (Lesson 3-8)

**70.** $\begin{bmatrix} 3 & -4 \\ 2 & -1 \end{bmatrix}$

**71.** $\begin{bmatrix} -4 & -1 \\ 0 & 6 \end{bmatrix}$

**72.** $\begin{bmatrix} 2 & 8 \\ -3 & -5 \end{bmatrix}$

**Evaluate each determinant.** (Lesson 3-7)

**73.** $\begin{vmatrix} 6 & -3 \\ -1 & 8 \end{vmatrix}$

**74.** $\begin{vmatrix} -3 & -5 \\ -1 & -9 \end{vmatrix}$

**75.** $\begin{vmatrix} 8 & 6 \\ 4 & 3 \end{vmatrix}$

**76. MANUFACTURING** The Community Service Committee is making canvas tote bags and leather tote bags for a fundraiser. They will line both types of bags with canvas and use leather handles on both. For the canvas bags, they need 4 yards of canvas and 1 yard of leather. For the leather bags, they need 3 yards of leather and 2 yards of canvas. The committee leader purchased 56 yards of leather and 104 yards of canvas. (Lesson 3-3)

**a.** Let $c$ represent the number of canvas bags, and let $\ell$ represent the number of leather bags. Write a system of inequalities for the number of bags that can be made.

**b.** Draw the graph showing the feasible region.

**c.** List the coordinates of the vertices of the feasible region.

**d.** If the club plans to sell the canvas bags at a profit of \$20 each and the leather bags at a profit of \$35 each, write a function for the total profit on the bags.

**e.** How can the club make the maximum profit?

**f.** What is the maximum profit?

**State whether each function is a linear function. Write *yes* or *no*. Explain.** (Lesson 2-2)

**77.** $y = 4x^2 - 3x$

**78.** $y = -2x - 4$

**79.** $y = 4$

## Skills Review

**Evaluate each function for the given value.**

**80.** $f(x) = 3x^2 - 4x + 6, x = -2$

**81.** $f(x) = -2x^2 + 6x - 5, x = 4$

**82.** $f(x) = 6x^2 + 18, x = -5$

# LESSON 4-2

# Solving Quadratic Equations by Graphing

### Then

- You solved systems of equations by graphing.

### Now

- **1** Solve quadratic equations by graphing.

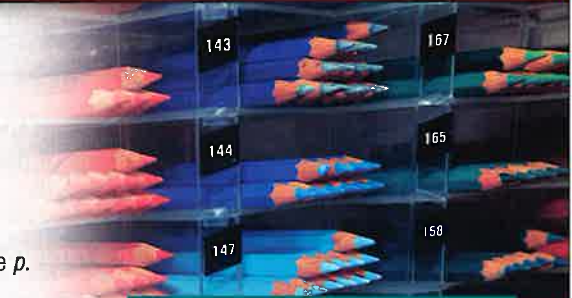
**2** Estimate solutions of quadratic equations by graphing.

### Why?

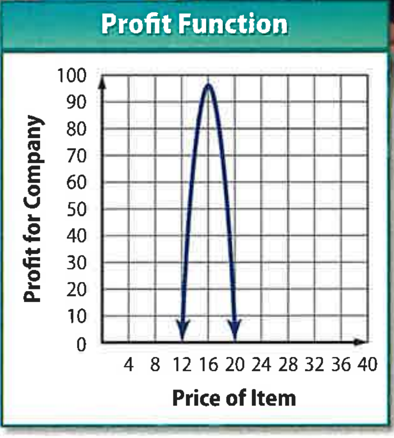
- Arielle works in the marketing department of a major retailer. Her job is to set prices for new products sold in the stores. Arielle determined that for a certain product, the function $f(p) = -6p^2 + 192p - 1440$ tells the profit $f(p)$ made at price $p$.

Arielle can determine the price range by finding the prices for which the profit is equal to \$0. This can be done by finding the solutions of the related quadratic equation $0 = -6p^2 + 192p - 1440$.

The graph of the function indicates that the profit is zero at 12 and 20, so the profitable price range of the item is between \$12 and \$20.



**Profit Function**



## NewVocabulary

quadratic equation
standard form
root
zero

## Common Core State Standards

**Content Standards**

**A.CED.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

**F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

**Mathematical Practices**

3 Construct viable arguments and critique the reasoning of others.

Jack Sullivan/Alamy

## 1 Solve Quadratic Equations

**Quadratic equations** are quadratic functions that are set equal to a value. The **standard form** of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ and $a$, $b$, and $c$ are integers.

The solutions of a quadratic equation are called the **roots** of the equation. One method for finding the roots of a quadratic equation is to find the **zeros** of the related quadratic function.

The zeros of the function are the $x$-intercepts of its graph.

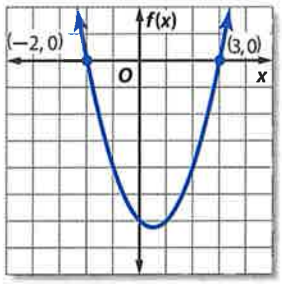
**Quadratic Function**

$$f(x) = x^2 - x - 6$$

$$f(-2) = (-2)^2 - (-2) - 6 \text{ or } 0$$
$$f(3) = 3^2 - 3 - 6 \text{ or } 0$$

**−2 and 3 are zeros of the function.**

**Graph of Function**



**The $x$-intercepts are −2 and 3.**

**Quadratic Equation**

$$x^2 - x - 6 = 0$$

$$(-2)^2 - (-2) - 6 \text{ or } 0$$
$$3^2 - 3 - 6 \text{ or } 0$$

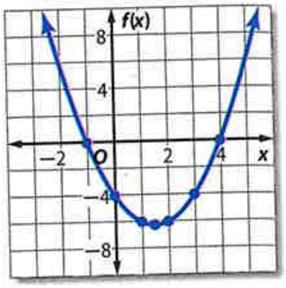
**−2 and 3 are roots of the equation.**

## Example 1 Two Real Solutions

**Solve $x^2 - 3x - 4 = 0$ by graphing.**

Graph the related function, $f(x) = x^2 - 3x - 4$. The equation of the axis of symmetry is $x = -\frac{-3}{2(1)}$ or 1.5. Make a table using $x$-values around 1.5. Then graph each point.

| $x$ | −1 | 0 | 1 | 1.5 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0 | −4 | −6 | −6.25 | −6 | −4 | 0 |



The zeros of the function are −1 and 4. Therefore, the solutions of the equation are −1 and 4 or $\{x \mid x = -1, 4\}$.

### Guided Practice

**Solve each equation by graphing.**

**1A.** $x^2 + 2x - 15 = 0$

**1B.** $x^2 - 8x = -12$

**StudyTip**

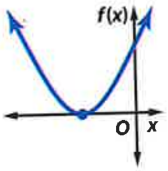
**Set-Builder Notation** In Lesson 1-5, you learned how to express the solution set of an inequality using set-builder notation. The solutions of an equation can also be expressed in set-builder notation. For example, the solutions of $x^2 = 25$ can be expressed as $\{x \mid x = -5, 5\}$.

The graph of the related function in Example 1 has two zeros; therefore, the quadratic equation has two real solutions. This is one of the three possible outcomes when solving a quadratic equation.

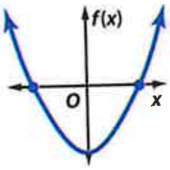
## KeyConcept Solutions of a Quadratic Equation

**Words** A quadratic equation can have one real solution, two real solutions, or no real solutions.

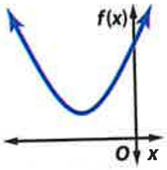
**Models**



one real solution



two real solutions



no real solution

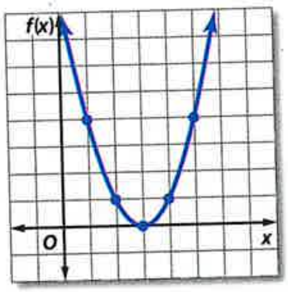
## Example 2 One Real Solution

**Solve $14 - x^2 = -6x + 23$ by graphing.**

| | |
|---|---|
| $14 - x^2 = -6x + 23$ | Original equation |
| $14 = x^2 - 6x + 23$ | Add $x^2$ to each side. |
| $0 = x^2 - 6x + 9$ | Subtract 14. |

Graph the related function $f(x) = x^2 - 6x + 9$.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x)$ | 4 | 1 | 0 | 1 | 4 |



The function has only one zero, 3. Therefore, the solution is 3 or $\{x \mid x = 3\}$.

### Guided Practice

**Solve each equation by graphing.**

**2A.** $x^2 + 5 = -8x - 11$

**2B.** $12 - x^2 = 48 - 12x$

**StudyTip**

**Optional Graph** $f(x) = -x^2 + 6x - 9$ could also have been graphed for this example. The graph would appear different, but it would have the same solution.

### Example 3 No Real Solution

**NUMBER THEORY Use a quadratic equation to find two real numbers with a sum of 15 and a product of 63.**

**Understand** Let $x$ represent one of the numbers. Then $15 - x$ is the other number.

**Plan**

$x(15 - x) = 63$ The product of the numbers is 63.

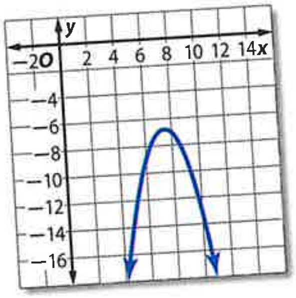
$15x - x^2 = 63$ Distributive Property

$-x^2 + 15x - 63 = 0$ Subtract 63.

**Solve** Graph the related function.

The graph has no $x$-intercepts. This means the original equation has no real solution. Thus, it is not possible for two real numbers to have a sum of 15 and a product of 63.

**Check** Try finding the product of several pairs of numbers with sums of 15. Is each product less than 63 as the graph suggests?



### Guided Practice

3. Find two real numbers with a sum of 6 and a product of −55, or show that no such numbers exist.

## 2 Estimate Solutions

Often exact roots cannot be found by graphing. You can estimate the solutions by stating the integers between which the roots are located.

**WatchOut!**

**Zeros** You will see in later chapters that many zeros can appear within small intervals.

When the value of the function is positive for one value and negative for a second value, then there is at least one zero between those two values.

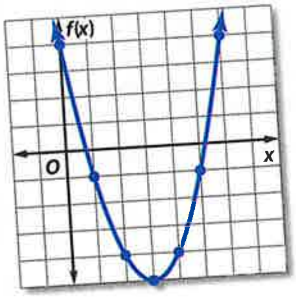
| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 12 | 3 | −6 | −2 | 4 | 8 | 14 |

zero zero

### Example 4 Estimate Roots

**Solve $x^2 - 6x + 4 = 0$ by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 4 | −1 | −4 | −5 | −4 | −1 | 4 |



The $x$-intercepts of the graph indicate that one solution is between 0 and 1, and the other solution is between 5 and 6.

### Guided Practice

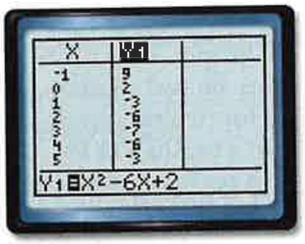
4. Solve $x^2 - x - 10 = 0$ by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

You can also use tables to solve quadratic equations. After entering the equation in your calculator, scroll through the table to locate the solutions.

## Example 5 Solve by Using a Table

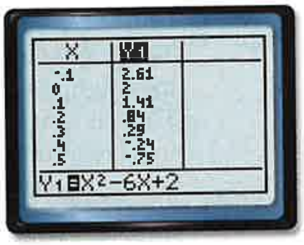
**Solve $x^2 - 6x + 2 = 0$.**

Enter **Y1** $= x^2 - 6x + 2$ in your graphing calculator. Use the **TABLE** window to find where the sign of **Y1** changes. Change **△Tbl** to 0.1 and look again for the sign change. Repeat the process with 0.01 and 0.001 to get a more accurate location of the zero.



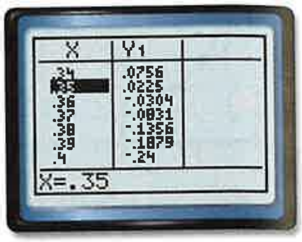
| X | Y1 |
|---|---|
| -1 | 9 |
| 0 | 2 |
| 1 | -3 |
| 2 | -6 |
| 3 | -7 |
| 4 | -6 |
| 5 | -3 |

Y1■X²-6X+2



| X | Y1 |
|---|---|
| -.1 | 2.61 |
| 0 | 2 |
| .1 | 1.41 |
| .2 | .84 |
| .3 | .29 |
| .4 | -.24 |
| .5 | -.75 |

Y1■X²-6X+2



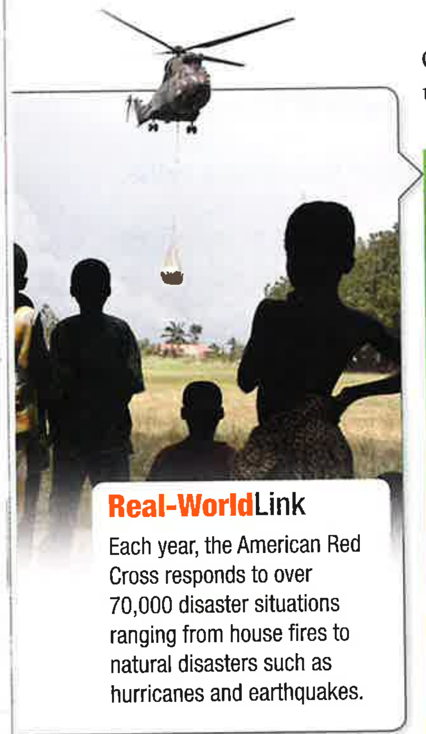
| X | Y1 |
|---|---|
| .34 | .0756 |
| .35 | .0225 |
| .36 | -.0304 |
| .37 | -.0831 |
| .38 | -.1356 |
| .39 | -.1879 |
| .4 | -.24 |

X=.35

One solution is approximately 0.354.

### GuidedPractice

**5.** Locate the second zero in the function above to the nearest thousandth.



**Real-WorldLink**

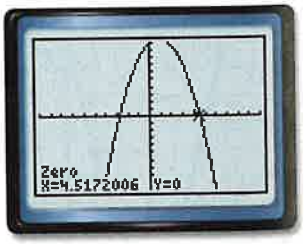
Each year, the American Red Cross responds to over 70,000 disaster situations ranging from house fires to natural disasters such as hurricanes and earthquakes.

Quadratic equations can be solved with a calculator as well. After entering the equation, use the **zero** feature in the **CALC** menu.

## Real-World Example 6 Solve by Using a Calculator

**RELIEF** **A package of supplies is tossed from a helicopter at an altitude of 200 feet. The package's height above the ground is modeled by $h(t) = -16t^2 + 28t + 200$, where $t$ is the time in seconds after it is tossed. How long will it take the package to reach the ground?**

We need to find $t$ when $h(t)$ is 0. Solve $0 = -16t^2 + 28t + 200$. Then graph the related function $f(t) = -16t^2 + 28t + 200$ on a graphing calculator.



[−10, 10] scl: 1 by [−200, 200] scl: 20

- Use the **zero** feature in the **CALC** menu to find the positive zero of the function, since time cannot be negative.
- Use the arrow keys to select a left bound and press ENTER.
- Locate a right bound and press ENTER twice.
- The positive zero of the function is about 4.52. The package would take about 4.52 seconds to reach the ground.

### GuidedPractice

**6.** How long would it take to reach the ground if the height was modeled by $h(t) = -16t^2 + 48t + 400$?

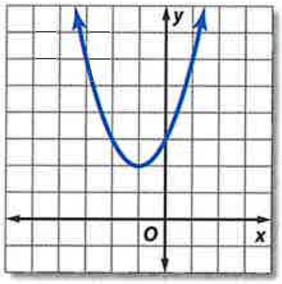
Emmanuel Duparcq/AFP/Getty Images

## Check Your Understanding

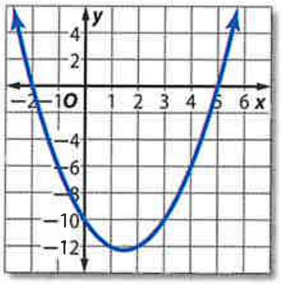
● = Step-by-Step Solutions begin on page R14.

**Example 1** Use the related graph of each equation to determine its solutions.

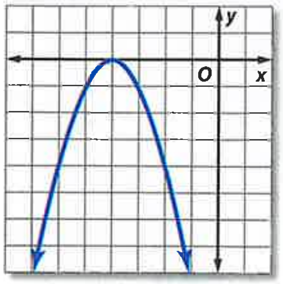
**1.** $x^2 + 2x + 3 = 0$



**2.** $x^2 - 3x - 10 = 0$



**3.** $-x^2 - 8x - 16 = 0$



**Examples 2-5** **CCSS PRECISION** Solve each equation. If exact roots cannot be found, state the consecutive integers between which the roots are located.

**4.** $x^2 + 8x = 0$

**⑤** $x^2 - 3x - 18 = 0$

**6.** $4x - x^2 + 8 = 0$

**7.** $-12 - 5x + 3x^2 = 0$

**8.** $x^2 - 6x + 4 = -8$

**9.** $9 - x^2 = 12$

**10.** $5x^2 + 10x - 4 = -6$

**11.** $x^2 - 20 = 2 + x$

**12.** **NUMBER THEORY** Use a quadratic equation to find two real numbers with a sum of 2 and a product of −24.

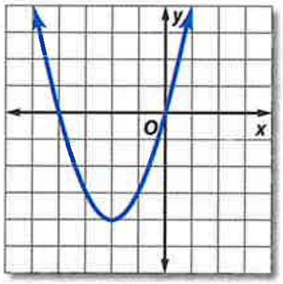
**Example 6** **13.** **PHYSICS** How long will it take an object to fall from the roof of a building 400 feet above the ground? Use the formula $h(x) = -16t^2 + h_0$, where $t$ is the time in seconds and the initial height $h_0$ is in feet.

## Practice and Problem Solving

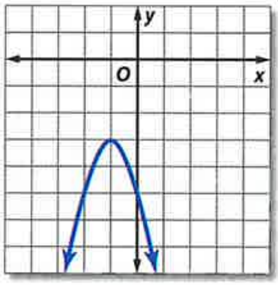
Extra Practice is on page R4.

**Example 1** Use the related graph of each equation to determine its solutions.

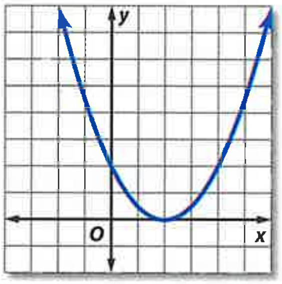
**14.** $x^2 + 4x = 0$



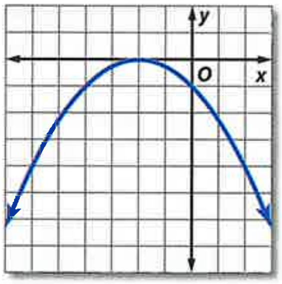
**15.** $-2x^2 - 4x - 5 = 0$



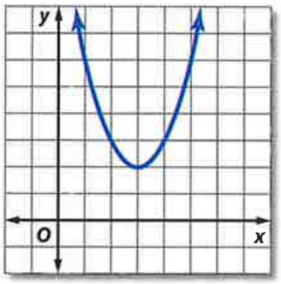
**16.** $0.5x^2 - 2x + 2 = 0$



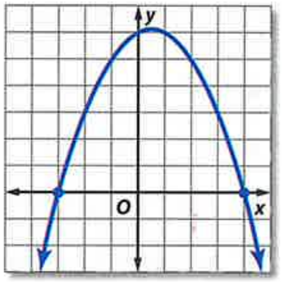
**17.** $-0.25x^2 - x - 1 = 0$



**18.** $x^2 - 6x + 11 = 0$



**19.** $-0.5x^2 + 0.5x + 6 = 0$



Emmanuel Duparcq/AFP/Getty Images

**Examples 2–4** **Solve each equation. If exact roots cannot be found, state the consecutive integers between which the roots are located.**

**20.** $x^2 = 5x$

**21.** $-2x^2 - 4x = 0$

**22.** $x^2 - 5x - 14 = 0$

**23.** $-x^2 + 2x + 24 = 0$

**24.** $x^2 - 18x = -81$

**25.** $2x^2 - 8x = -32$

**26.** $2x^2 - 3x - 15 = 4$

**27.** $-3x^2 - 7 + 2x = -11$

**28.** $-0.5x^2 + 3 = -5x - 2$

**29.** $-2x + 12 = x^2 + 16$

**Example 5** **Use the tables to determine the location of the zeros of each quadratic function.**

**30.**

| x | −7 | −6 | −5 | −4 | −3 | −2 | −1 | 0 |
|---|---|---|---|---|---|---|---|---|
| f(x) | −8 | −1 | 4 | 4 | −1 | −8 | −22 | −48 |

**31.**

| x | −2 | −1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|
| f(x) | 32 | 14 | 2 | −3 | −3 | 2 | 14 | 32 |

**32.**

| x | −6 | −3 | 0 | 3 | 6 | 9 | 12 | 15 |
|---|---|---|---|---|---|---|---|---|
| f(x) | −6 | −1 | 3 | 5 | 3 | −1 | −6 | −14 |

**Example 6** **NUMBER THEORY** **Use a quadratic equation to find two real numbers that satisfy each situation, or show that no such numbers exist.**

**33** Their sum is −15, and their product is −54.

**34.** Their sum is 4, and their product is −117.

**35.** Their sum is 12, and their product is −84.

**36.** Their sum is −13, and their product is 42.

**37.** Their sum is −8 and their product is −209.

**CCSS MODELING** **For Exercises 38–40, use the formula $h(t) = v_0t - 16t^2$, where $h(t)$ is the height of an object in feet, $v_0$ is the object's initial velocity in feet per second, and $t$ is the time in seconds.**

**38.** **BASEBALL** A baseball is hit with an initial velocity of 80 feet per second. Ignoring the height of the baseball player, how long does it take for the ball to hit the ground?

**39.** **CANNONS** A cannonball is shot with an initial velocity of 55 feet per second. Ignoring the height of the cannon, how long does it take for the cannonball to hit the ground?

**40.** **GOLF** A golf ball is hit with an initial velocity of 100 feet per second. How long will it take for it to hit the ground?

**Solve each equation. If exact roots cannot be found, state the consecutive integers between which the roots are located.**

**41.** $2x^2 + x = 15$

**42.** $-5x - 12 = -2x^2$

**43.** $4x^2 - 15 = -4x$

**44.** $-35 = -3x - 2x^2$

**45.** $-3x^2 + 11x + 9 = 1$

**46.** $13 - 4x^2 = -3x$

**47.** $-0.5x^2 + 18 = -6x + 33$

**48.** $0.5x^2 + 0.75 = 0.25x$

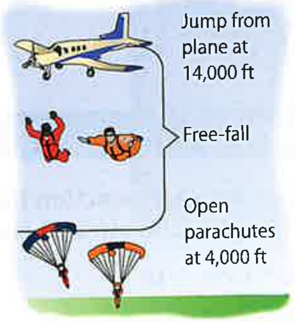
**49** **WATER BALLOONS** Tony wants to drop a water balloon so that it splashes on his brother. Use the formula $h(t) = -16t^2 + h_0$, where $t$ is the time in seconds and the initial height $h_0$ is in feet, to determine how far his brother should be from the target when Tony lets go of the balloon.

**50.** **WATER HOSES** A water hose can spray water at an initial velocity of 40 feet per second. Use the formula $h(t) = v_0t - 16t^2$, where $h(t)$ is the height of the water in feet, $v_0$ is the initial velocity in feet per second, and $t$ is the time in seconds.

**a.** How long will it take the water to hit the nozzle on the way down?

**b.** Assuming the nozzle is 5 feet up, what is the maximum height of the water?

**51.** **SKYDIVING** In 2003, John Fleming and Dan Rossi became the first two blind skydivers to be in free fall together. They jumped from an altitude of 14,000 feet and free fell to an altitude of 4000 feet before their parachutes opened. Ignoring air resistance and using the formula $h(t) = -16t^2 + h_0$, where $t$ is the time in seconds and the initial height $h_0$ is in feet, determine how long they were in free fall.



## H.O.T. Problems Use Higher-Order Thinking Skills

**52.** **CCSS CRITIQUE** Hakeem and Tanya were asked to find the location of the roots of the quadratic function represented by the table. Is either of them correct? Explain.

| $x$ | −4 | −2 | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 52 | 26 | 8 | −2 | −4 | 2 | 16 | 38 |

**Hakeem**

*The roots are between 4 and 6 because f(x) stops decreasing and begins to increase between x = 4 and x = 6.*

**Tanya**

*The roots are between −2 and 0 because x changes signs at that location.*

**53.** **CHALLENGE** Find the value of a positive integer $k$ such that $f(x) = x^2 - 2kx + 55$ has roots at $k + 3$ and $k - 3$.

**54.** **REASONING** If a quadratic function has a minimum at $(-6, -14)$ and a root at $x = -17$, what is the other root? Explain your reasoning.

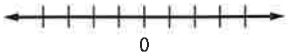
**55.** **OPEN ENDED** Write a quadratic function with a maximum at $(3, 125)$ and roots at $-2$ and 8.

**56.** **WRITING IN MATH** Explain how to solve a quadratic equation by graphing its related quadratic function.

## Standardized Test Practice

**57. SHORT RESPONSE** A bag contains five different colored marbles. The colors of the marbles are black, silver, red, green, and blue. A student randomly chooses a marble. Then, without replacing it, he chooses a second marble. What is the probability that the student chooses the red and then the green marble?

**58.** Which number would be closest to zero on the number line?



**A** $-0.6$ **C** $0.5$

**B** $\frac{2}{5}$ **D** $\frac{\sqrt{2}}{2}$

**59. SAT/ACT** A salesman's monthly gross pay consists of $3500 plus 20 percent of the dollar amount of his sales. If his gross pay for one month was $15,500, what was the dollar amount of his sales for that month?

**F** $12,000 **J** $70,000

**G** $16,000 **K** $77,500

**H** $60,000

**60.** Find the next term in the sequence below.

$$\frac{2x}{5}, \frac{3x}{5}, \frac{4x}{5}, \ldots$$

**A** $x$ **C** $\frac{x}{5}$

**B** $5x$ **D** $\frac{5x}{4}$

## Spiral Review

**Determine whether each function has a *maximum* or *minimum* value, and find that value. Then state the domain and range of the function.** (Lesson 4-1)

**61.** $f(x) = -4x^2 + 8x - 16$ **62.** $f(x) = 3x^2 + 12x - 18$ **63.** $f(x) = 4x + 13 - 2x^2$

**Determine whether each pair of matrices are inverses of each other.** (Lesson 3-8)

**64.** $\begin{bmatrix} 4 & -3 \\ -1 & -6 \end{bmatrix}$ and $\begin{bmatrix} \frac{3}{13} & -\frac{1}{18} \\ -\frac{1}{26} & -\frac{2}{13} \end{bmatrix}$ **65.** $\begin{bmatrix} 6 & -3 \\ 4 & 8 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{10} & \frac{1}{20} \\ -\frac{1}{15} & \frac{2}{15} \end{bmatrix}$ **66.** $\begin{bmatrix} 2 & 4 \\ -3 & -2 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{3}{8} & \frac{1}{4} \end{bmatrix}$

**67. SALES** Alex is in charge of stocking shirts for the concession stand at the high school football game. The numbers of shirts needed for a regular season game are listed in the matrix. Alex plans to double the number of shirts stocked for a playoff game. (Lesson 3-6)

| Size | small | medium | large |
|---|---|---|---|
| Child | 10 | 10 | 15 |
| Adult | 25 | 35 | 45 |

**a.** Write a matrix $A$ to represent the regular season stock.

**b.** What scalar can be used to determine a matrix $M$ to represent the new numbers? Find $M$.

**c.** What is $M - A$? What does this represent in this situation?

**Solve each system of equations.** (Lesson 3-1)

**68.** $4x - 7y = -9$
$5x + 2y = -22$

**69.** $8y - 2x = 38$
$5x - 3y = -27$

**70.** $3x + 8y = 24$
$-16y - 6x = 48$

**Solve each inequality.** (Lesson 1-5)

**71.** $3x - 6 \leq -14$ **72.** $6 - 4x \leq 2$ **73.** $-6x + 3 \geq 3x - 16$

## Skills Review

**Find the GCF of each set of numbers.**

**74.** 16, 48, 128 **75.** 15, 21, 49 **76.** 12, 28, 36

LESSON 4-3

# Solving Quadratic Equations by Factoring

## Then

- You found the greatest common factors of sets of numbers.

## Now

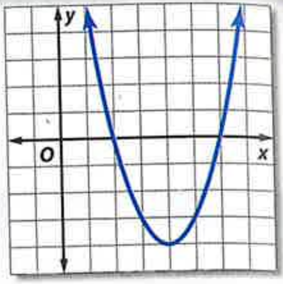
- **1** Write quadratic equations in standard form.

**2** Solve quadratic equations by factoring.

## Why?

- The **factored form** of a quadratic equation is $0 = a(x - p)(x - q)$. In the equation, $p$ and $q$ represent the $x$-intercepts of the graph of the equation.

The $x$-intercepts of the graph at the right are 2 and 6. In this lesson, you will learn how to change a quadratic equation in factored form into standard form and vice versa.



Related Graph
2 and 6 are $x$-intercepts.

Standard Form
$0 = x^2 - 8x + 12$

Factored Form
$0 = (x - 6)(x - 2)$
Factors

## NewVocabulary

factored form
FOIL method

## Common Core State Standards

**Content Standards**

**A.SSE.2** Use the structure of an expression to identify ways to rewrite it.

**F.IF.8.a** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

**Mathematical Practices**

2 Reason abstractly and quantitatively.

## 1 Standard Form

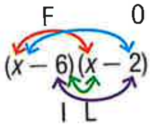
You can use the FOIL method to write a quadratic equation that is in factored form in standard form. The **FOIL method** uses the Distributive Property to multiply binomials.

### KeyConcept FOIL Method for Multiplying Binomials

**Words** To multiply two binomials, find the sum of the products of ***F*** the *First terms*, ***O*** the *Outer terms*, ***I*** the *Inner terms*, and ***L*** the *Last terms*.

**Examples**

| | Product of First Terms | | Product of Outer Terms | | Product of Inner Terms | | Product of Last Terms |
|---|---|---|---|---|---|---|---|
| | ↓ | | ↓ | | ↓ | | ↓ |
| $(x - 6)(x - 2) =$ | $(x)(x)$ | + | $(x)(-2)$ | + | $(-6)(x)$ | + | $(-6)(-2)$ |



$= x^2 - 2x - 6x + 12$ or $x^2 - 8x + 12$

### Example 1 Translate Sentences into Equations

**Write a quadratic equation in standard form with $-\frac{1}{3}$ and 6 as its roots.**

$(x - p)(x - q) = 0$ Write the pattern.

$\left[x - \left(-\frac{1}{3}\right)\right](x - 6) = 0$ Replace $p$ with $-\frac{1}{3}$ and $q$ with 6.

$\left(x + \frac{1}{3}\right)(x - 6) = 0$ Simplify.

$x^2 - \frac{17}{3}x - 2 = 0$ Multiply.

$3x^2 - 17x - 6 = 0$ Multiply each side by 3 so that $b$ and $c$ are integers.

**GuidedPractice**

**1.** Write a quadratic equation in standard form with $\frac{3}{4}$ and $-5$ as its roots.

## 2 Solve Equations by Factoring

Solving quadratic equations by factoring is an application of the Zero Product Property.

### KeyConcept Zero Product Property

**Words** For any real numbers $a$ and $b$, if $ab = 0$, then either $a = 0$, $b = 0$, or both $a$ and $b$ equal zero.

**Example** If $(x + 3)(x - 5) = 0$, then $x + 3 = 0$ or $x - 5 = 0$.

### Example 2 Factor the GCF

**Solve $16x^2 + 8x = 0$.**

$16x^2 + 8x = 0$ Original equation.

$8x(2x) + 8x(1) = 0$ Factor the GCF.

$8x(2x + 1) = 0$ Distributive Property

$8x = 0$ or $2x + 1 = 0$ Zero Product Property

$x = 0$ $\quad 2x = -1$ Solve both equations.

$x = -\frac{1}{2}$

**GuidedPractice** **Solve each equation.**

**2A.** $20x^2 + 15x = 0$

**2B.** $4y^2 + 16y = 0$

**2C.** $6a^5 + 18a^4 = 0$

**ReviewVocabulary**

**perfect square** a number with a positive square root that is a whole number

Trinomials and binomials that are perfect squares have special factoring rules. In order to use these rules, the first and last terms need to be perfect squares and the middle term needs to be twice the product of the square roots of the first and last terms.

### Example 3 Perfect Squares and Differences of Squares

**Solve each equation.**

**a.** $x^2 + 16x + 64 = 0$

$x^2 = (x)^2; 64 = (8)^2$ First and last terms are perfect squares.

$16x = 2(x)(8)$ Middle term equals $2ab$.

$x^2 + 16x + 64$ is a perfect square trinomial.

$x^2 + 16x + 64 = 0$ Original equation

$(x + 8)^2 = 0$ Factor using the pattern.

$x + 8 = 0$ Take the square root of each side.

$x = -8$ Solve.

**StudyTip**

**Square Roots** By inspection, notice that the square root of 64 is −8 and 8. Also, for $x^2 = 4$, the solutions would be −2 and 2.

**b.** $x^2 = 64$

$x^2 = 64$ Original equation

$x^2 - 64 = 0$ Subtract 64 from each side.

$x^2 - (8)^2 = 0$ Write in the form $a^2 - b^2$.

$(x + 8)(x - 8) = 0$ Factor the difference of squares.

$x + 8 = 0$ or $x - 8 = 0$ Zero Product Property

$x = -8$ $\quad x = 8$ Solve.

**GuidedPractice**

**3A.** $4x^2 - 12x + 9 = 0$

**3B.** $81x^2 - 9x = 0$

**3C.** $6a^2 - 3a = 0$

**StudyTip**

**Structure** If values for $m$ and $p$ exist, then the trinomial can always be factored.

A special pattern is used when factoring trinomials of the form $ax^2 + bx + c$. First, multiply the values of $a$ and $c$. Then, find two values, $m$ and $p$, such that their product equals $ac$ and their sum equals $b$.

Consider $6x^2 + 13x - 5$: $ac = 6(-5) = -30$.

| Factors of −30 | Sum | Factors of −30 | Sum |
|---|---|---|---|
| 1, −30 | −29 | −1, 30 | 29 |
| 2, −15 | −13 | **−2, 15** | **13** |
| 3, −10 | −7 | −3, 10 | 7 |
| 5, −6 | −1 | −5, 6 | 1 |

Now the middle term, $13x$, can be rewritten as $-2x + 15x$.

This polynomial can now be factored by grouping.

| | |
|---|---|
| $6x^2 + 13x - 5 = 6x^2 + mx + px - 5$ | Write the pattern. |
| $= 6x^2 - 2x + 15x - 5$ | $m = -2$ and $p = 15$ |
| $= (6x^2 - 2x) + (15x - 5)$ | Group terms. |
| $= 2x(3x - 1) + 5(3x - 1)$ | Factor the GCF. |
| $= (2x + 5)(3x - 1)$ | Distributive Property |

## Example 4 Factor Trinomials

**Solve each equation.**

**a.** $\mathbf{x^2 + 9x + 20 = 0}$

$ac = 20$ $a = 1, c = 20$

| Factors of 20 | Sum | Factors of 20 | Sum |
|---|---|---|---|
| 1, 20 | 21 | −1, −20 | −21 |
| 2, 10 | 12 | −2, −10 | −12 |
| **4, 5** | **9** | −4, −5 | −9 |

| | |
|---|---|
| $x^2 + 9x + 20 = 0$ | Original expression |
| $x^2 + mx + px + 20 = 0$ | Write the pattern. |
| $x^2 + 4x + 5x + 20 = 0$ | $m = 4$ and $p = 5$ |
| $(x^2 + 4x) + (5x + 20) = 0$ | Group terms with common factors. |
| $x(x + 4) + 5(x + 4) = 0$ | Factor the GCF from each grouping. |
| $(x + 5)(x + 4) = 0$ | Distributive Property |
| $x + 5 = 0$ or $x + 4 = 0$ | Zero Product Property |
| $x = -5$ $\quad$ $x = -4$ | Solve each equation. |

**StudyTip**

**Trinomials** It does not matter if the values of $m$ and $p$ are switched when grouping.

**b.** $\mathbf{6y^2 - 23y + 20 = 0}$

| | |
|---|---|
| $ac = 120$ | $a = 6, c = 20$ |
| $m = -8, p = -15$ | $-8(-15) = 120; -8 + (-15) = -23$ |
| $6y^2 - 23y + 20 = 0$ | Original equation |
| $6y^2 + my + py + 20 = 0$ | Write the pattern. |
| $6y^2 - 8y - 15y + 20 = 0$ | $m = -8$ and $p = -15$ |
| $(6y^2 - 8y) + (-15y + 20) = 0$ | Group terms with common factors. |
| $2y(3y - 4) - 5(3y - 4) = 0$ | Factor the GCF from each grouping. |
| $(2y - 5)(3y - 4) = 0$ | Distributive Property |
| $2y - 5 = 0$ or $3y - 4 = 0$ | Zero Product Property |
| $2y = 5$ $\quad$ $3y = 4$ | Solve both equations. |
| $y = \frac{5}{2}$ $\quad$ $y = \frac{4}{3}$ | |

## GuidedPractice

**4A.** $x^2 - 11x + 30 = 0$

**4B.** $x^2 - 4x - 21 = 0$

**4C.** $15x^2 - 8x + 1 = 0$

**4D.** $-12x^2 + 8x + 15 = 0$

### Real-WorldLink

Cuba's Osleidys Menendez broke the javelin world record in 2002 with a distance of 234 feet 8 inches.

**Source:** *New York Times*

Stu Forster/Getty Images Sport/Getty Images

## Real-World Example 5 Solve Equations by Factoring

PT

**TRACK AND FIELD** **The height of a javelin in feet is modeled by $h(t) = -16t^2 + 79t + 5$, where $t$ is the time in seconds after the javelin is thrown. How long is it in the air?**

To determine how long the javelin is in the air, we need to find when the height equals 0. We can do this by solving $-16t^2 + 79t + 5 = 0$.

| | |
|---|---|
| $-16t^2 + 79t + 5 = 0$ | Original equation |
| $m = 80; p = -1$ | $-16(5) = -80, 80 \cdot (-1) = -80, 80 + (-1) = 79$ |
| $-16t^2 + 80t - t + 5 = 0$ | Write the pattern. |
| $(-16t^2 + 80t) + (-t + 5) = 0$ | Group terms with common factors. |
| $16t(-t + 5) + 1(-t + 5) = 0$ | Factor GCF from each group. |
| $(16t + 1)(-t + 5) = 0$ | Distributive Property |
| $16t + 1 = 0$ or $-t + 5 = 0$ | Zero Product Property |
| $16t = -1$ $\quad -t = -5$ | Solve both equations. |
| $t = -\frac{1}{16}$ $\quad t = 5$ | Solve. |

**CHECK** We have two solutions.

- The first solution is negative and since time cannot be negative, this solution can be eliminated.
- The second solution of 5 seconds seems reasonable for the time a javelin spends in the air.
- The answer can be confirmed by substituting back into the original equation.

$$-16t^2 + 79t + 5 = 0$$

$$-16(5)^2 + 79(5) + 5 \stackrel{?}{=} 0$$

$$-400 + 395 + 5 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

The javelin is in the air for 5 seconds.

## GuidedPractice

**5.** **BUNGEE JUMPING** Juan recorded his brother bungee jumping from a height of 1100 feet. At the time the cord lifted his brother back up, he was 76 feet above the ground. If Juan started recording as soon as his brother fell, how much time elapsed when the cord snapped back? Use $f(t) = -16t^2 + c$, where $c$ is the height in feet.

## Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

**Example 1** Write a quadratic equation in standard form with the given root(s).

**1.** $-8, 5$
**2.** $\frac{3}{2}, \frac{1}{4}$
**3.** $-\frac{2}{3}, \frac{5}{2}$

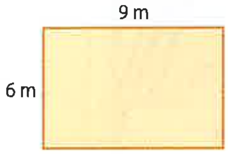
**Examples 2–4** Factor each polynomial.

**4.** $35x^2 - 15x$
**5.** $18x^2 - 3x + 24x - 4$
**6.** $x^2 - 12x + 32$
**7.** $x^2 - 4x - 21$
**8.** $2x^2 + 7x - 30$
**9.** $16x^2 - 16x + 3$

**Example 5** Solve each equation.

**10.** $x^2 - 36 = 0$
**11.** $12x^2 - 18x = 0$
**12.** $12x^2 - 2x - 2 = 0$
**13.** $x^2 - 9x = 0$
**14.** $x^2 - 3x - 28 = 0$
**15.** $2x^2 - 24x = -72$

**16.** CCSS **SENSE-MAKING** Tamika wants to double the area of her garden by increasing the length and width by the same amount. What will be the dimensions of her garden then?



## Practice and Problem Solving

Extra Practice is on page R4.

**Example 1** Write a quadratic equation in standard form with the given root(s).

**17.** $7$
**18.** $-5, \frac{1}{2}$
**19.** $\frac{1}{5}, 6$

**Examples 2–4** Factor each polynomial.

**20.** $40a^2 - 32a$
**21.** $51c^3 - 34c$
**22.** $32xy + 40bx - 12ay - 15ab$
**23.** $3x^2 - 12$
**24.** $15y^2 - 240$
**㉕** $48cg + 36cf - 4dg - 3df$
**26.** $x^2 + 13x + 40$
**27.** $x^2 - 9x - 22$
**28.** $3x^2 + 12x - 36$
**29.** $15x^2 + 7x - 2$
**30.** $4x^2 + 29x + 30$
**31.** $18x^2 + 15x - 12$
**32.** $8x^2z^2 - 4xz^2 - 12z^2$
**33.** $9x^2 - 25$
**34.** $18x^2y^2 - 24xy^2 + 36y^2$

**Example 5** Solve each equation.

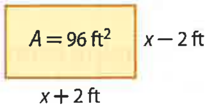
**35.** $15x^2 - 84x - 36 = 0$
**36.** $12x^2 + 13x - 14 = 0$
**37.** $12x^2 - 108x = 0$
**38.** $x^2 + 4x - 45 = 0$
**39.** $x^2 - 5x - 24 = 0$
**40.** $x^2 = 121$
**41.** $x^2 + 13 = 17$
**42.** $-3x^2 - 10x + 8 = 0$
**43.** $-8x^2 + 46x - 30 = 0$

**44.** **GEOMETRY** The hypotenuse of a right triangle is 1 centimeter longer than one side and 4 centimeters longer than three times the other side. Find the dimensions of the triangle.

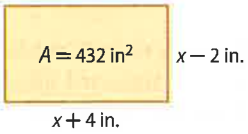
**45.** **NUMBER THEORY** Find two consecutive even integers with a product of 624.

**GEOMETRY** Find $x$ and the dimensions of each rectangle.

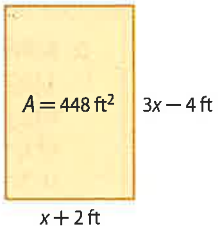
**46.**



**47.**



**48.**



**Solve each equation by factoring.**

**49.** $12x^2 - 4x = 5$

**50.** $5x^2 = 15x$

**51.** $16x^2 + 36 = -48x$

**52.** $75x^2 - 60x = -12$

**53.** $4x^2 - 144 = 0$

**54.** $-7x + 6 = 20x^2$

**55** **MOVIE THEATER** A company plans to build a large multiplex theater. The financial analyst told her manager that the profit function for their theater was $P(x) = -x^2 + 48x - 512$, where $x$ is the number of movie screens, and $P(x)$ is the profit earned in thousands of dollars. Determine the range of production of movie screens that will guarantee that the company will not lose money.

**Write a quadratic equation in standard form with the given root(s).**

**56.** $-\frac{4}{7}, \frac{3}{8}$

**57.** 3.4, 0.6

**58.** $\frac{2}{11}, \frac{5}{9}$

**Solve each equation by factoring.**

**59.** $10x^2 + 25x = 15$

**60.** $27x^2 + 5 = 48x$

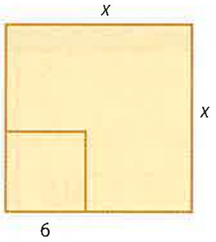
**61.** $x^2 + 0.25x = 1.25$

**62.** $48x^2 - 15 = -22x$

**63.** $3x^2 + 2x = 3.75$

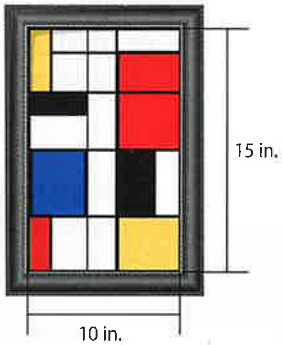
**64.** $-32x^2 + 56x = 12$

**65.** **DESIGN** A square is cut out of the figure at the right. Write an expression for the area of the figure that remains, and then factor the expression.



**66.** **CCSS PERSEVERANCE** After analyzing the market, a company that sells Web sites determined the profitability of their product was modeled by $P(x) = -16x^2 + 368x - 2035$, where $x$ is the price of each Web site and $P(x)$ is the company's profit. Determine the price range of the Web sites that will be profitable for the company.

**67.** **PAINTINGS** Enola wants to add a border to her painting, distributed evenly, that has the same area as the painting itself. What are the dimensions of the painting with the border included?



**68.** **MULTIPLE REPRESENTATIONS** In this problem, you will consider $a(x - p)(x - q) = 0$.

**a. Graphical** Graph the related function for $a = 1$, $p = 2$, and $q = -3$.

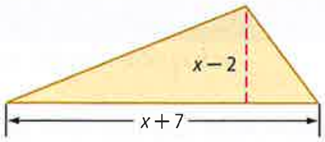
**b. Analytical** What are the solutions of the equation?

**c. Graphical** Graph the related functions for $a = 4, -3$, and $\frac{1}{2}$ on the same graph.

**d. Verbal** What are the similarities and differences between the graphs?

**e. Verbal** What conclusion can you make about the relationship between the factored form of a quadratic equation and its solutions?

**69.** **GEOMETRY** The area of the triangle is 26 square centimeters. Find the length of the base.



**70.** **SOCCER** When a ball is kicked in the air, its height in meters above the ground can be modeled by $h(t) = -4.9t^2 + 14.7t$ and the distance it travels can be modeled by $d(t) = 16t$, where $t$ is the time in seconds.

- **a.** How long is the ball in the air?
- **b.** How far does it travel before it hits the ground? (*Hint*: Ignore air resistance.)
- **c.** What is the maximum height of the ball?

**Factor each polynomial.**

**71.** $18a - 24ay + 48b - 64by$

**72.** $3x^2 + 2xy + 10y + 15x$

**73** $6a^2b^2 - 12ab^2 - 18b^3$

**74.** $12a^2 - 18ab + 30ab^3$

**75.** $32ax + 12bx - 48ay - 18by$

**76.** $30ac + 80bd + 40ad + 60bc$

**77.** $5ax^2 - 2by^2 - 5ay^2 + 2bx^2$

**78.** $12c^2x + 4d^2y - 3d^2x - 16c^2y$

## H.O.T. Problems Use Higher-Order Thinking Skills

**79.** **ERROR ANALYSIS** Gwen and Morgan are solving $-12x^2 + 5x + 2 = 0$. Is either of them correct? Explain your reasoning.

*Gwen*

$$-12x^2 + 5x + 2 = 0$$
$$-12x^2 + 8x - 3x + 2 = 0$$
$$4x(-3x + 2) - (3x + 2) = 0$$
$$(4x - 1)(3x + 2) = 0$$
$$x = \frac{1}{4} \text{ or } -\frac{2}{3}$$

*Morgan*

$$-12x^2 + 5x + 2 = 0$$
$$-12x^2 + 8x - 3x + 2 = 0$$
$$4x(-3x + 2) + (-3x + 2) = 0$$
$$(4x + 1)(-3x + 2) = 0$$
$$x = -\frac{1}{4} \text{ or } \frac{2}{3}$$

**80.** **CHALLENGE** Solve $3x^6 - 39x^4 + 108x^2 = 0$ by factoring.

**81.** **CHALLENGE** The rule for factoring a difference of cubes is shown below. Use this rule to factor $40x^5 - 135x^2y^3$.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

**82.** **OPEN ENDED** Choose two integers. Then write an equation in standard form with those roots. How would the equation change if the signs of the two roots were switched?

**83.** **CHALLENGE** For a quadratic equation of the form $(x - p)(x - q) = 0$, show that the axis of symmetry of the related quadratic function is located halfway between the $x$-intercepts $p$ and $q$.

**84.** **WRITE A QUESTION** A classmate is using the guess-and-check strategy to factor trinomials of the form $x^2 + bx + c$. Write a question to help him think of a way to use that strategy for $ax^2 + bx + c$.

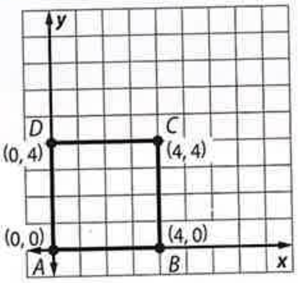
**85.** **CCSS ARGUMENTS** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

> *In a quadratic equation in standard form where a, b, and c are integers, if b is odd, then the quadratic cannot be a perfect square trinomial.*

**86.** **WRITING IN MATH** Explain how to factor a trinomial in standard form with $a > 1$.

## Standardized Test Practice

**87. SHORT RESPONSE** If $ABCD$ is transformed by $(x, y) \rightarrow (3x, 4y)$, determine the area of $A'B'C'D'$.



**88.** For $y = 2|6 - 3x| + 4$, which set describes $x$ when $y < 6$?

A $\left\{x \,\middle|\, \frac{5}{3} < x < \frac{7}{3}\right\}$

B $\left\{x \,\middle|\, x < \frac{5}{3} \text{ or } x > \frac{7}{3}\right\}$

C $\left\{x \,\middle|\, x < \frac{5}{3}\right\}$

D $\left\{x \,\middle|\, x > \frac{7}{3}\right\}$

**89. PROBABILITY** A 5-character password can contain the numbers 0 through 9 and 26 letters of the alphabet. None of the characters can be repeated. What is the probability that the password begins with a consonant?

F $\frac{21}{26}$

G $\frac{21}{35}$

H $\frac{21}{36}$

J $\frac{5}{36}$

**90. SAT/ACT** If $c = \frac{8a^3}{b}$, what happens to the value of $c$ when both $a$ and $b$ are doubled?

A $c$ is unchanged.

B $c$ is halved.

C $c$ is doubled.

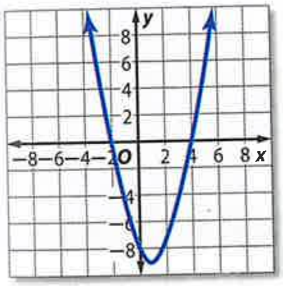
D $c$ is multiplied by 4.

E $c$ is multiplied by 8.

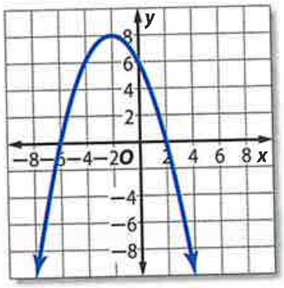
## Spiral Review

**Use the related graph of each equation to determine its solutions.** (Lesson 4-2)

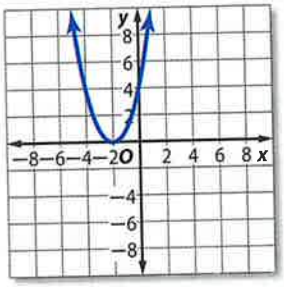
**91.** $x^2 - 2x - 8 = 0$



**92.** $x^2 + 4x = 12$



**93.** $x^2 + 4x + 4 = 0$



**Graph each function.** (Lesson 4-1)

**94.** $f(x) = x^2 - 6x + 2$

**95.** $f(x) = -2x^2 + 4x + 1$

**96.** $f(x) = (x - 3)(x + 4)$

**97. FUNDRAISING** Lawrence High School sold wrapping paper and boxed cards for their fundraising event. The school gets $1.00 for each roll of wrapping paper sold and $0.50 for each box of cards sold. (Lesson 3-6)

**a.** Write a matrix that represents the amounts sold for each class and a matrix that represents the amount of money the school earns for each item sold.

**b.** Write a matrix that shows how much each class earned.

**c.** Which class earned the most money?

**d.** What is the total amount of money the school made from the fundraiser?

**Total Amounts for Each Class**

| Class | Wrapping Paper | Cards |
|---|---|---|
| freshmen | 72 | 49 |
| sophomores | 68 | 63 |
| juniors | 90 | 56 |
| seniors | 86 | 62 |

## Skills Review

**Simplify.**

**98.** $\sqrt{5} \cdot \sqrt{15}$

**99.** $\sqrt{8} \cdot \sqrt{32}$

**100.** $2\sqrt{3} \cdot \sqrt{27}$

# LESSON 4-4 Complex Numbers

**Then**

- You simplified square roots.

**Now**

- **1** Perform operations with pure imaginary numbers.

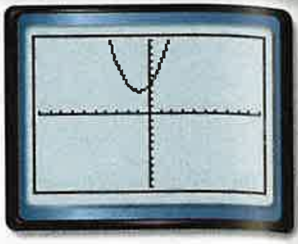
  **2** Perform operations with complex numbers.

**Why?**

- Consider the graph of $y = x^2 + 2x + 4$ at the right. Notice how this graph has no $x$-intercepts and therefore does not have any roots. Does this mean there are no solutions to $0 = x^2 + 2x + 4$?

  Use the **Solver** function located in the MATH menu of a graphing calculator. Enter the equation and select $x = 2$ as your guess to a solution.

  Press ALPHA ENTER and the calculator will attempt to solve the equation. The calculator indicates there is no solution with the error message. So there are no real solutions. However, there are imaginary solutions.



[−10, 10] scl: 1 by [−10, 10] scl: 1

```
ERR:NO SIGN CHNG
1:Quit
2:Goto
```

**NewVocabulary**

imaginary unit
pure imaginary number
complex number
complex conjugates

**Common Core State Standards**

**Content Standards**

**N.CN.1** Know there is a complex number $i$ such that $i^2 = -1$, and every complex number has the form $a + bi$ with $a$ and $b$ real.

**N.CN.2** Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

**Mathematical Practices**

6 Attend to precision.

## 1 Pure Imaginary Numbers

In your math studies so far, you have worked with real numbers. Equations like the one above led mathematicians to define imaginary numbers. The **imaginary unit $i$** is defined to be $i^2 = -1$. The number $i$ is the principal square root of $-1$; that is, $i = \sqrt{-1}$.

Numbers of the form $6i$, $-2i$, and $i\sqrt{3}$ are called **pure imaginary numbers**. Pure imaginary numbers are square roots of negative real numbers. For any positive real number $b$, $\sqrt{-b^2} = \sqrt{b^2} \cdot \sqrt{-1}$ or $bi$.

### Example 1 Square Roots of Negative Numbers

**Simplify.**

**a.** $\sqrt{-27}$

$$\begin{aligned}\sqrt{-27} &= \sqrt{-1 \cdot 3^2 \cdot 3}\\ &= \sqrt{-1} \cdot \sqrt{3^2} \cdot \sqrt{3}\\ &= i \cdot 3 \cdot \sqrt{3} \text{ or } 3i\sqrt{3}\end{aligned}$$

**b.** $\sqrt{-216}$

$$\begin{aligned}\sqrt{-216} &= \sqrt{-1 \cdot 6^2 \cdot 6}\\ &= \sqrt{-1} \cdot \sqrt{6^2} \cdot \sqrt{6}\\ &= i \cdot 6 \cdot \sqrt{6} \text{ or } 6i\sqrt{6}\end{aligned}$$

**GuidedPractice**

**1A.** $\sqrt{-18}$

**1B.** $\sqrt{-125}$

The Commutative and Associative Properties of Multiplication hold true for pure imaginary numbers. The first few powers of $i$ are shown below.

| | | | |
|---|---|---|---|
| $i^1 = i$ | $i^2 = -1$ | $i^3 = i^2 \cdot i$ or $-i$ | $i^4 = (i^2)^2$ or 1 |
| $i^5 = i^4 \cdot i$ or $i$ | $i^6 = i^4 \cdot i^2$ or $-1$ | $i^7 = i^4 \cdot i^3$ or $-i$ | $i^8 = (i^2)^4$ or 1 |

### Example 2 Products of Pure Imaginary Numbers

**Simplify.**

**a.** $-5i \cdot 3i$

$$\begin{aligned} -5i \cdot 3i &= -15i^2 && \text{Multiply.} \\ &= -15(-1) && i^2 = -1 \\ &= 15 && \text{Simplify.} \end{aligned}$$

**b.** $\sqrt{-6} \cdot \sqrt{-15}$

$$\begin{aligned} \sqrt{-6} \cdot \sqrt{-15} &= i\sqrt{6} \cdot i\sqrt{15} && i = \sqrt{-1} \\ &= i^2\sqrt{90} && \text{Multiply.} \\ &= -1 \cdot \sqrt{9} \cdot \sqrt{10} && \text{Simplify.} \\ &= -3\sqrt{10} && \text{Multiply.} \end{aligned}$$

**GuidedPractice**

**2A.** $3i \cdot 4i$

**2B.** $\sqrt{-20} \cdot \sqrt{-12}$

**2C.** $i^{31}$

You can solve some quadratic equations by using the **Square Root Property**. Similar to a difference of squares, the sum of squares can be factored over the complex numbers.

### Example 3 Equation with Pure Imaginary Solutions

**Solve $x^2 + 64 = 0$.**

**Method 1** Square Root Property

$$\begin{aligned} x^2 + 64 &= 0 \\ x^2 &= -64 \\ x &= \pm\sqrt{-64} \\ x &= \pm 8i \end{aligned}$$

**Method 2** Factoring

$$\begin{aligned} x^2 + 64 &= 0 \\ x^2 + 8^2 &= 0 \\ x^2 - (-8^2) &= 0 \\ (x + 8i)(x - 8i) &= 0 \\ (x + 8i) = 0 \text{ or } (x - 8i) &= 0 \\ x = -8i \qquad x &= 8i \end{aligned}$$

**GuidedPractice**

**Solve each equation.**

**3A.** $4x^2 + 100 = 0$

**3B.** $x^2 + 4 = 0$

**Real-WorldCareer**

**Electrical Engineer**
Electrical engineers design, develop, test, and supervise the making of electrical equipment such as digital music players, electric motors, lighting, and radar and navigation systems. A bachelor's degree in engineering is required for almost all entry-level engineering jobs.

FABRICE COFFRINI/AFP/Getty Images

## 2 Operations with Complex Numbers

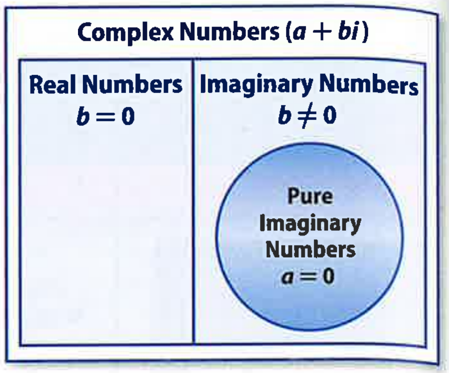
Consider $2 + 3i$. Since 2 is a real number and $3i$ is a pure imaginary number, the terms are not like terms and cannot be combined. This type of expression is called a **complex number**.

### KeyConcept Complex Numbers

| | |
|---|---|
| Words | A complex number is any number that can be written in the form $a + bi$, where $a$ and $b$ are real numbers and $i$ is the imaginary unit. $a$ is called the real part, and $b$ is called the imaginary part. |
| Examples | $5 + 2i$ $\qquad$ $1 - 3i = 1 + (-3)i$ |

The Venn diagram shows the set of complex numbers.

- If $b = 0$, the complex number is a real number.
- If $b \neq 0$, the complex number is imaginary.
- If $a = 0$, the complex number is a pure imaginary number.



Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is, $a + bi = c + di$ if and only if $a = c$ and $b = d$.

**StudyTip**

**Complex Numbers** Whereas all real numbers are also complex, the term *complex number* usually refers to a number that is not real.

### Example 4 Equate Complex Numbers

**Find the values of $x$ and $y$ that make $3x - 5 + (y - 3)i = 7 + 6i$ true.**

Set the real parts equal to each other and the imaginary parts equal to each other.

$3x - 5 = 7$ Real parts

$3x = 12$ Add 5 to each side.

$x = 4$ Divide each side by 3.

$y - 3 = 6$ Imaginary parts

$y = 9$ Add 3 to each side.

**GuidedPractice**

**4.** Find the values of $x$ and $y$ that make $5x + 1 + (3 + 2y)i = 2x - 2 + (y - 6)i$ true.

The Commutative, Associative, and Distributive Properties of Multiplication and Addition hold true for complex numbers. To add or subtract complex numbers, combine like terms. That is, combine the real parts, and combine the imaginary parts.

### Example 5 Add and Subtract Complex Numbers

**Simplify.**

**a.** $(5 - 7i) + (2 + 4i)$

$(5 - 7i) + (2 + 4i) = (5 + 2) + (-7 + 4)i$ Commutative and Associative Properties

$= 7 - 3i$ Simplify.

**b.** $(4 - 8i) - (3 - 6i)$

$(4 - 8i) - (3 - 6i) = (4 - 3) + [-8 - (-6)]i$ Commutative and Associative Properties

$= 1 - 2i$ Simplify.

**GuidedPractice**

**5A.** $(-2 + 5i) + (1 - 7i)$

**5B.** $(4 + 6i) - (-1 + 2i)$

**StudyTip**

**Reading Math** Electrical engineers use $j$ as the imaginary unit to avoid confusion with the $i$ for current.

Complex numbers are used with electricity. In these problems, $j$ usually represents the imaginary unit. In a circuit with alternating current, the voltage, current, and impedance, or hindrance to current, can be represented by complex numbers. To multiply these numbers, use the FOIL method.

## Real-WorldLink

An example of a series circuit is a string of holiday lights. The number of bulbs on a circuit affects the strength of the current, which in turn affects the brightness of the lights.

**Source:** *Popular Science*

### Real-World Example 6 Multiply Complex Numbers

**ELECTRICITY In an AC circuit, the voltage $V$, current $C$, and impedance $I$ are related by the formula $V = C \cdot I$. Find the voltage in a circuit with current $2 + 4j$ amps and impedance $9 - 3j$ ohms.**

| | |
|---|---|
| $V = C \cdot I$ | Electricity formula |
| $= (2 + 4j) \cdot (9 - 3j)$ | $C = 2 + 4j$ and $I = 9 - 3j$ |
| $= 2(9) + 2(-3j) + 4j(9) + 4j(-3j)$ | FOIL Method |
| $= 18 - 6j + 36j - 12j^2$ | Multiply. |
| $= 18 + 30j - 12(-1)$ | $j^2 = -1$ |
| $= 30 + 30j$ | Add. |

The voltage is $30 + 30j$ volts.

### GuidedPractice

**6.** Find the voltage in a circuit with current $2 - 4j$ amps and impedance $3 - 2j$ ohms.

Two complex numbers of the form $a + bi$ and $a - bi$ are called **complex conjugates**. The product of complex conjugates is always a real number. You can use this fact to simplify the quotient of two complex numbers.

### Example 7 Divide Complex Numbers

**Simplify.**

**a.** $\dfrac{2i}{3 + 6i}$

| | |
|---|---|
| $\dfrac{2i}{3 + 6i} = \dfrac{2i}{3 + 6i} \cdot \dfrac{3 - 6i}{3 - 6i}$ | $3 + 6i$ and $3 - 6i$ are complex conjugates. |
| $= \dfrac{6i - 12i^2}{9 - 36i^2}$ | Multiply. |
| $= \dfrac{6i - 12(-1)}{9 - 36(-1)}$ | $i^2 = -1$ |
| $= \dfrac{6i + 12}{45}$ | Simplify. |
| $= \dfrac{4}{15} + \dfrac{2}{15}i$ | $a + bi$ form |

**b.** $\dfrac{4 + i}{5i}$

| | |
|---|---|
| $\dfrac{4 + i}{5i} = \dfrac{4 + i}{5i} \cdot \dfrac{i}{i}$ | Multiply by $\frac{i}{i}$. |
| $= \dfrac{4i + i^2}{5i^2}$ | Multiply. |
| $= \dfrac{4i - 1}{-5}$ | $i^2 = -1$ |
| $= \dfrac{1}{5} - \dfrac{4}{5}i$ | $a + bi$ form |

## StudyTip

**Technology** Operations with complex numbers can be preformed with a TI-83/84 Plus graphing calculator. Use the [2nd] [*i*] function to enter the expression. Then press [MATH] [ENTER] [ENTER] to view the answer.

```
2i/(3+6i)
.2666666667+.13…
Ans▶Frac
      4/15+2/15i
```

### GuidedPractice

**7A.** $\dfrac{-2i}{3 + 5i}$

**7B.** $\dfrac{2 + i}{1 - i}$

Design Pics Inc./Alamy

## Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

**Examples 1–2** **Simplify.**

**1.** $\sqrt{-81}$

**2.** $\sqrt{-32}$

**3.** $(4i)(-3i)$

**4.** $3\sqrt{-24} \cdot 2\sqrt{-18}$

**5.** $i^{40}$

**6.** $i^{63}$

**Example 3** **Solve each equation.**

**7.** $4x^2 + 32 = 0$

**8.** $x^2 + 1 = 0$

**Example 4** **Find the values of $a$ and $b$ that make each equation true.**

**9.** $3a + (4b + 2)i = 9 - 6i$

**10.** $4b - 5 + (-a - 3)i = 7 - 8i$

**Examples 5 and 7** **Simplify.**

**11.** $(-1 + 5i) + (-2 - 3i)$

**12.** $(7 + 4i) - (1 + 2i)$

**13.** $(6 - 8i)(9 + 2i)$

**14.** $(3 + 2i)(-2 + 4i)$

**15.** $\frac{3 - i}{4 + 2i}$

**16.** $\frac{2 + i}{5 + 6i}$

**Example 6**

**17.** **ELECTRICITY** The current in one part of a series circuit is $5 - 3j$ amps. The current in another part of the circuit is $7 + 9j$ amps. Add these complex numbers to find the total current in the circuit.

## Practice and Problem Solving

Extra Practice is on page R4.

**Examples 1–2** **STRUCTURE** **Simplify.**

**18.** $\sqrt{-121}$

**19.** $\sqrt{-169}$

**20.** $\sqrt{-100}$

**21.** $\sqrt{-81}$

**22.** $(-3i)(-7i)(2i)$

**23.** $4i(-6i)^2$

**24.** $i^{11}$

**25.** $i^{25}$

**26.** $(10 - 7i) + (6 + 9i)$

**27.** $(-3 + i) + (-4 - i)$

**28.** $(12 + 5i) - (9 - 2i)$

**29.** $(11 - 8i) - (2 - 8i)$

**30.** $(1 + 2i)(1 - 2i)$

**31.** $(3 + 5i)(5 - 3i)$

**32.** $(4 - i)(6 - 6i)$

**33.** $\frac{2i}{1 + i}$

**34.** $\frac{5}{2 + 4i}$

**35.** $\frac{5 + i}{3i}$

**Example 3** **Solve each equation.**

**36.** $4x^2 + 4 = 0$

**㊲** $3x^2 + 48 = 0$

**38.** $2x^2 + 50 = 0$

**39.** $2x^2 + 10 = 0$

**40.** $6x^2 + 108 = 0$

**41.** $8x^2 + 128 = 0$

**Example 4** **Find the values of $x$ and $y$ that make each equation true.**

**42.** $9 + 12i = 3x + 4yi$

**43.** $x + 1 + 2yi = 3 - 6i$

**44.** $2x + 7 + (3 - y)i = -4 + 6i$

**45.** $5 + y + (3x - 7)i = 9 - 3i$

**46.** $a + 3b + (3a - b)i = 6 + 6i$

**47.** $(2a - 4b)i + a + 5b = 15 + 58i$

**Examples 5 and 7**

**Simplify.**

**48.** $\sqrt{-10} \cdot \sqrt{-24}$

**49.** $4i\left(\frac{1}{2}i\right)^2(-2i)^2$

**50.** $i^{41}$

**51.** $(4 - 6i) + (4 + 6i)$

**52.** $(8 - 5i) - (7 + i)$

**53.** $(-6 - i)(3 - 3i)$

**54.** $\frac{(5 + i)^2}{3 - i}$

**55.** $\frac{6 - i}{2 - 3i}$

**56.** $(-4 + 6i)(2 - i)(3 + 7i)$

**57.** $(1 + i)(2 + 3i)(4 - 3i)$

**58.** $\frac{4 - i\sqrt{2}}{4 + i\sqrt{2}}$

**59.** $\frac{2 - i\sqrt{3}}{2 + i\sqrt{3}}$

**Example 6**

**60.** **ELECTRICITY** The impedance in one part of a series circuit is $7 + 8j$ ohms, and the impedance in another part of the circuit is $13 - 4j$ ohms. Add these complex numbers to find the total impedance in the circuit.

**ELECTRICITY** **Use the formula $V = C \cdot I$.**

**61** The current in a circuit is $3 + 6j$ amps, and the impedance is $5 - j$ ohms. What is the voltage?

**62.** The voltage in a circuit is $20 - 12j$ volts, and the impedance is $6 - 4j$ ohms. What is the current?

**63.** Find the sum of $ix^2 - (4 + 5i)x + 7$ and $3x^2 + (2 + 6i)x - 8i$.

**64.** Simplify $[(2 + i)x^2 - ix + 5 + i] - [(-3 + 4i)x^2 + (5 - 5i)x - 6]$.

**65.** **MULTIPLE REPRESENTATIONS** In this problem, you will explore quadratic equations that have complex roots.

- **a. Algebraic** Write a quadratic equation in standard form with $3i$ and $-3i$ as its roots.
- **b. Graphical** Graph the quadratic equation found in part **a** by graphing its related function.
- **c. Algebraic** Write a quadratic equation in standard form with $2 + i$ and $2 - i$ as its roots.
- **d. Graphical** Graph the quadratic equation found in part **c** by graphing its related function.
- **e. Analytical** How do you know when a quadratic equation will have only complex solutions?

## H.O.T. Problems Use Higher-Order Thinking Skills

**66.** CCSS **CRITIQUE** Joe and Sue are simplifying $(2i)(3i)(4i)$. Is either of them correct? Explain your reasoning.

| Joe | Sue |
|---|---|
| $24i^3 = -24$ | $24i^3 = -24i$ |

**67.** **CHALLENGE** Simplify $(1 + 2i)^3$.

**68.** **REASONING** Determine whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

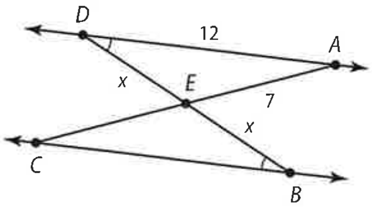
*Every complex number has both a real part and an imaginary part.*

**69.** **OPEN ENDED** Write two complex numbers with a product of 20.

**70.** **WRITING IN MATH** Explain how complex numbers are related to quadratic equations.

## Standardized Test Practice

**71. EXTENDED RESPONSE** Refer to the figure to answer the following.



**a.** Name two congruent triangles with vertices in correct order.

**b.** Explain why the triangles are congruent.

**c.** What is the length of $\overline{EC}$? Explain your procedure.

**72.** $(3 + 6)^2 =$

A $2 \times 3 + 2 \times 6$

B $9^2$

C $3^2 + 6^2$

D $3^2 \times 6^2$

**73. SAT/ACT** A store charges $49 for a pair of pants. This price is 40% more than the amount it costs the store to buy the pants. After a sale, any employee is allowed to purchase any remaining pairs of pants at 30% off the store's cost. How much would it cost an employee to purchase the pants after the sale?

F $10.50

G $12.50

H $13.72

J $24.50

K $35.00

**74.** What are the values of $x$ and $y$ when $(5 + 4i) - (x + yi) = (-1 - 3i)$?

A $x = 6, y = 7$

B $x = 4, y = i$

C $x = 6, y = i$

D $x = 4, y = 7$

## Spiral Review

**Solve each equation by factoring.** (Lesson 4-3)

**75.** $2x^2 + 7x = 15$

**76.** $4x^2 - 12 = 22x$

**77.** $6x^2 = 5x + 4$

**NUMBER THEORY** **Use a quadratic equation to find two real numbers that satisfy each situation, or show that no such numbers exist.** (Lesson 4-2)

**78.** Their sum is −3, and their product is −40.

**79.** Their sum is 19, and their product is 48.

**80.** Their sum is −15, and their product is 56.

**81.** Their sum is −21, and their product is 108.

**82. RECREATION** Refer to the table. (Lesson 3-5)

**a.** Write a matrix that represents the cost of admission for residents and a matrix that represents the cost of admission for nonresidents.

**b.** Write the matrix that represents the additional cost for nonresidents.

**c.** Write a matrix that represents the difference in cost if a child or adult goes after 6:00 P.M. instead of before 6:00 P.M.

| Daily Admission Fees | | |
|---|---|---|
| **Residents Time of day** | **Child** | **Adult** |
| Before 6:00 p.m. | $3.00 | $4.50 |
| After 6:00 p.m. | $2.00 | $3.50 |
| **Nonresidents Time of day** | **Child** | **Adult** |
| Before 6:00 p.m. | $4.50 | $6.75 |
| After 6:00 p.m. | $3.00 | $5.25 |

**83. PART-TIME JOBS** Terrell makes $10 per hour cutting grass and $12 per hour for raking leaves. He cannot work more than 15 hours per week. Graph two inequalities that Terrell can use to determine how many hours he needs to work at each job if he wants to earn at least $120 per week. (Lesson 3-2)

## Skills Review

**Determine whether each trinomial is a perfect square trinomial. Write *yes* or *no*.**

**84.** $x^2 + 16x + 64$

**85.** $x^2 - 12x + 36$

**86.** $x^2 + 8x - 16$

**87.** $x^2 - 14x - 49$

**88.** $x^2 + x + 0.25$

**89.** $x^2 + 5x + 6.25$

LESSON 4-5

# Completing the Square

## Then

- You factored perfect square trinomials.

## Now

- **1** Solve quadratic equations by using the Square Root Property.

  **2** Solve quadratic equations by completing the square.

## Why?

- When going through a school zone, drivers must slow to a speed of 20 miles per hour. Once they are out of the school zone, the drivers can increase their speed.

  Suppose Arturo is leaving school to go home for lunch, and he lives 5000 feet from the school zone. If Arturo accelerates at a constant rate of 8 feet per second squared, the equation $t^2 + 2t + 8 = 16$ represents the time $t$ it takes him to reach home.

  To solve this equation, you can use the Square Root Property.

**New Vocabulary**
completing the square

**Common Core State Standards**

**Content Standards**

**N.CN.7** Solve quadratic equations with real coefficients that have complex solutions.

**F.IF.8.a** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

**Mathematical Practices**

7 Look for and make use of structure.

## 1 Square Root Property

You have solved equations like $x^2 - 25 = 0$ by factoring. You have also used the Square Root Property to solve such equations. This method can be useful with equations like the one above that describes the car's speed.

PT

### Example 1 Equation with Rational Roots

**Solve $x^2 + 6x + 9 = 36$ by using the Square Root Property.**

$x^2 + 6x + 9 = 36$ — Original equation

$(x + 3)^2 = 36$ — Factor the perfect square trinomial.

$x + 3 = \pm\sqrt{36}$ — Square Root Property

$x + 3 = \pm 6$ — $\sqrt{36} = 6$

$x = -3 \pm 6$ — Subtract 3 from each side.

$x = -3 + 6$ or $x = -3 - 6$ — Write as two equations.

$= 3$ $\quad = -9$ — Simplify.

The solution set is $\{-9, 3\}$ or $\{x \mid x = -9, 3\}$.

**CHECK** Substitute both values into the original equation.

| | | |
|---|---|---|
| $x^2 + 6x + 9 = 36$ | Original equation | $x^2 + 6x + 9 = 36$ |
| $3^2 + 6(3) + 9 \stackrel{?}{=} 36$ | Substitute 3 and −9. | $(-9)^2 + 6(-9) + 9 \stackrel{?}{=} 36$ |
| $9 + 18 + 9 \stackrel{?}{=} 36$ | Simplify. | $81 - 54 + 9 \stackrel{?}{=} 36$ |
| $36 = 36$ ✓ | Both solutions are correct. | $36 = 36$ ✓ |

**Guided Practice**

**Solve each equation by using the Square Root Property.**

**1A.** $x^2 - 12x + 36 = 25$

**1B.** $x^2 - 16x + 64 = 49$

Arthur S. Aubry/Photodisc/Getty Images

Roots that are irrational numbers may be written as exact answers in radical form or as *approximate* answers in decimal form when a calculator is used.

### Example 2 Equation with Irrational Roots

**Solve $x^2 - 10x + 25 = 27$ by using the Square Root Property.**

$x^2 - 10x + 25 = 27$ Original equation

$(x - 5)^2 = 27$ Factor the perfect square trinomial.

$x - 5 = \pm\sqrt{27}$ Square Root Property

$x = 5 \pm 3\sqrt{3}$ Add 5 to each side; $\sqrt{27} = 3\sqrt{3}$.

$x = 5 + 3\sqrt{3}$ or $x = 5 - 3\sqrt{3}$ Write as two equations.

$\approx 10.2$ $\approx -0.2$ Use a calculator.

**Study Tip**

**Plus or Minus** When using the Square Root Property, remember to put a ± sign before the radical.

The exact solutions of this equation are $5 + 3\sqrt{3}$ and $5 - 3\sqrt{3}$. The approximate solutions are $-0.2$ and 10.2. Check these results by finding and graphing the related quadratic function.

$x^2 - 10x + 25 = 27$ Original equation

$x^2 - 10x - 2 = 0$ Subtract 27 from each side.

$y = x^2 - 10x - 2$ Related quadratic function

**CHECK** Use the **zero** function of a graphing calculator. The approximate zeros of the related function are $-0.2$ and 10.2.

[−5, 15] scl: 1 by [−30, 20] scl: 2

### Guided Practice

**Solve each equation by using the Square Root Property.**

**2A.** $x^2 + 8x + 16 = 20$

**2B.** $x^2 - 6x + 9 = 32$

## 2 Complete the Square

All quadratic equations can be solved using the Square Root Property by manipulating the equation until one side is a perfect square. This method is called **completing the square**.

Consider $x^2 + 16x = 9$. Remember to perform each operation on each side of the equation.

$x^2 + 16x + \blacksquare = 9$ What value is needed for the perfect square?

$x^2 + 16x + 64 = 9 + 64$ $\left(\frac{16}{2}\right)^2 = 64$; add 64 to each side.

$x^2 + 16x + 64 = 73$ Simplify.

$(x + 8)^2 = 73$ We can now use the Square Root Property.

Use this pattern of coefficients to complete the square of a quadratic expression.

Arthur S. Aubry/Photodisc/Getty Images

## KeyConcept Completing the Square

**Words** To complete the square for any quadratic expression of the form $x^2 + bx$, follow the steps below.

|  |  |
|---|---|
| **Step 1** | Find one half of $b$, the coefficient of $x$. |
| **Step 2** | Square the result in Step 1. |
| **Step 3** | Add the result of Step 2 to $x^2 + bx$. |

**Symbols** $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

## Example 3 Complete the Square

**Find the value of $c$ that makes $x^2 + 16x + c$ a perfect square. Then write the trinomial as a perfect square.**

|  |  |  |
|---|---|---|
| **Step 1** | Find one half of 16. | $\frac{16}{2} = 8$ |
| **Step 2** | Square the result in Step 1. | $8^2 = 64$ |
| **Step 3** | Add the result of Step 2 to $x^2 + 16x$. | $x^2 + 16x + 64$ |

The trinomial $x^2 + 16x + 64$ can be written as $(x + 8)^2$.

### GuidedPractice

**3.** Find the value of $c$ that makes $x^2 - 14x + c$ a perfect square. Then write the trinomial as a perfect square.

You can solve any quadratic equation by completing the square. Because you are solving an equation, add the value you use to complete the square to each side.

**WatchOut!**

*Each Side* When solving equations by completing the square, don't forget to add $\left(\frac{b}{2}\right)^2$ to *each* side of the equation.

## Example 4 Solve an Equation by Completing the Square

**Solve $x^2 + 10x - 11 = 0$ by completing the square.**

|  |  |
|---|---|
| $x^2 + 10x - 11 = 0$ | Notice that $x^2 + 10x - 11$ is not a perfect square. |
| $x^2 + 10x = 11$ | Rewrite so the left side is of the form $x^2 + bx$. |
| $x^2 + 10x + 25 = 11 + 25$ | Since $\left(\frac{10}{2}\right)^2 = 25$, add 25 to each side. |
| $(x + 5)^2 = 36$ | Write the left side as a perfect square. |
| $x + 5 = \pm 6$ | Square Root Property |
| $x = -5 \pm 6$ | Subtract 5 from each side. |
| $x = -5 + 6$ or $x = -5 - 6$ | Write as two equations. |
| $= 1$ $\quad = -11$ | Simplify. |

The solution set is $\{-11, 1\}$ or $\{x \mid x = -11, 1\}$. Check the result by using factoring.

### GuidedPractice

**Solve each equation by completing the square.**

**4A.** $x^2 - 10x + 24 = 0$

**4B.** $x^2 + 10x + 9 = 0$

When the coefficient of the quadratic term is not 1, you must divide the equation by that coefficient before completing the square.

### Example 5 Equation with $a \neq 1$

**Solve $2x^2 - 7x + 5 = 0$ by completing the square.**

| | |
|---|---|
| $2x^2 - 7x + 5 = 0$ | Notice that $2x^2 - 7x + 5$ is not a perfect square. |
| $x^2 - \frac{7}{2}x + \frac{5}{2} = 0$ | Divide by the coefficient of the quadratic term, 2. |
| $x^2 - \frac{7}{2}x = -\frac{5}{2}$ | Subtract $\frac{5}{2}$ from each side. |
| $x^2 - \frac{7}{2}x + \frac{49}{16} = -\frac{5}{2} + \frac{49}{16}$ | Since $\left(-\frac{7}{2} \div 2\right)^2 = \frac{49}{16}$, add $\frac{49}{16}$ to each side. |
| $\left(x - \frac{7}{4}\right)^2 = \frac{9}{16}$ | Write the left side as a perfect square by factoring.<br>Simplify the right side. |
| $x - \frac{7}{4} = \pm\frac{3}{4}$ | Square Root Property |
| $x = \frac{7}{4} \pm \frac{3}{4}$ | Add $\frac{7}{4}$ to each side. |
| $x = \frac{7}{4} + \frac{3}{4}$ or $x = \frac{7}{4} - \frac{3}{4}$ | Write as two equations. |
| $= \frac{5}{2}$ $\quad\quad = 1$ | |

The solution set is $\left\{1, \frac{5}{2}\right\}$. or $\left\{x \mid x = 1, \frac{5}{2}\right\}$.

**Guided Practice**

**Solve each equation by completing the square.**

**5A.** $3x^2 + 10x - 8 = 0$

**5B.** $3x^2 + 14x - 16 = 0$

Not all solutions of quadratic equations are real numbers. In some cases, the solutions are complex numbers of the form $a + bi$, where $b \neq 0$.

**Study Tip**

**CCSS Perseverance** A graph of the related function shows that the equation has no real solutions since the graph has no $x$-intercepts. Imaginary solutions must be checked algebraically by substituting them in the original equation.

### Example 6 Equation with Imaginary Solutions

**Solve $x^2 + 8x + 22 = 0$ by completing the square.**

| | |
|---|---|
| $x^2 + 8x + 22 = 0$ | Notice that $x^2 + 8x + 22$ is not a perfect square. |
| $x^2 + 8x = -22$ | Rewrite so the left side is of the form $x^2 + bx$. |
| $x^2 + 8x + 16 = -22 + 16$ | Since $\left(\frac{8}{2}\right)^2 = 16$, add 16 to each side. |
| $(x + 4)^2 = -6$ | Write the left side as a perfect square. |
| $x + 4 = \pm\sqrt{-6}$ | Square Root Property |
| $x + 4 = \pm i\sqrt{6}$ | $\sqrt{-1} = i$ |
| $x = -4 \pm i\sqrt{6}$ | Subtract 4 from each side. |

The solution set is $\{-4 + i\sqrt{6}, -4 - i\sqrt{6}\}$ or $\{x \mid x = -4 + i\sqrt{6}, -4 - i\sqrt{6}\}$.

**Guided Practice**

**Solve each equation by completing the square.**

**6A.** $x^2 + 2x + 2 = 0$

**6B.** $x^2 - 6x + 25 = 0$

## Check Your Understanding

🟢 = Step-by-Step Solutions begin on page R14.

**Examples 1–2** Solve each equation by using the Square Root Property. Round to the nearest hundredth if necessary.

**1.** $x^2 + 12x + 36 = 6$
**2.** $x^2 - 8x + 16 = 13$
**3.** $x^2 + 18x + 81 = 15$
**4.** $9x^2 + 30x + 25 = 11$

**5. LASER LIGHT SHOW** The area $A$ in square feet of a projected laser light show is given by $A = 0.16d^2$, where $d$ is the distance from the laser to the screen in feet. At what distance will the projected laser light show have an area of 100 square feet?

**Example 3** Find the value of $c$ that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

**6.** $x^2 - 10x + c$
**7.** $x^2 - 5x + c$

**Examples 4–6** Solve each equation by completing the square.

**8.** $x^2 + 2x - 8 = 0$
**9.** $x^2 - 4x + 9 = 0$
**10.** $2x^2 - 3x - 3 = 0$
**11.** $2x^2 + 6x - 12 = 0$
**12.** $x^2 + 4x + 6 = 0$
**13.** $x^2 + 8x + 10 = 0$

## Practice and Problem Solving

Extra Practice is on page R4.

**Examples 1–2** Solve each equation by using the Square Root Property. Round to the nearest hundredth if necessary.

**14.** $x^2 + 4x + 4 = 10$
**15.** $x^2 - 6x + 9 = 20$
**16.** $x^2 + 8x + 16 = 18$
**17.** $x^2 + 10x + 25 = 7$
**18.** $x^2 + 12x + 36 = 5$
**19.** $x^2 - 2x + 1 = 4$
**20.** $x^2 - 5x + 6.25 = 4$
**21.** $x^2 - 15x + 56.25 = 8$
**22.** $x^2 + 32x + 256 = 1$
**23.** $x^2 - 3x + \frac{9}{4} = 6$
**24.** $x^2 + 7x + \frac{49}{4} = 4$
**25.** $x^2 - 9x + \frac{81}{4} = \frac{1}{4}$

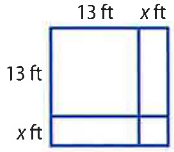
**Example 3** Find the value of $c$ that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

**26.** $x^2 + 8x + c$
**27.** $x^2 + 16x + c$
**28.** $x^2 - 11x + c$
**29.** $x^2 + 9x + c$

**Examples 4–6** Solve each equation by completing the square.

**30.** $x^2 - 4x + 12 = 0$
**31.** $x^2 + 2x - 12 = 0$
**32.** $x^2 + 6x + 8 = 0$
**33.** $x^2 - 4x + 3 = 0$
**34.** $2x^2 + x - 3 = 0$
**35.** $2x^2 - 3x + 5 = 0$
**36.** $2x^2 + 5x + 7 = 0$
**㊲** $3x^2 - 6x - 9 = 0$
**38.** $x^2 - 2x + 3 = 0$
**39.** $x^2 + 4x + 11 = 0$
**40.** $x^2 - 6x + 18 = 0$
**41.** $x^2 - 10x + 29 = 0$
**42.** $3x^2 - 4x = 2$
**43.** $2x^2 - 7x = -12$
**44.** $x^2 - 2.4x = 2.2$
**45.** $x^2 - 5.3x = -8.6$
**46.** $x^2 - \frac{1}{5}x - \frac{11}{5} = 0$
**47.** $x^2 - \frac{9}{2}x - \frac{24}{5} = 0$

**48. CCSS MODELING** An architect's blueprints call for a dining room measuring 13 feet by 13 feet. The customer would like the dining room to be a square, but with an area of 250 square feet. How much will this add to the dimensions of the room?



**Solve each equation. Round to the nearest hundredth if necessary.**

**49.** $4x^2 - 28x + 49 = 5$
**50.** $9x^2 + 30x + 25 = 11$ $9x^2+30x=-14$
**51.** $x^2 + x + \frac{1}{3} = \frac{2}{3}$
**52.** $x^2 + 1.2x + 0.56 = 0.91$

**53.** **FIREWORKS** A firework's distance $d$ meters from the ground is given by $d = -1.5t^2 + 25t$, where $t$ is the number of seconds after the firework has been lit.

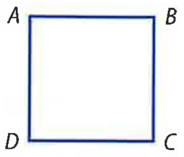
**a.** How many seconds have passed since the firework was lit when the firework explodes if it explodes at the maximum height of its path?

**b.** What is the height of the firework when it explodes?

**Find the value of $c$ that makes each trinomial a perfect square. Then write the trinomial as a perfect square.**

**54.** $x^2 + 0.7x + c$ **55.** $x^2 - 3.2x + c$ **56.** $x^2 - 1.8x + c$

**57.** **MULTIPLE REPRESENTATIONS** In this problem, you will use quadratic equations to investigate golden rectangles and the golden ratio.



**a. Geometric**

- Draw square $ABCD$.
- Locate the midpoint of $\overline{CD}$. Label the midpoint $P$. Draw $\overline{PB}$.
- Construct an arc with a radius of $\overline{PB}$ from $B$ clockwise past the bottom of the square.
- Extend $\overline{CD}$ until it intersects the arc. Label this point $Q$.
- Construct rectangle $ARQD$.

**b. Algebraic** Let $AD = x$ and $CQ = 1$. Use completing the square to solve $\frac{DQ}{AD} = \frac{QR}{CQ}$ for $x$.

**c. Tabular** Make a table of $x$ and values for $CQ = 2, 3,$ and $4$.

**d. Verbal** What do you notice about the $x$-values? Write an equation you could use to determine $x$ for $CQ = n$, where $n$ is a nonzero real number.

## H.O.T. Problems Use Higher-Order Thinking Skills

**58.** **ERROR ANALYSIS** Alonso and Aida are solving $x^2 + 8x - 20 = 0$ by completing the square. Is either of them correct? Explain your reasoning.

*Alonso*

$x^2 + 8x - 20 = 0$

$x^2 + 8x = 20$

$x^2 + 8x + 16 = 20 + 16$

$(x + 4)^2 = 36$

$x + 4 = \pm 6$

$x = -4 \pm 6$

*Aida*

$x^2 + 8x - 20 = 0$

$x^2 + 8x = 20$

$x^2 + 8x + 16 = 20$

$(x + 4)^2 = 20$

$x + 4 = \pm\sqrt{20}$

$x = -4 \pm \sqrt{20}$

**59.** **CHALLENGE** Solve $x^2 + bx + c = 0$ by completing the square. Your answer will be an expression for $x$ in terms of $b$ and $c$.

**60.** CCSS **ARGUMENTS** Without solving, determine how many unique solutions there are for each equation. Are they rational, real, or complex? Justify your reasoning.

**a.** $(x + 2)^2 = 16$ **b.** $(x - 2)^2 = 16$ **c.** $-(x - 2)^2 = 16$

**d.** $36 - (x - 2)^2 = 16$ **e.** $16(x + 2)^2 = 0$ **f.** $(x + 4)^2 = (x + 6)^2$

**61.** **OPEN ENDED** Write a perfect square trinomial equation in which the linear coefficient is negative and the constant term is a fraction. Then solve the equation.

**62.** **WRITING IN MATH** Explain what it means to complete the square. Describe each step.

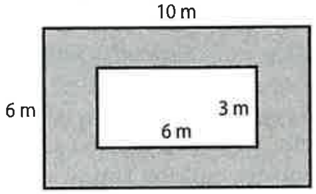
## Standardized Test Practice

**63. SAT/ACT** If $x^2 + y^2 = 2xy$, then $y$ must equal

**A** $-1$ **C** $1$ **E** $x$

**B** $0$ **D** $-x$

**64. GEOMETRY** Find the area of the shaded region.



**F** 14 $m^2$ **G** 18 $m^2$ **H** 42 $m^2$ **J** 60 $m^2$

**65. SHORT RESPONSE** What value of $c$ should be used to solve the following equation by completing the square?

$$5x^2 - 50x + c = 12 + c$$

**66.** If $5 - 3i$ is a solution for $x^2 + ax + b = 0$, where $a$ and $b$ are real numbers, what is the value of $b$?

**A** 10 **C** 34

**B** 14 **D** 40

## Spiral Review

**Simplify.** (Lesson 4-4)

**67.** $(8 + 5i)^2$

**68.** $4(3 - i) + 6(2 - 5i)$

**69.** $\frac{5 - 2i}{6 + 9i}$

**Write a quadratic equation in standard form with the given root(s).** (Lesson 4-3)

**70.** $\frac{4}{5}, \frac{3}{4}$

**71.** $-\frac{2}{5}, 6$

**72.** $-\frac{1}{4}, -\frac{6}{7}$

**73. TRAVEL** Yoko is going with the Spanish Club to Costa Rica. She buys 10 traveler's checks in denominations of \$20, \$50, and \$100, totaling \$370. She has twice as many \$20 checks as \$50 checks. How many of each denomination of traveler's checks does she have? (Lesson 3-4)

**74. SHOPPING** Main St. Media sells all DVDs for one price and all books for another price. Alex bought 4 DVDs and 6 books for \$170, while Matt bought 3 DVDs and 8 books for \$180. What is the cost of a DVD and the cost of a book? (Lesson 3-1)

**Graph each inequality.** (Lesson 2-8)

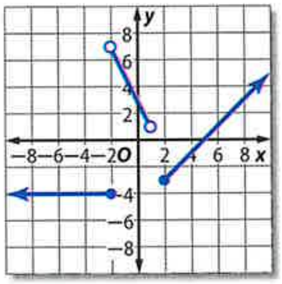
**75.** $y \geq 4x - 3$

**76.** $2x - 3y < 6$

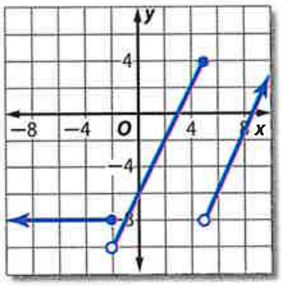
**77.** $5x + 2y + 3 \leq 0$

**Write the piecewise function shown in each graph.** (Lesson 2-6)

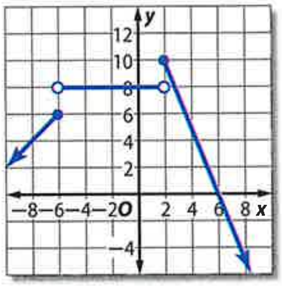
**78.**



**79.**



**80.**



## Skills Review

**Evaluate $b^2 - 4ac$ for the given values of $a$, $b$, and $c$.**

**81.** $a = 5, b = 6, c = 2$

**82.** $a = -2, b = -7, c = 3$

**83.** $a = -5, b = -8, c = -10$

# LESSON 4-6 The Quadratic Formula and the Discriminant

## Then

- You solved equations by completing the square.

## Now

- **1** Solve quadratic equations by using the Quadratic Formula.

**2** Use the discriminant to determine the number and type of roots of a quadratic equation.

## Why?

- Pumpkin catapult is an event in which a contestant builds a catapult and launches a pumpkin at a target.

The path of the pumpkin can be modeled by the quadratic function $h = -4.9t^2 + 117t + 42$, where $h$ is the height of the pumpkin and $t$ is the number of seconds.

To predict when the pumpkin will hit the target, you can solve the equation $0 = -4.9t^2 + 117t + 42$. This equation would be difficult to solve using factoring, graphing, or completing the square.

Izzy Schwartz/Photodisc/Getty Images

## NewVocabulary

Quadratic Formula

discriminant

## Common Core State Standards

**Content Standards**

**N.CN.7** Solve quadratic equations with real coefficients that have complex solutions.

**A.SSE.1.b** Interpret complicated expressions by viewing one or more of their parts as a single entity.

**Mathematical Practices**

8 Look for and express regularity in repeated reasoning.

## 1 Quadratic Formula

You have found solutions of some quadratic equations by graphing, by factoring, and by using the Square Root Property. There is also a formula that can be used to solve any quadratic equation. This formula can be derived by solving the standard form of a quadratic equation.

| General Case | | Specific Case |
|---|---|---|
| $ax^2 + bx + c = 0$ | Standard quadratic equation | $2x^2 + 8x + 1 = 0$ |
| $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ | Divide each side by $a$. | $x^2 + 4x + \frac{1}{2} = 0$ |
| $x^2 + \frac{b}{a}x = -\frac{c}{a}$ | Subtract $\frac{c}{a}$ from each side. | $x^2 + 4x = -\frac{1}{2}$ |
| $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$ | Complete the square. | $x^2 + 4x + \left(\frac{4}{2}\right)^2 = -\frac{1}{2} + \left(\frac{4}{2}\right)^2$ |
| $\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$ | Factor the left side. | $(x + 2)^2 = -\frac{1}{2} + \left(\frac{4}{2}\right)^2$ |
| $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ | Simplify the right side. | $(x + 2)^2 = \frac{7}{2}$ |
| $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$ | Square Root Property | $x + 2 = \pm\sqrt{\frac{7}{2}}$ |
| $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ | Subtract $\frac{b}{2a}$ from each side. | $x = -2 \pm \sqrt{\frac{7}{2}}$ |
| $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | Simplify. | $x = \frac{-4 \pm \sqrt{14}}{2}$ |

The equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as the **Quadratic Formula**.

**StudyTip**

**Quadratic Formula** Although factoring may be an easier method to solve some of the equations, the Quadratic Formula can be used to solve any quadratic equation.

## KeyConcept Quadratic Formula

**Words** The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the following formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Example** $x^2 + 5x + 6 = 0 \rightarrow x = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2(1)}$

### Example 1 Two Rational Roots

**Solve $x^2 - 10x = 11$ by using the Quadratic Formula.**

First, write the equation in the form $ax^2 + bx + c = 0$ and identify $a$, $b$, and $c$.

$$\begin{array}{ccc} & & ax^2 + bx + c = 0 \\ & & \downarrow \qquad \downarrow \qquad \downarrow \\ x^2 - 10x = 11 & \rightarrow & 1x^2 - 10x - 11 = 0 \end{array}$$

Then, substitute these values into the Quadratic Formula.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic Formula

$= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-11)}}{2(1)}$ Replace $a$ with 1, $b$ with $-10$, and $c$ with $-11$.

$= \frac{10 \pm \sqrt{100 + 44}}{2}$ Multiply.

$= \frac{10 \pm \sqrt{144}}{2}$ Simplify.

$= \frac{10 \pm 12}{2}$ $\sqrt{144} = 12$

$x = \frac{10 + 12}{2}$ or $x = \frac{10 - 12}{2}$ Write as two equations.

$= 11$ $= -1$ Simplify.

The solutions are $-1$ and 11.

**CHECK** Substitute both values into the original equation.

$$\begin{array}{rr} x^2 - 10x = 11 & x^2 - 10x = 11 \\ (-1)^2 - 10(-1) \stackrel{?}{=} 11 & (11)^2 - 10(11) \stackrel{?}{=} 11 \\ 1 + 10 \stackrel{?}{=} 11 & 121 - 110 \stackrel{?}{=} 11 \\ 11 = 11 \checkmark & 11 = 11 \checkmark \end{array}$$

**GuidedPractice**

**Solve each equation by using the Quadratic Formula.**

**1A.** $x^2 + 6x = 16$

**1B.** $2x^2 + 25x + 33 = 0$

**ReviewVocabulary**

**radicand** the value underneath the radical symbol

When the value of the radicand in the Quadratic Formula is 0, the quadratic equation has exactly one rational root.

Izzy Schwartz/Photodisc/Getty Images

**Math HistoryLink**

**Brahmagupta (598–668)** Indian mathematician Brahmagupta offered the first general solution of the quadratic equation $ax^2 + bx = c$, now known as the Quadratic Formula.

### Example 2 One Rational Root

**Solve $x^2 + 8x + 16 = 0$ by using the Quadratic Formula.**

Identify $a$, $b$, and $c$. Then, substitute these values into the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

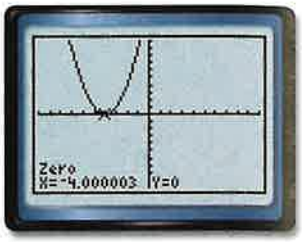
$$= \frac{-(8) \pm \sqrt{(8)^2 - 4(1)(16)}}{2(1)}$$ Replace $a$ with 1, $b$ with 8, and $c$ with 16.

$$= \frac{-8 \pm \sqrt{0}}{2}$$ Simplify.

$$= \frac{-8}{2} \text{ or } -4$$ $\sqrt{0} = 0$

The solution is −4.

**CHECK** A graph of the related function shows that there is one solution at $x = -4$.



Zero
X=-4.000003 Y=0

[−10, 10] scl: 1 by [−10, 10] scl: 1

#### GuidedPractice

**Solve each equation by using the Quadratic Formula.**

**2A.** $x^2 - 16x + 64 = 0$

**2B.** $x^2 + 34x + 289 = 0$

You can express irrational roots exactly by writing them in radical form.

### Example 3 Irrational Roots

**Solve $2x^2 + 6x - 7 = 0$ by using the Quadratic Formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

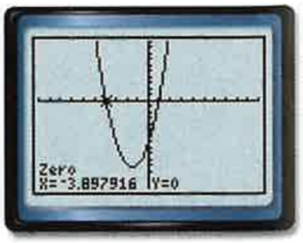
$$= \frac{-(6) \pm \sqrt{(6)^2 - 4(2)(-7)}}{2(2)}$$ Replace $a$ with 2, $b$ with 6, and $c$ with −7.

$$= \frac{-6 \pm \sqrt{92}}{4}$$ Simplify.

$$= \frac{-6 \pm 2\sqrt{23}}{4} \text{ or } \frac{-3 \pm \sqrt{23}}{2}$$ $\sqrt{92} = \sqrt{4 \cdot 23}$ or $2\sqrt{23}$

The approximate solutions are −3.9 and 0.9.

**CHECK** Check these results by graphing the related quadratic function, $y = 2x^2 + 6x - 7$. Using the **zero** function of a graphing calculator, the approximate zeros of the related function are −3.9 and 0.9.



Zero
X=-3.897916 Y=0

[−10, 10] scl: 1 by [−10, 10] scl: 1

#### GuidedPractice

**Solve each equation by using the Quadratic Formula.**

**3A.** $3x^2 + 5x + 1 = 0$

**3B.** $x^2 - 8x + 9 = 0$

Dinodia/age fotostock

**StudyTip**

**Complex Numbers** Remember to write your solutions in the form $a + bi$, sometimes called the *standard form* of a complex number.

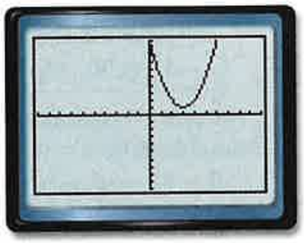
When using the Quadratic Formula, if the value of the radicand is negative, the solutions will be complex. Complex solutions always appear in conjugate pairs.

### Example 4 Complex Roots

**Solve $x^2 - 6x = -10$ by using the Quadratic Formula.**

| | |
|---|---|
| $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | Quadratic Formula |
| $= \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$ | Replace $a$ with 1, $b$ with −6, and $c$ with 10. |
| $= \dfrac{6 \pm \sqrt{-4}}{2}$ | Simplify. |
| $= \dfrac{6 \pm 2i}{2}$ | $\sqrt{-4} = \sqrt{4 \cdot (-1)}$ or $2i$ |
| $= 3 \pm i$ | Simplify. |

The solutions are the complex numbers $3 + i$ and $3 - i$.



[−10, 10] scl: 1 by [−10, 10] scl: 1

**CHECK** A graph of the related function shows that the solutions are complex, but it cannot help you find them. To check complex solutions, substitute them into the original equation.

| | |
|---|---|
| $x^2 - 6x = -10$ | Original equation |
| $(3 + i)^2 - 6(3 + i) \stackrel{?}{=} -10$ | $x = 3 + i$ |
| $9 + 6i + i^2 - 18 - 6i \stackrel{?}{=} -10$ | Square of a sum; Distributive Property |
| $-9 + i^2 \stackrel{?}{=} -10$ | Simplify. |
| $-9 - 1 = -10$ ✓ | $i^2 = -1$ |

| | |
|---|---|
| $x^2 - 6x = -10$ | Original equation |
| $(3 - i)^2 - 6(3 - i) \stackrel{?}{=} -10$ | $x = 3 - i$ |
| $9 - 6i + i^2 - 18 + 6i \stackrel{?}{=} -10$ | Square of a sum; Distributive Property |
| $-9 + i^2 \stackrel{?}{=} -10$ | Simplify. |
| $-9 - 1 = -10$ ✓ | $i^2 = -1$ |

**GuidedPractice**

**Solve each equation by using the Quadratic Formula.**

**4A.** $3x^2 + 5x + 4 = 0$

**4B.** $x^2 - 4x = -13$

## 2 Roots and the Discriminant

**Roots and the Discriminant** In the previous examples, observe the relationship between the value of the expression under the radical and the roots of the quadratic equation. The expression $b^2 - 4ac$ is called the **discriminant**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \leftarrow \text{discriminant}$$

The value of the discriminant can be used to determine the number and type of roots of a quadratic equation. The table on the following page summarizes the possible types of roots.

The discriminant can also be used to confirm the number and type of solutions after you solve the quadratic equation.

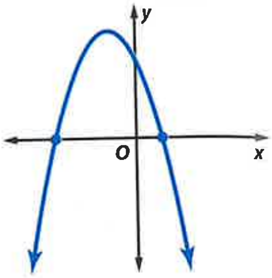
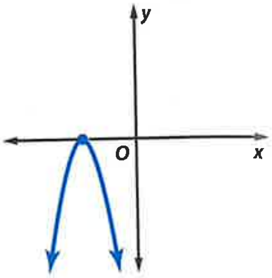
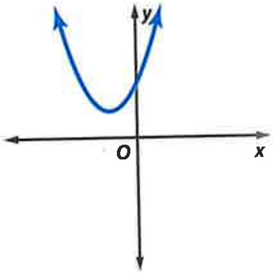
Dinodia/age fotostock

**StudyTip**

**Roots** Remember that the solutions of an equation are called *roots* or *zeros* and are the value(s) where the graph crosses the *x*-axis.

### KeyConcept Discriminant

Consider $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are rational numbers and $a \neq 0$.

| Value of Discriminant | Type and Number of Roots | Example of Graph of Related Function |
|---|---|---|
| $b^2 - 4ac > 0$;<br>$b^2 - 4ac$ is a perfect square. | 2 real, rational roots |  |
| $b^2 - 4ac > 0$;<br>$b^2 - 4ac$ is *not* a perfect square. | 2 real, irrational roots | |
| $b^2 - 4ac = 0$ | 1 real rational root |  |
| $b^2 - 4ac < 0$ | 2 complex roots |  |

### Example 5 Describe Roots

**Find the value of the discriminant for each quadratic equation. Then describe the number and type of roots for the equation.**

**a.** $7x^2 - 11x + 5 = 0$

$a = 7, b = -11, c = 5$

$$\begin{aligned} b^2 - 4ac &= (-11)^2 - 4(7)(5) \\ &= 121 - 140 \\ &= -19 \end{aligned}$$

The discriminant is negative, so there are two complex roots.

**b.** $x^2 + 22x + 121 = 0$

$a = 1, b = 22, c = 121$

$$\begin{aligned} b^2 - 4ac &= (22)^2 - 4(1)(121) \\ &= 484 - 484 \\ &= 0 \end{aligned}$$

The discriminant is 0, so there is one rational root.

**GuidedPractice**

**5A.** $-5x^2 + 8x - 1 = 0$

**5B.** $-7x + 15x^2 - 4 = 0$

You have studied a variety of methods for solving quadratic equations. The table below summarizes these methods.

**StudyTip**

**Study Notebook** You may wish to copy this list of methods to your math notebook or Foldable to keep as a reference as you study.

### ConceptSummary Solving Quadratic Equations

| Method | Can be Used | When to Use |
|---|---|---|
| graphing | sometimes | Use only if an exact answer is not required. Best used to check the reasonableness of solutions found algebraically. |
| factoring | sometimes | Use if the constant term is 0 or if the factors are easily determined.<br>**Example** $x^2 - 7x = 0$ |
| Square Root Property | sometimes | Use for equations in which a perfect square is equal to a constant.<br>**Example** $(x - 5)^2 = 18$ |
| completing the square | always | Useful for equations of the form $x^2 + bx + c = 0$, where $b$ is even.<br>**Example** $x^2 + 6x - 14 = 0$ |
| Quadratic Formula | always | Useful when other methods fail or are too tedious.<br>**Example** $2.3x^2 - 1.8x + 9.7 = 0$ |

## Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

**Examples 1–4** **Solve each equation by using the Quadratic Formula.**

① $x^2 + 12x - 9 = 0$

**2.** $x^2 + 8x + 5 = 0$

**3.** $4x^2 - 5x - 2 = 0$

**4.** $9x^2 + 6x - 4 = 0$

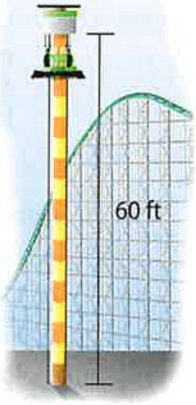
**5.** $10x^2 - 3 = 13x$

**6.** $22x = 12x^2 + 6$

**7.** $-3x^2 + 4x = -8$

**8.** $x^2 + 3 = -6x + 8$

**Examples 3–4** **9.** **CCSS MODELING** An amusement park ride takes riders to the top of a tower and drops them at speeds reaching 80 feet per second. A function that models this ride is $h = -16t^2 - 64t + 60$, where $h$ is the height in feet and $t$ is the time in seconds. About how many seconds does it take for riders to drop from 60 feet to 0 feet?



**Example 5** **Complete parts a and b for each quadratic equation.**

**a.** **Find the value of the discriminant.**

**b.** **Describe the number and type of roots.**

**10.** $3x^2 + 8x + 2 = 0$

**11.** $2x^2 - 6x + 9 = 0$

**12.** $-16x^2 + 8x - 1 = 0$

**13.** $5x^2 + 2x + 4 = 0$

## Practice and Problem Solving

Extra Practice is on page R4.

**Examples 1–4** **Solve each equation by using the Quadratic Formula.**

**14.** $x^2 + 45x = -200$

**15.** $4x^2 - 6 = -12x$

**16.** $3x^2 - 4x - 8 = -6$

**17.** $4x^2 - 9 = -7x - 4$

**18.** $5x^2 - 9 = 11x$

**19.** $12x^2 + 9x - 2 = -17$

**20.** **DIVING** Competitors in the 10-meter platform diving competition jump upward and outward before diving into the pool below. The height $h$ of a diver in meters above the pool after $t$ seconds can be approximated by the equation $h = -4.9t^2 + 3t + 10$.

**a.** Determine a domain and range for which this function makes sense.

**b.** When will the diver hit the water?

**Example 5** **Complete parts a–c for each quadratic equation.**

**a. Find the value of the discriminant.**

**b. Describe the number and type of roots.**

**c. Find the exact solutions by using the Quadratic Formula.**

**㉑** $2x^2 + 3x - 3 = 0$

**22.** $4x^2 - 6x + 2 = 0$

**23.** $6x^2 + 5x - 1 = 0$

**24.** $6x^2 - x - 5 = 0$

**25.** $3x^2 - 3x + 8 = 0$

**26.** $2x^2 + 4x + 7 = 0$

**27.** $-5x^2 + 4x + 1 = 0$

**28.** $x^2 - 6x = -9$

**29.** $-3x^2 - 7x + 2 = 6$

**30.** $-8x^2 + 5 = -4x$

**31.** $x^2 + 2x - 4 = -9$

**32.** $-6x^2 + 5 = -4x + 8$

**33.** **VIDEO GAMES** While Darnell is grounded his friend Jack brings him a video game. Darnell stands at his bedroom window, and Jack stands directly below the window. If Jack tosses a game cartridge to Darnell with an initial velocity of 35 feet per second, an equation for the height $h$ feet of the cartridge after $t$ seconds is $h = -16t^2 + 35t + 5$.

**a.** If the window is 25 feet above the ground, will Darnell have 0, 1, or 2 chances to catch the video game cartridge?

**b.** If Darnell is unable to catch the video game cartridge, when will it hit the ground?

**34.** **CCSS SENSE-MAKING** Civil engineers are designing a section of road that is going to dip below sea level. The road's curve can be modeled by the equation $y = 0.00005x^2 - 0.06x$, where $x$ is the horizontal distance in feet between the points where the road is at sea level and $y$ is the elevation. The engineers want to put stop signs at the locations where the elevation of the road is equal to sea level. At what horizontal distances will they place the stop signs?

**Complete parts a–c for each quadratic equation.**

**a. Find the value of the discriminant.**

**b. Describe the number and type of roots.**

**c. Find the exact solutions by using the Quadratic Formula.**

**35.** $5x^2 + 8x = 0$

**36.** $8x^2 = -2x + 1$

**37.** $4x - 3 = -12x^2$

**38.** $0.8x^2 + 2.6x = -3.2$

**39.** $0.6x^2 + 1.4x = 4.8$

**40.** $-4x^2 + 12 = -6x - 8$

41. **SMOKING** A decrease in smoking in the United States has resulted in lower death rates caused by lung cancer. The number of deaths per 100,000 people $y$ can be approximated by $y = -0.26x^2 - 0.55x + 91.81$, where $x$ represents the number of years after 2000.

| Year | Deaths per 100,000 |
|---|---|
| 2000 | 91.8 |
| 2002 | 89.7 |
| 2004 | 85.5 |
| 2010 | 60.3 |
| 2015 | ? |
| 2017 | ? |

**a.** Calculate the number of deaths per 100,000 people for 2015 and 2017.

**b.** Use the Quadratic Formula to solve for $x$ when $y = 50$.

**c.** According to the quadratic function, when will the death rate be 0 per 100,000? Do you think that this prediction is reasonable? Why or why not?

42. **NUMBER THEORY** The sum $S$ of consecutive integers 1, 2, 3, …, $n$ is given by the formula $S = \frac{1}{2}n(n + 1)$. How many consecutive integers, starting with 1, must be added to get a sum of 666?

## H.O.T. Problems Use Higher-Order Thinking Skills

43. **CCSS CRITIQUE** Tama and Jonathan are determining the number of solutions of $3x^2 - 5x = 7$. Is either of them correct? Explain your reasoning.

**Tama**

$3x^2 - 5x = 7$

$b^2 - 4ac = (-5)^2 - 4(3)(7)$

$= -59$

*Since the discriminant is negative, there are no real solutions.*

**Jonathan**

$3x^2 - 5x = 7$

$3x^2 - 5x - 7 = 0$

$b^2 - 4ac = (-5)^2 - 4(3)(-7)$

$= 109$

*Since the discriminant is positive, there are two real roots.*

44. **CHALLENGE** Find the solutions of $4ix^2 - 4ix + 5i = 0$ by using the Quadratic Formula.

45. **REASONING** Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

**a.** In a quadratic equation in standard form, if $a$ and $c$ are different signs, then the solutions will be real.

**b.** If the discriminant of a quadratic equation is greater than 1, the two roots are real irrational numbers.

46. **OPEN ENDED** Sketch the corresponding graph and state the number and type of roots for each of the following.

**a.** $b^2 - 4ac = 0$

**b.** A quadratic function in which $f(x)$ never equals zero.

**c.** A quadratic function in which $f(a) = 0$ and $f(b) = 0$; $a \neq b$.

**d.** The discriminant is less than zero.

**e.** $a$ and $b$ are both solutions and can be represented as fractions.

47. **CHALLENGE** Find the value(s) of $m$ in the quadratic equation $x^2 + x + m + 1 = 0$ such that it has one solution.

48. **WRITING IN MATH** Describe three different ways to solve $x^2 - 2x - 15 = 0$. Which method do you prefer, and why?

## Standardized Test Practice

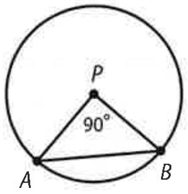
**49.** A company determined that its monthly profit $P$ is given by $P = -8x^2 + 165x - 100$, where $x$ is the selling price for each unit of product. Which of the following is the best estimate of the maximum price per unit that the company can charge without losing money?

**A** \$10 **B** \$20 **C** \$30 **D** \$40

**50.** **SAT/ACT** For which of the following sets of numbers is the mean greater than the median?

**F** {4, 5, 6, 7, 8}
**G** {4, 6, 6, 6, 8}
**H** {4, 5, 6, 7, 9}
**J** {3, 5, 6, 7, 8}
**K** {2, 6, 6, 6, 6}

**51.** **SHORT RESPONSE** In the figure below, $P$ is the center of the circle with radius 15 inches. What is the area of $\triangle APB$?



**52.** 75% of 88 is the same as 60% of what number?

**A** 100 **B** 101 **C** 108 **D** 110

## Spiral Review

**Find the value of $c$ that makes each trinomial a perfect square. Then write the trinomial as a perfect square.** (Lesson 4-5)

**53.** $x^2 + 13x + c$

**54.** $x^2 + 2.4x + c$

**55.** $x^2 + \frac{4}{5}x + c$

**Simplify.** (Lesson 4-4)

**56.** $i^{26}$

**57.** $\sqrt{-16}$

**58.** $4\sqrt{-9} \cdot 2\sqrt{-25}$

**59.** **PILOT TRAINING** Evita is training for her pilot's license. Flight instruction costs \$105 per hour, and the simulator costs \$45 per hour. She spent 4 more hours in airplane training than in the simulator. If Evita spent \$3870, how much time did she spend training in an airplane and in a simulator? (Lesson 3-8)

**60.** **BUSINESS** Ms. Larson owns three fruit farms on which she grows apples, peaches, and apricots. She sells apples for \$22 a case, peaches for \$25 a case, and apricots for \$18 a case. (Lesson 3-6)

**a.** Write an inventory matrix for the number of cases for each type of fruit for each farm and a cost matrix for the price per case for each type of fruit.

**b.** Find the total income of the three fruit farms expressed as a matrix.

**c.** What is the total income from all three fruit farms?

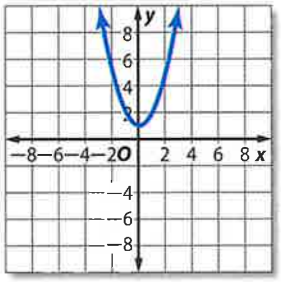
**Number of Cases in Stock of Each Type of Fruit**

| Fruit | Farm 1 | Farm 2 | Farm 3 |
|---|---|---|---|
| apples | 290 | 175 | 110 |
| peaches | 165 | 240 | 75 |
| apricots | 210 | 190 | 0 |

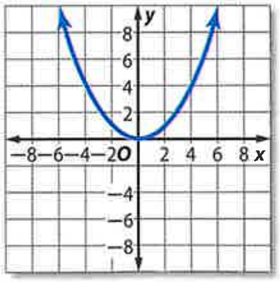
## Skills Review

**Write an equation for each graph.**

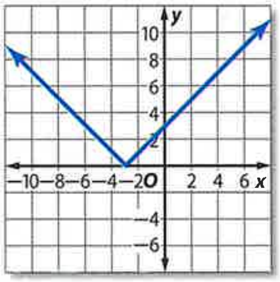
**61.**



**62.**



**63.**



LESSON 4-7

# Transformations of Quadratic Graphs

## Then

- You transformed graphs of functions.

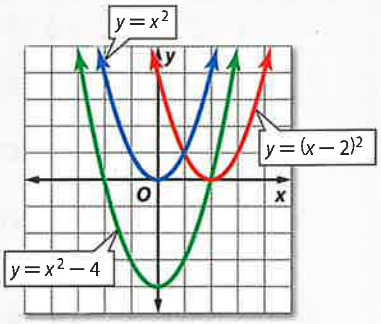
## Now

- 1 Write a quadratic function in the form $y = a(x - h)^2 + k$.
- 2 Transform graphs of quadratic functions of the form $y = a(x - h)^2 + k$.

## Why?

- Recall that a family of graphs is a group of graphs that display one or more similar characteristics. The parent graph is the simplest graph in the family. For the family of quadratic functions, $y = x^2$ is the parent graph.

Other graphs in the family of quadratic functions, such as $y = (x - 2)^2$ and $y = x^2 - 4$, can be drawn by transforming the graph of $y = x^2$.



## New Vocabulary

vertex form

## Common Core State Standards

### Content Standards

**F.IF.8.a** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

**F.BF.3** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

### Mathematical Practices

7 Look for and make use of structure.

## 1 Write Quadratic Functions in Vertex Form

Each function above is written in **vertex form**, $y = a(x - h)^2 + k$, where $(h, k)$ is the vertex of the parabola, $x = h$ is the axis of symmetry, and $a$ determines the shape of the parabola and the direction in which it opens.

When a quadratic function is in the form $y = ax^2 + bx + c$, you can complete the square to write the function in vertex form. If the coefficient of the quadratic term is not 1, then factor the coefficient from the quadratic and linear terms *before* completing the square. After completing the square and writing the function in vertex form, the value of $k$ indicated a minimum value  if $a < 0$ or a maximum value if $a > 0$.

### Example 1 Write Functions in Vertex Form

**Write each function in vertex form.**

**a.** $y = x^2 + 6x - 5$

| | |
|---|---|
| $y = x^2 + 6x - 5$ | Original function |
| $y = (x^2 + 6x + 9) - 5 - 9$ | Complete the square. |
| $y = (x + 3)^2 - 14$ | Simplify. |

**b.** $y = -2x^2 + 8x - 3$

| | |
|---|---|
| $y = -2x^2 + 8x - 3$ | Original function |
| $y = -2(x^2 - 4x) - 3$ | Group $ax^2 + bx$ and factor, dividing by $a$. |
| $y = -2(x^2 - 4x + 4) - 3 - (-2)(4)$ | Complete the square. |
| $y = -2(x - 2)^2 + 5$ | Simplify. |

### Guided Practice

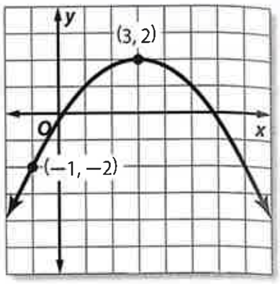
**1A.** $y = x^2 + 4x + 6$

**1B.** $y = 2x^2 - 12x + 17$

If the vertex and one additional point on the graph of a parabola are known, you can write the equation of the parabola in vertex form.

### Standardized Test Example 2 Write an Equation Given a Graph

**Which is an equation of the function shown in the graph?**



**A** $y = -4(x - 3)^2 + 2$

**B** $y = -\frac{1}{4}(x - 3)^2 + 2$

**C** $y = \frac{1}{4}(x + 3)^2 - 2$

**D** $y = 4(x + 3)^2 - 2$

**Read the Test Item**

You are given a graph of a parabola with the vertex and a point on the graph labeled. You need to find an equation of the parabola.

**Test-Taking Tip**

**The Meaning of *a*** The sign of *a* in vertex form does not determine the width of the parabola. The sign indicates whether the parabola opens up or down. The width of a parabola is determined by the absolute value of *a*.

**Solve the Test Item**

The vertex of the parabola is at (3, 2), so $h = 3$ and $k = 2$. Since $(-1, -2)$ is a point on the graph, let $x = -1$ and $y = -2$. Substitute these values into the vertex form of the equation and solve for $a$.

$y = a(x - h)^2 + k$ Vertex form

$-2 = a(-1 - 3)^2 + 2$ Substitute −2 for *y*, −1 for *x*, 3 for *h* and 2 for *k*.

$-2 = a(16) + 2$ Simplify.

$-4 = 16a$ Subtract 2 from each side.

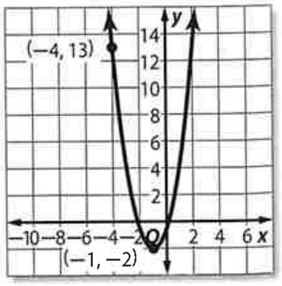
$-\frac{1}{4} = a$ Divide each side by 16.

The equation of the parabola in vertex form is $y = -\frac{1}{4}(x - 3)^2 + 2$.

The answer is B.

**Guided Practice**

2. Which is an equation of the function shown in the graph?



**F** $y = \frac{9}{25}(x - 1)^2 + 2$

**G** $y = \frac{3}{5}(x + 1)^2 - 2$

**H** $y = \frac{5}{3}(x + 1)^2 - 2$

**J** $y = \frac{25}{9}(x - 1)^2 + 2$

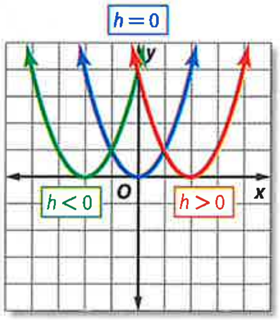
**2 Transformations of Quadratic Graphs** In Lesson 2-7, you learned how different transformations affect the graphs of parent functions. The following summarizes these transformations for quadratic functions.

## **Concept**Summary Transformations of Quadratic Functions

$f(x) = a(x - h)^2 + k$

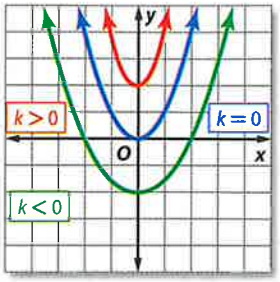
### *h*, Horizontal Translation

*h* units to the right if *h* is positive
$|h|$ units to the left if *h* is negative



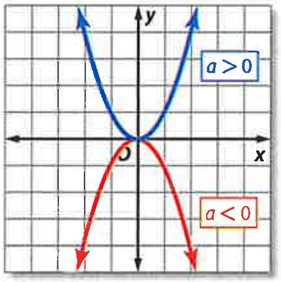
### *k*, Vertical Translation

*k* units up if *k* is positive
$|k|$ units down if *k* is negative



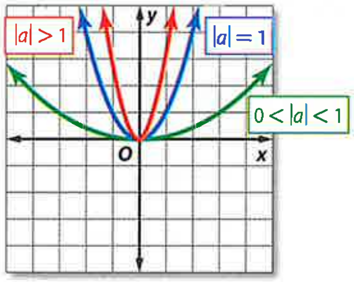
### *a*, Reflection

If $a > 0$, the graph opens up.
If $a < 0$, the graph opens down.



### *a*, Dilation

If $|a| > 1$, the graph is stretched vertically. If $0 < |a| < 1$, the graph is compressed vertically.



**Study**Tip

**Absolute Value**
$0 < |a| < 1$ means that *a* is a rational number between 0 and 1, such as $\frac{3}{4}$, or a rational number between −1 and 0, such as −0.3.

## **Example 3** Graph Equations in Vertex Form

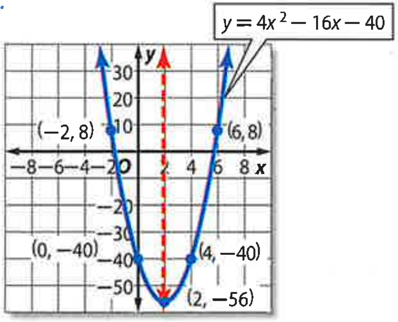
Graph $y = 4x^2 - 16x - 40$.

**Step 1** Rewrite the equation in vertex form.

| | |
|---|---|
| $y = 4x^2 - 16x - 40$ | Original equation |
| $y = 4(x^2 - 4x) - 40$ | Distributive Property |
| $y = 4(x^2 - 4x + 4) - 40 - 4(4)$ | Complete the square. |
| $y = 4(x - 2)^2 - 56$ | Simplify. |

**Step 2** The vertex is at (2, −56). The axis of symmetry is $x = 2$. Because $a = 4$, the graph is narrower than the graph of $y = x^2$.

**Step 3** Plot additional points to help you complete the graph.



### **Guided**Practice

**3A.** $y = (x - 3)^2 - 2$

**3B.** $y = 0.25(x + 1)^2$

## Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

**Example 1** Write each function in vertex form.

**1.** $y = x^2 + 6x + 2$

**2.** $y = -2x^2 + 8x - 5$

**3.** $y = 4x^2 + 24x + 24$

**Example 2**

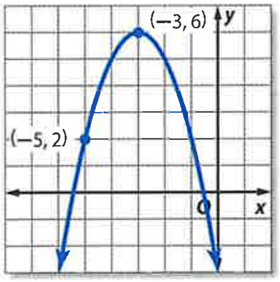
**4.** **MULTIPLE CHOICE** Which function is shown in the graph?

A $y = -(x + 3)^2 + 6$

B $y = -(x - 3)^2 - 6$

C $y = -2(x + 3)^2 + 6$

D $y = -2(x - 3)^2 - 6$



**Example 3** Graph each function.

**5.** $y = (x - 3)^2 - 4$

**6.** $y = -2x^2 + 5$

**7.** $y = \frac{1}{2}(x + 6)^2 - 8$

## Practice and Problem Solving

Extra Practice is on page R4.

**Example 1** Write each function in vertex form.

**8.** $y = x^2 + 9x + 8$

**9.** $y = x^2 - 6x + 3$

**10.** $y = -2x^2 + 5x$

**⑪** $y = x^2 + 2x + 7$

**12.** $y = -3x^2 + 12x - 10$

**13.** $y = x^2 + 8x + 16$

**14.** $y = 2x^2 - 4x - 3$

**15.** $y = 3x^2 + 10x$

**16.** $y = x^2 - 4x + 9$

**17.** $y = -4x^2 - 24x - 15$

**18.** $y = x^2 - 12x + 36$

**19.** $y = -x^2 - 4x - 1$

**Example 2**

**20.** **FIREWORKS** During an Independence Day fireworks show, the height $h$ in meters of a specific rocket after $t$ seconds can be modeled by $h = -4.9(t - 4)^2 + 80$. Graph the function.

**21.** **FINANCIAL LITERACY** A bicycle rental shop rents an average of 120 bicycles per week and charges $25 per day. The manager estimates that there will be 15 additional bicycles rented for each $1 reduction in the rental price. The maximum income the manager can expect can be modeled by $y = -15x^2 + 255x + 3000$, where $y$ is the weekly income and $x$ is the number of bicycles rented. Write this function in vertex form. Then graph.

**Example 3** Graph each function.

**22.** $y = (x - 5)^2 + 3$

**23.** $y = 9x^2 - 8$

**24.** $y = -2(x - 5)^2$

**25.** $y = \frac{1}{10}(x + 6)^2 + 6$

**26.** $y = -3(x - 5)^2 - 2$

**27.** $y = -\frac{1}{4}x^2 - 5$

**28.** $y = 2x^2 + 10$

**29.** $y = -(x + 3)^2$

**30.** $y = \frac{1}{6}(x - 3)^2 - 10$

**31.** $y = (x - 9)^2 - 7$

**32.** $y = -\frac{5}{8}x^2 - 8$

**33.** $y = -4(x - 10)^2 - 10$

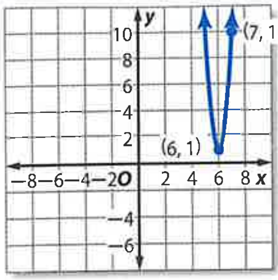
**34.** CCSS **MODELING** A sailboard manufacturer uses an automated process to manufacture the masts for its sailboards. The function $f(x) = \frac{1}{250}x^2 + \frac{3}{5}x$ is programmed into a computer to make one such mast.

**a.** Write the quadratic function in vertex form. Then graph the function.

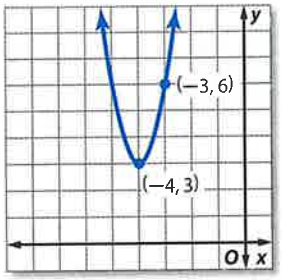
**b.** Describe how the manufacturer can adjust the function to make its masts with a greater or smaller curve.

**Write an equation in vertex form for each parabola.**

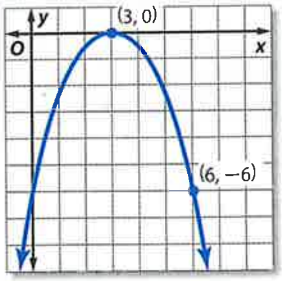
**35.**



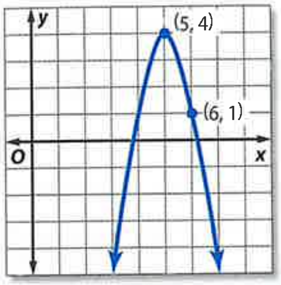
**36.**



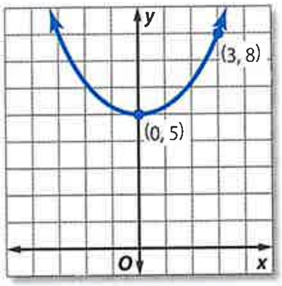
**37.**



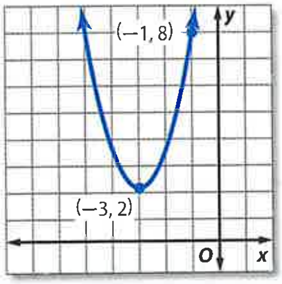
**38.**



**39.**



**40.**



**Write each function in vertex form. Then identify the vertex, axis of symmetry, and direction of opening.**

**41.** $3x^2 - 4x = 2 + y$

**42.** $-2x^2 + 7x = y - 12$

**43.** $-x^2 - 4.7x = y - 2.8$

**44.** $x^2 + 1.4x - 1.2 = y$

**45.** $x^2 - \frac{2}{3}x - \frac{26}{9} = y$

**46.** $x^2 + 7x + \frac{49}{4} = y$

**47** **CARS** The formula $S(t) = \frac{1}{2}at^2 + v_0t$ can be used to determine the position $S(t)$ of an object after $t$ seconds at a rate of acceleration $a$ with initial velocity $v_0$. Valerie's car can accelerate 0.002 miles per second squared.

**a.** Express $S(t)$ in vertex form as she accelerates from 35 miles per hour to enter highway traffic.

**b.** How long will it take Valerie to match the average speed of highway traffic of 68 miles per hour? (*Hint*: Use *acceleration* • *time* = *velocity*.)

**c.** If the entrance ramp is $\frac{1}{8}$-mile long, will Valerie have sufficient time to match the average highway speed? Explain.

## H.O.T. Problems Use Higher-Order Thinking Skills

**48.** **OPEN ENDED** Write an equation for a parabola that has been translated, compressed, and reflected in the $x$-axis.

**49.** **CHALLENGE** Explain how you can find an equation of a parabola using the coordinates of three points on the graph.

**50.** **CHALLENGE** Write the standard form of a quadratic function $ax^2 + bx + c = y$ in vertex form. Identify the vertex and the axis of symmetry.

**51.** **REASONING** Describe the graph of $f(x) = a(x - h)^2 + k$ when $a = 0$. Is the graph the same as that of $g(x) = ax^2 + bx + c$ when $a = 0$? Explain.

**52.** **CCSS ARGUMENTS** Explain how the graph of $y = x^2$ can be used to graph any quadratic function. Include a description of the effects produced by changing $a$, $h$, and $k$ in the equation $y = a(x - h)^2 + k$, and a comparison of the graph of $y = x^2$ and the graph of $y = a(x - h)^2 + k$ using values you choose for $a$, $h$, and $k$.

## Standardized Test Practice

**53.** Flowering bushes need a mixture of 70% soil and 30% vermiculite. About how many buckets of vermiculite should you add to 20 buckets of soil?

**A** 6.0 **C** 14.0

**B** 8.0 **D** 24.0

**54. SAT/ACT** The sum of the integers $x$ and $y$ is 495. The units digit of $x$ is 0. If $x$ is divided by 10, the result is equal to $y$. What is the value of $x$?

**F** 40 **J** 250

**G** 45 **K** 450

**H** 245

**55.** What is the solution set of the inequality $|4x - 1| < 9$?

**A** $\{x \mid 2.5 < x \text{ or } x < -2\}$

**B** $\{x \mid x < 2.5\}$

**C** $\{x \mid x > -2\}$

**D** $\{x \mid -2 < x < 2.5\}$

**56. SHORT RESPONSE** At your store, you buy wrenches for \$30.00 a dozen and sell them for \$3.50 each. What is the percent markup for the wrenches?

## Spiral Review

**Solve each equation by using the method of your choice. Find exact solutions.** (Lesson 4-6)

**57.** $4x^2 + 15x = 21$ **58.** $-3x^2 + 19 = 5x$ **59.** $6x - 5x^2 + 9 = 3$

**Find the value of $c$ that makes each trinomial a perfect square.** (Lesson 4-5)

**60.** $x^2 - 12x + c$ **61.** $x^2 + 0.1x + c$ **62.** $x^2 - 0.45x + c$

**Determine whether each function has a maximum or minimum value, and find that value.** (Lesson 4-1)

**63.** $f(x) = 6x^2 - 8x + 12$ **64.** $f(x) = -4x^2 + x - 18$ **65.** $f(x) = 3x^2 - 9 + 6x$

**66. ARCHAEOLOGY** A coordinate grid is laid over an archaeology dig to identify the location of artifacts. Three corners of a building have been partially unearthed at $(-1, 6)$, $(4, 5)$, and $(-1, -2)$. If each square on the grid measures one square foot, estimate the area of the floor of the building. (Lesson 3-7)

**67. ART** Rai can spend no more than \$225 on the art club's supply of brushes and paint. She needs at least 20 brushes and 56 tubes of paint. Graph the region that shows how many boxes of each item can be purchased. (Lesson 3-2)

**Solve each system of equations by graphing.** (Lesson 3-1)

**68.** $y = 3x - 4$
$y = -2x + 16$

**69.** $2x + 5y = 1$
$6y - 5x = 16$

**70.** $4x + 3y = -30$
$3x - 2y = 3$

**Evaluate each function.** (Lesson 2-1)

**71.** $f(3)$ if $f(x) = x^2 - 4x + 12$ **72.** $f(-2)$ if $f(x) = -4x^2 + x - 8$ **73.** $f(4)$ if $f(x) = 3x^2 + x$

## Skills Review

**Determine whether the given value satisfies the inequality.**

**74.** $3x^2 - 5 > 6; x = 2$ **75.** $-2x^2 + x - 1 < 4; x = -2$ **76.** $4x^2 + x - 3 \leq 36; x = 3$

LESSON

# 4-8 Quadratic Inequalities

## :: Then

- You solved linear inequalities.

## :: Now

- **1** Graph quadratic inequalities in two variables.

**2** Solve quadratic inequalities in one variable.

## :: Why?

- A water balloon launched from a slingshot can be represented by several different quadratic equations and inequalities.

Suppose the height of a water balloon $h(t)$ in meters above the ground $t$ seconds after being launched is modeled by the quadratic function $h(t) = -4.9t^2 + 32t + 1.2$. You can solve a quadratic inequality to determine how long the balloon will be a certain distance above the ground.

### NewVocabulary

quadratic inequality

### Common Core State Standards

**Content Standards**

**A.CED.1** Create equations and inequalities in one variable and use them to solve problems.

**A.CED.3** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

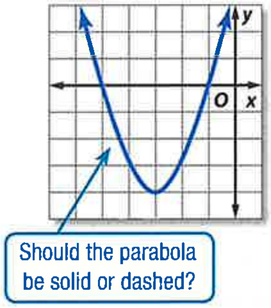
**Mathematical Practices**

1 Make sense of problems and persevere in solving them.

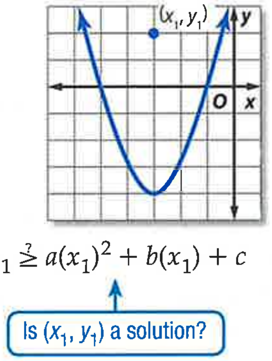
## 1 Graph Quadratic Inequalities

You can graph **quadratic inequalities** in two variables by using the same techniques used to graph linear inequalities in two variables.

**Step 1** Graph the related function.

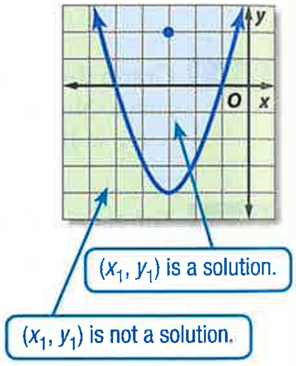


**Step 2** Test a point not on the parabola.



$$y_1 \overset{?}{\geqq} a(x_1)^2 + b(x_1) + c$$

**Step 3** Shade accordingly.



### Example 1 Graph a Quadratic Inequality

PT

Graph $y > x^2 + 2x + 1$.

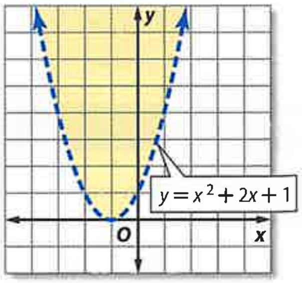
**Step 1** Graph the related function, $y = x^2 + 2x + 1$. The parabola should be dashed.

**Step 2** Test a point not on the graph of the parabola.

$$y > x^2 + 2x + 1$$
$$-1 \overset{?}{>} 0^2 + 2(0) + 1$$
$$-1 \not> 1$$

So, $(0, -1)$ is *not* a solution of the inequality.

**Step 3** Shade the region that does not contain the point $(0, -1)$.



### GuidedPractice

**Graph each inequality.**

**1A.** $y \leq x^2 + 2x + 4$

**1B.** $y < -2x^2 + 3x + 5$

madelyn mulvaney/Flickr/Getty Images

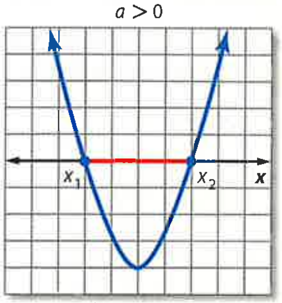
## 2 Solve Quadratic Inequalities

Quadratic inequalities in one variable can be solved using the graphs of the related quadratic functions.

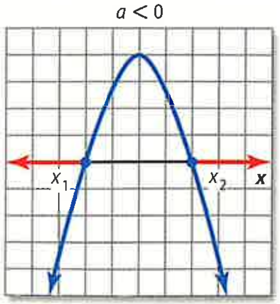
**$ax^2 + bx + c < 0$**

Graph $y = ax^2 + bx + c$ and identify the $x$-values for which the graph lies *below* the $x$-axis.

For $\leq$, include the $x$-intercepts in the solution.



$\{x \mid x_1 < x < x_2\}$

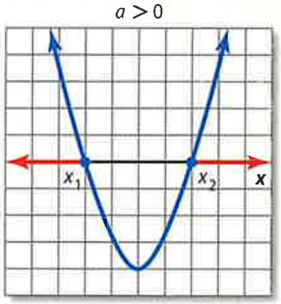


$\{x \mid x < x_1 \text{ or } x > x_2\}$

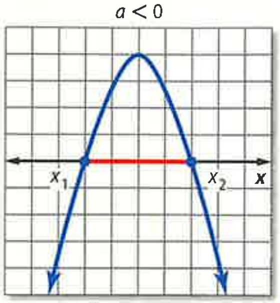
**$ax^2 + bx + c > 0$**

Graph $y = ax^2 + bx + c$ and identify the $x$-values for which the graph lies *above* the $x$-axis.

For $\geq$, include the $x$-intercepts in the solution.



$\{x \mid x < x_1 \text{ or } x > x_2\}$



$\{x \mid x_1 < x < x_2\}$

> **StudyTip**
>
> **Solving Quadratic Inequalities by Graphing**
>
> A precise graph of the related quadratic function is not necessary since the zeros of the function were found algebraically.

### Example 2 Solve $ax^2 + bx + c < 0$ by Graphing

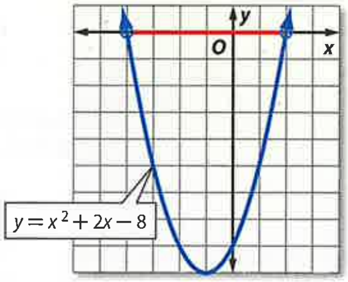
**Solve $x^2 + 2x - 8 < 0$ by graphing.**

The solution consists of $x$-values for which the graph of the related function lies *below* the $x$-axis. Begin by finding the roots of the related function.

| | |
|---|---|
| $x^2 + 2x - 8 = 0$ | Related equation |
| $(x - 2)(x + 4) = 0$ | Factor. |
| $x - 2 = 0 \quad \text{or} \quad x + 4 = 0$ | Zero Product Property |
| $x = 2 \qquad\qquad x = -4$ | Solve each equation. |

Sketch the graph of a parabola that has $x$-intercepts at $-4$ and 2. The graph should open up because $a > 0$.

The graph lies below the $x$-axis between $x = -4$ and $x = 2$. Thus, the solution set is $\{x \mid -4 < x < 2\}$ or $(-4, 2)$.



**CHECK** Test one value of $x$ less than $-4$, one between $-4$ and 2, and one greater than 2 in the original inequality.

**Test $x = -6$.**

$x^2 + 2x - 8 < 0$

$(-6)^2 + 2(-6) - 8 \overset{?}{<} 0$

$16 < 0$ ✗

**Test $x = 0$.**

$x^2 + 2x - 8 < 0$

$0^2 + 2(0) - 8 \overset{?}{<} 0$

$-8 < 0$ ✓

**Test $x = 5$.**

$x^2 + 2x - 8 < 0$

$5^2 + 2(5) - 8 \overset{?}{<} 0$

$27 < 0$ ✗

### GuidedPractice

**Solve each inequality by graphing.**

**2A.** $0 > x^2 + 5x - 6$

**2B.** $-x^2 + 3x + 10 \leq 0$

madelyn mulvaney/Flickr/Getty Images

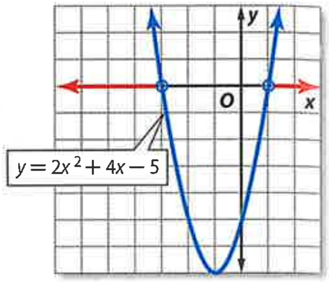
## Example 3 Solve $ax^2 + bx + c \geq 0$ by Graphing

**Solve $2x^2 + 4x - 5 \geq 0$ by graphing.**

The solution consists of $x$-values for which the graph of the related function lies *on and above* the $x$-axis. Begin by finding the roots of the related function.

| | |
|---|---|
| $2x^2 + 4x - 5 = 0$ | Related equation |
| $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | Use the Quadratic Formula |
| $x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-5)}}{2(2)}$ | Replace $a$ with 4, $b$ with 2, and $c$ with −5. |
| $x = \frac{-4 + \sqrt{56}}{4}$ or $x = \frac{-4 - \sqrt{56}}{4}$ | Simplify and write as two equations. |
| $\approx 0.87$ $\qquad \approx -2.87$ | Simplify. |

Sketch the graph of a parabola with $x$-intercepts at $-2.87$ and $0.87$. The graph opens up since $a > 0$. The graph lies on and above the $x$-axis at about $x \leq -2.87$ and $x \geq 0.87$. Therefore, the solution is approximately $\{x \mid x \leq -2.87 \text{ or } x \geq 0.87\}$ or $(-\infty, -2.87] \cup [0.87, \infty)$.



### GuidedPractice

**Solve each inequality by graphing.**

**3A.** $x^2 - 6x + 2 > 0$

**3B.** $-4x^2 + 5x + 7 \geq 0$

Real-world problems can be solved by graphing quadratic inequalities.

**Real-WorldLink**

It takes just milliseconds for a water balloon to break. A high-speed camera can capture the impact on the fluid before gravity makes it fall.

**Source:** NASA

## Real-World Example 4 Solve a Quadratic Inequality

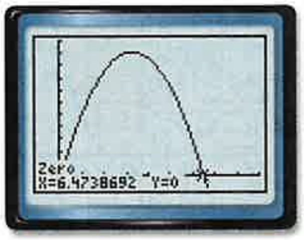
**WATER BALLOONS** **Refer to the beginning of the lesson. At what time will a water balloon be within 3 meters of the ground after it has been launched?**

The function $h(t) = -4.9t^2 + 32t + 1.2$ describes the height of the water balloon. Therefore, you want to find the values of $t$ for which $h(t) \leq 3$.

| | |
|---|---|
| $h(t) \leq 3$ | Original inequality |
| $-4.9t^2 + 32t + 1.2 \leq 3$ | $h(t) = -4.9t^2 + 32t + 1.2$ |
| $-4.9t^2 + 32t - 1.8 \leq 0$ | Subtract 3 from each side. |

Graph the related function $y = -4.9x^2 + 32x - 1.8$ using a graphing calculator. The zeros of the function are about 0.06 and 6.47, and the graph lies below the $x$-axis when $x < 0.06$ and $x > 6.47$.

So, the water balloon is within 3 meters of the ground during the first 0.06 second after being launched and again after about 6.47 seconds until it hits the ground.



Zero
X=6.4738692 Y=0

[−1, 9] scl: 1 by [−5, 55] scl: 5

### GuidedPractice

4. **ROCKETS** The height $h(t)$ of a model rocket in feet $t$ seconds after its launch can be represented by the function $h(t) = -16t^2 + 82t + 0.25$. During what interval is the rocket at least 100 feet above the ground?

RubberBall/Alamy

**Study Tip**

**Solving Quadratic Inequalities Algebraically** If the roots of the related quadratic equation are complex, the solution of the inequality will be all real numbers or the empty set. Testing one point will help you determine the solution.

## Example 5 Solve a Quadratic Inequality Algebraically

**Solve $x^2 - 3x \leq 18$ algebraically.**

**Step 1** Solve the related quadratic equation $x^2 - 3x = 18$.

$x^2 - 3x = 18$ Related quadratic equation

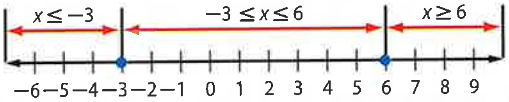
$x^2 - 3x - 18 = 0$ Subtract 18 from each side.

$(x + 3)(x - 6) = 0$ Factor.

$x + 3 = 0$ or $x - 6 = 0$ Zero Product Property

$x = -3$ $x = 6$ Solve each equation.

**Step 2** Plot −3 and 6 on a number line. Use dots since these values are solutions of the original inequality. Notice that the number line is divided into three intervals.



**Step 3** Test a value from each interval to see if it satisfies the original inequality.

| $x \leq -3$ | $-3 \leq x \leq 6$ | $x \geq 6$ |
|---|---|---|
| Test $x = -5$. | Test $x = 0$. | Test $x = 8$. |
| $x^2 - 3x \leq 18$ | $x^2 - 3x \leq 18$ | $x^2 - 3x \leq 18$ |
| $(-5)^2 - 3(-5) \overset{?}{\leq} 18$ | $(0)^2 - 3(0) \overset{?}{\leq} 18$ | $(8)^2 - 3(8) \overset{?}{\leq} 18$ |
| $40 \nleq 18$ | $0 \leq 18$ | $40 \nleq 18$ |

The solution set is $\{x \mid -3 \leq x \leq 6\}$ or $[-3, 6]$.

### Guided Practice

**Solve each inequality algebraically.**

**5A.** $x^2 + 5x < -6$

**5B.** $x^2 + 11x + 30 \geq 0$

## Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

**Example 1** **Graph each inequality.**

**1.** $y \leq x^2 - 8x + 2$

**2.** $y > x^2 + 6x - 2$

**3.** $y \geq -x^2 + 4x + 1$

**Examples 2–3** CCSS **SENSE-MAKING** **Solve each inequality by graphing.**

**4.** $0 < x^2 - 5x + 4$

**5.** $x^2 + 8x + 15 < 0$

**6.** $-2x^2 - 2x + 12 \geq 0$

**7.** $0 \geq 2x^2 - 4x + 1$

**Example 4**

**8.** **SOCCER** A midfielder kicks a ball toward the goal during a match. The height of the ball in feet above the ground $h(t)$ at time $t$ can be represented by $h(t) = -0.1t^2 + 2.4t + 1.5$. If the height of the goal is 8 feet, at what time during the kick will the ball be able to enter the goal?

**Example 5** **Solve each inequality algebraically.**

**9.** $x^2 + 6x - 16 < 0$

**10.** $x^2 - 14x > -49$

**11** $-x^2 + 12x \geq 28$

**12.** $x^2 - 4x \leq 21$

RubberBall/Alamy

## Practice and Problem Solving

Extra Practice is on page R4.

**Example 1** Graph each inequality.

**13.** $y \geq x^2 + 5x + 6$

**14.** $x^2 - 2x - 8 < y$

**15.** $y \leq -x^2 - 7x + 8$

**16.** $-x^2 + 12x - 36 > y$

**17.** $y > 2x^2 - 2x - 3$

**18.** $y \geq -4x^2 + 12x - 7$

**Examples 2–3** Solve each inequality by graphing.

**19.** $x^2 - 9x + 9 < 0$

**20.** $x^2 - 2x - 24 \leq 0$

**21.** $x^2 + 8x + 16 \geq 0$

**22.** $x^2 + 6x + 3 > 0$

**23.** $0 > -x^2 + 7x + 12$

**24.** $-x^2 + 2x - 15 < 0$

**25.** $4x^2 + 12x + 10 \leq 0$

**26.** $-3x^2 - 3x + 9 > 0$

**27.** $0 > -2x^2 + 4x + 4$

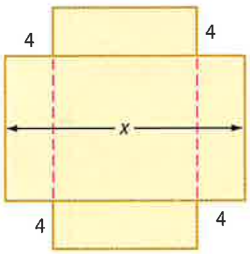
**28.** $3x^2 + 12x + 36 \leq 0$

**29.** $0 \leq -4x^2 + 8x + 5$

**30.** $-2x^2 + 3x + 3 \leq 0$

**Example 4** **31** **ARCHITECTURE** An arched entry of a room is shaped like a parabola that can be represented by the equation $f(x) = -x^2 + 6x + 1$. How far from the sides of the arch is its height at least 7 feet?

**32.** **MANUFACTURING** A box is formed by cutting 4-inch squares from each corner of a square piece of cardboard and then folding the sides. If $V(x) = 4x^2 - 64x + 256$ represents the volume of the box, what should the dimensions of the original piece of cardboard be if the volume of the box cannot exceed 750 cubic inches?



**Example 5** Solve each inequality algebraically.

**33.** $x^2 - 9x < -20$

**34.** $x^2 + 7x \geq -10$

**35.** $2 > x^2 - x$

**36.** $-3 \leq -x^2 - 4x$

**37.** $-x^2 + 2x \leq -10$

**38.** $-6 > x^2 + 4x$

**39.** $2x^2 + 4 \geq 9$

**40.** $3x^2 + x \geq -3$

**41.** $-4x^2 + 2x < 3$

**42.** $-11 \geq -2x^2 - 5x$

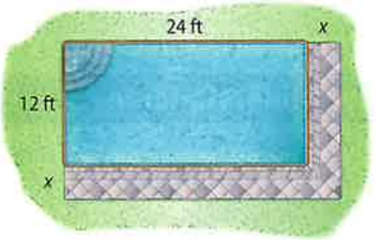
**43.** $-12 < -5x^2 - 10x$

**44.** $-3x^2 - 10x > -1$

**45.** **CCSS PERSEVERANCE** The Sanchez family is adding a deck along two sides of their swimming pool. The deck width will be the same on both sides and the total area of the pool and deck cannot exceed 750 square feet.

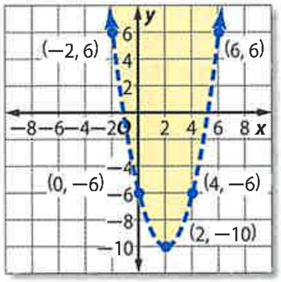
**a.** Graph the quadratic inequality.

**b.** Determine the possible widths of the deck.

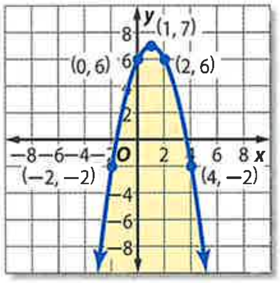


**Write a quadratic inequality for each graph.**

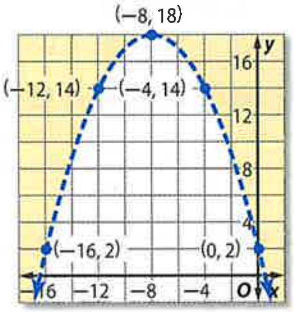
**46.**



**47.**



**48.**



**Solve each quadratic inequality by using a graph, a table, or algebraically.**

**49.** $-2x^2 + 12x < -15$

**50.** $5x^2 + x + 3 \geq 0$

**(51)** $11 \leq 4x^2 + 7x$

**52.** $x^2 - 4x \leq -7$

**53.** $-3x^2 + 10x < 5$

**54.** $-1 \geq -x^2 - 5x$

**55.** **BUSINESS** An electronics manufacturer uses the function $P(x) = x(-27.5x + 3520) + 20{,}000$ to model their monthly profits when selling $x$ thousand digital audio players.

**a.** Graph the quadratic inequality for a monthly profit of at least \$100,000.

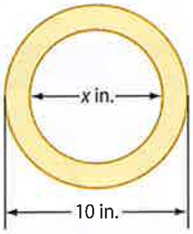
**b.** How many digital audio players must the manufacturer sell to earn a profit of at least \$100,000 in a month?

**c.** Suppose the manufacturer has an additional monthly expense of \$25,000. Explain how this affects the graph of the profit function. Then determine how many digital audio players the manufacturer needs to sell to have at least \$100,000 in profits.

**56.** **UTILITIES** A contractor is installing drain pipes for a shopping center's parking lot. The outer diameter of the pipe is to be 10 inches. The cross sectional area of the pipe must be at least 35 square inches and should not be more than 42 square inches.

**a.** Graph the quadratic inequalities.

**b.** What thickness of drain pipe can the contractor use?



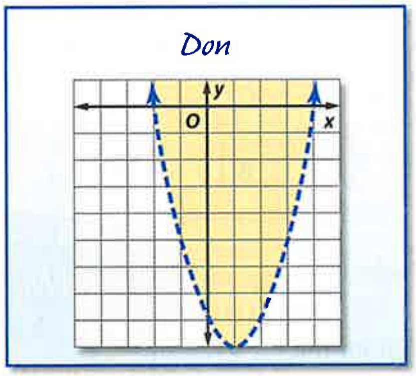
## H.O.T. Problems Use Higher-Order Thinking Skills

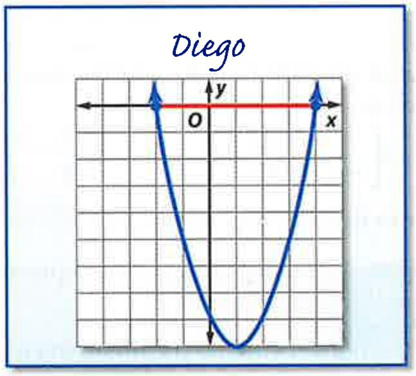
**57.** **OPEN ENDED** Write a quadratic inequality for each condition.

**a.** The solution set is all real numbers.

**b.** The solution set is the empty set.

**58.** **CCSS CRITIQUE** Don and Diego used a graph to solve the quadratic inequality $x^2 - 2x - 8 > 0$. Is either of them correct? Explain.





**59.** **REASONING** Are the boundaries of the solution set of $x^2 + 4x - 12 \leq 0$ twice the value of the boundaries of $\frac{1}{2}x^2 + 2x - 6 \leq 0$? Explain.

**60.** **REASONING** Determine if the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

*The intersection of* $y \leq -ax^2 + c$ *and* $y \geq ax^2 - c$ *is the empty set.*

**61.** **CHALLENGE** Graph the intersection of the graphs of $y \leq -x^2 + 4$ and $y \geq x^2 - 4$.

**62.** **WRITING IN MATH** How are the techniques used when solving quadratic inequalities and quadratic equations similar? different?

## Standardized Test Practice

**63. GRIDDED RESPONSE** You need to seed an area that is 80 feet by 40 feet. Each bag of seed can cover 25 square yards of land. How many bags of seed will you need?

**64. SAT/ACT** The product of two integers is between 107 and 116. Which of the following *cannot* be one of the integers?

A 5
B 10
C 12
D 15
E 23

**65 PROBABILITY** Five students are to be arranged side by side with the tallest student in the center and the two shortest students on the ends. If no two students are the same height, how many different arrangements are possible?

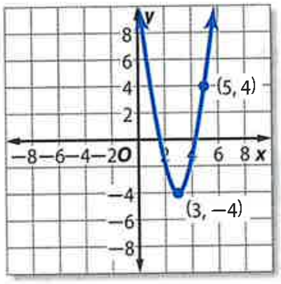
F 2
G 4
H 5
J 6

**66. SHORT RESPONSE** Simplify $\frac{5 + i}{6 - 3i}$.

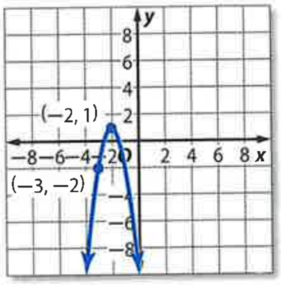
## Spiral Review

**Write an equation in vertex form for each parabola.** (Lesson 4-7)

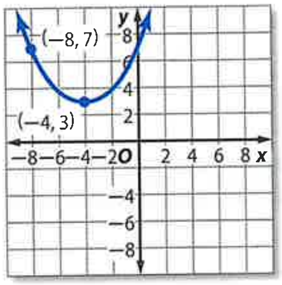
**67.**



**68.**



**69.**



**Complete parts a and b for each quadratic equation.**

**a. Find the value of the discriminant.**

**b. Describe the number and type of roots.** (Lesson 4-6)

**70.** $4x^2 + 7x - 3 = 0$

**71.** $-3x^2 + 2x - 4 = 9$

**72.** $6x^2 + x - 4 = 12$

**Perform the indicated operation. If the matrix does not exist, write *impossible*.** (Lesson 3-5)

**73.** $4\begin{bmatrix} 3 & -6 \\ -5 & 2 \end{bmatrix} - 3\begin{bmatrix} 4 & -1 \\ -2 & 8 \end{bmatrix}$

**74.** $-2\begin{bmatrix} 5 & -9 \\ 5 & 11 \end{bmatrix} - 6\begin{bmatrix} 3 & -7 \\ -5 & 8 \end{bmatrix}$

**75.** $\begin{bmatrix} 2 & -6 \\ -4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix}$

**76. EXERCISE** Refer to the graphic. (Lesson 3-1)

**a.** For each option, write an equation that represents the cost of belonging to the gym.

**b.** Graph the equations. Estimate the break-even point for the gym memberships.

**c.** Explain what the break-even point means.

**d.** If you plan to visit the gym at least once per week during the year, which option should you chose?

## Skills Review

**Use the Distributive Property to find each product.**

**77.** $-6(x - 4)$

**78.** $8(w + 3x)$

**79.** $-4(-2y + 3z)$

**80.** $-1(c - d)$

**81.** $0.5(5x + 6y)$

**82.** $-3(-6y - 4z)$

# CHAPTER 4 Study Guide and Review

## Study Guide

### KeyConcepts

**Graphing Quadratic Functions** (Lesson 4-1)

- The graph of $y = ax^2 + bx + c$, $a \neq 0$, opens up, and the function has a minimum value when $a > 0$. The graph opens down, and the function has a maximum value when $a < 0$.

**Solving Quadratic Equations** (Lessons 4-2 and 4-3)

- Roots of a quadratic equation are the zeros of the related quadratic function. You can find the zeros of a quadratic function by finding the $x$-intercepts of the graph.

**Complex Numbers** (Lesson 4-4)

- $i$ is the imaginary unit; $i^2 = -1$ and $i = \sqrt{-1}$.

**Solving Quadratic Equations** (Lessons 4-5 and 4-6)

- Completing the square: **Step 1** Find one half of $b$, the coefficient of $x$. **Step 2** Square the result in Step 1. **Step 3** Add the result of Step 2 to $x^2 + bx$.
- Quadratic Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

**Transformations of Quadratic Graphs** (Lesson 4-7)

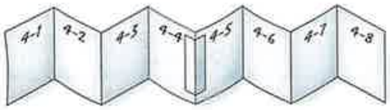
- The graph of $y = (x - h)^2 + k$ is the graph of $y = x^2$ translated $|h|$ units left if $h$ is negative or $h$ units right if $h$ is positive and $k$ units up if $k$ is positive or $|k|$ units down if $k$ is negative.
- Consider $y = a(x - h)^2 + k$, $a \neq 0$. If $a > 0$, the graph opens up; if $a < 0$ the graph opens down. If $|a| > 1$, the graph is narrower than the graph of $y = x^2$. If $|a| < 1$, the graph is wider than the graph of $y = x^2$.

**Quadratic Inequalities** (Lesson 4-8)

- Graph the related function, test a point *not* on the parabola and determine if it is a solution, and shade accordingly.

### FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.



### KeyVocabulary

**axis of symmetry** (p. 220)
**completing the square** (p. 257)
**complex conjugates** (p. 249)
**complex number** (p. 247)
**constant term** (p. 219)
**discriminant** (p. 267)
**factored form** (p. 238)
**FOIL method** (p. 238)
**imaginary unit** (p. 246)
**linear term** (p. 219)
**maximum value** (p. 222)
**minimum value** (p. 222)
**parabola** (p. 219)
**pure imaginary number** (p. 246)
**quadratic equation** (p. 229)
**Quadratic Formula** (p. 264)
**quadratic function** (p. 219)
**quadratic inequality** (p. 282)
**quadratic term** (p. 219)
**root** (p. 229)
**Square Root Property** (p. 247)
**standard form** (p. 229)
**vertex** (p. 220)
**vertex form** (p. 275)
**zero** (p. 229)

### VocabularyCheck

**State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.**

1. The factored form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ and $a$, $b$, and $c$ are integers.
2. The graph of a quadratic function is called a parabola.
3. The vertex form of a quadratic function is $y = a(x - p)(x - q)$.
4. The axis of symmetry will intersect a parabola in one point called the vertex.
5. A method called FOIL method is used to make a quadratic expression a perfect square in order to solve the related equation.
6. The equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as the discriminant.
7. The number $6i$ is called a pure imaginary number.
8. The two numbers $2 + 3i$ and $2 - 3i$ are called complex conjugates.

# Lesson-by-Lesson Review

## 4-1 Graphing Quadratic Functions

Complete parts **a–c** for each quadratic function.

**a.** Find the *y*-intercept, the equation of the axis of symmetry, and the *x*-coordinate of the vertex.

**b.** Make a table of values that includes the vertex.

**c.** Use this information to graph the function.

**9.** $f(x) = x^2 + 5x + 12$ **10.** $f(x) = x^2 - 7x + 15$

**11.** $f(x) = -2x^2 + 9x - 5$ **12.** $f(x) = -3x^2 + 12x - 1$

Determine whether each function has a maximum or minimum value and find the maximum or minimum value. Then state the domain and range of the function.

**13.** $f(x) = -x^2 + 3x - 1$ **14.** $f(x) = -3x^2 - 4x + 5$

**15.** **BUSINESS** Sal's Shirt Store sells 100 T-shirts per week at a rate of \$10 per shirt. Sal estimates that he will sell 5 fewer shirts for each \$1 increase in price. What price will maximize Sal's T-shirt income?

### Example 1

**Consider the quadratic function $f(x) = x^2 - 4x + 11$. Find the *y*-intercept, the equation for the axis of symmetry, and the *x*-coordinate of the vertex.**

In the function, $a = 1$, $b = -4$, and $c = 11$. The *y*-intercept is $c = 11$.

Use $a$ and $b$ to find the equation of the axis of symmetry.

$x = -\frac{b}{2a}$ Equation of the axis of symmetry

$= -\frac{-4}{2(1)}$ $a = 1$ and $b = -4$

$= 2$ Simplify.

The equation of the axis of symmetry is $x = 2$. Therefore, the *x*-coordinate of the vertex is 2.

## 4-2 Solving Quadratic Equations by Graphing

Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

**16.** $x^2 - x - 20 = 0$

**17.** $2x^2 - x - 3 = 0$

**18.** $4x^2 - 6x - 15 = 0$

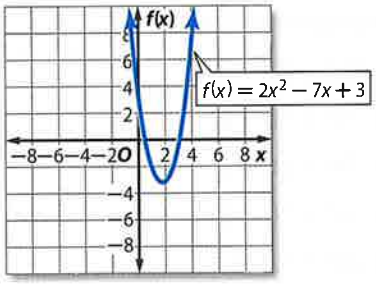
**19.** **BASEBALL** A baseball is hit upward at 120 feet per second. Use the formula $h(t) = v_0t - 16t^2$, where $h(t)$ is the height of an object in feet, $v_0$ is the object's initial velocity in feet per second, and $t$ is the time in seconds. Ignoring the height of the ball when it was hit, how long does it take for the ball to hit the ground?

### Example 2

**Solve $2x^2 - 7x + 3 = 0$ by graphing.**

The equation of the axis of symmetry is $-\frac{-7}{2(2)}$ or $x = \frac{7}{4}$.

| $x$ | 0 | 1 | $\frac{7}{4}$ | 2 | 3 |
|---|---|---|---|---|---|
| $f(x)$ | 3 | −2 | $-2\frac{5}{8}$ | −3 | 0 |



The zeros of the related function are $\frac{1}{2}$ and 3. Therefore, the solutions of the equation are $\frac{1}{2}$ and 3.

# Study Guide and Review *Continued*

## 4-3 Solving Quadratic Equations by Factoring

Write a quadratic equation in standard form with the given roots.

**20.** 5, 6

**21.** −3, −7

**22.** −4, 2

**23.** $-\frac{2}{3}, 1$

**24.** $\frac{1}{6}, 5$

**25.** $-\frac{1}{4}, -1$

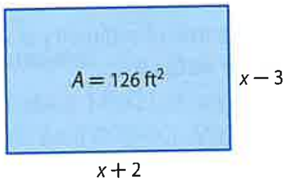
Solve each equation by factoring.

**26.** $2x^2 - 2x - 24 = 0$

**27.** $2x^2 - 5x - 3 = 0$

**28.** $3x^2 - 16x + 5 = 0$

**29.** Find $x$ and the dimensions of the rectangle below.



### Example 3

Write a quadratic equation in standard form with $-\frac{1}{2}$ and 4 as its roots.

| | |
|---|---|
| $(x - p)(x - q) = 0$ | Write the pattern. |
| $\left[x - \left(-\frac{1}{2}\right)\right](x - 4) = 0$ | Replace $p$ with $-\frac{1}{2}$ and $q$ with 4. |
| $\left(x + \frac{1}{2}\right)(x - 4) = 0$ | Simplify. |
| $x^2 - \frac{7}{2}x - 2 = 0$ | Multiply. |
| $2x^2 - 7x - 4 = 0$ | Multiply each side by 2 so that $b$ and $c$ are integers. |

### Example 4

**Solve $2x^2 - 3x - 5 = 0$ by factoring.**

| | |
|---|---|
| $2x^2 - 3x - 5 = 0$ | Original equation |
| $(2x - 5)(x + 1) = 0$ | Factor the trinomial. |
| $2x - 5 = 0$ or $x + 1 = 0$ | Zero Product Property |
| $x = \frac{5}{2}$ $\quad$ $x = -1$ | |

The solution set is $\left\{-1, \frac{5}{2}\right\}$ or $\left\{x \mid x = -1, \frac{5}{2}\right\}$.

## 4-4 Complex Numbers

Simplify.

**30.** $\sqrt{-8}$

**31.** $(2 - i) + (13 + 4i)$

**32.** $(6 + 2i) - (4 - 3i)$

**33.** $(6 + 5i)(3 - 2i)$

**34.** **ELECTRICITY** The impedance in one part of a series circuit is $3 + 2j$ ohms, and the impedance in the other part of the circuit is $4 - 3j$ ohms. Add these complex numbers to find the total impedance in the circuit.

Solve each equation.

**35.** $2x^2 + 50 = 0$

**36.** $4x^2 + 16 = 0$

**37.** $3x^2 + 15 = 0$

**38.** $8x^2 + 16 = 0$

**39.** $4x^2 + 1 = 0$

### Example 5

**Simplify $(12 + 3i) - (-5 + 2i)$.**

| | |
|---|---|
| $(12 + 3i) - (-5 + 2i)$ | |
| $= [12 - (-5)] + (3 - 2)i$ | Group the real and imaginary parts. |
| $= 17 + i$ | Simplify. |

### Example 6

**Solve $3x^2 + 12 = 0$.**

| | |
|---|---|
| $3x^2 + 12 = 0$ | Original equation |
| $3x^2 = -12$ | Subtract 12 from each side. |
| $x^2 = -4$ | Divide each side by 3. |
| $x = \pm\sqrt{-4}$ | Square Root Property |
| $x = \pm 2i$ | $\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1}$ |

## 4-5 Completing the Square

Find the value of *c* that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

**40.** $x^2 + 18x + c$

**41.** $x^2 - 4x + c$

**42.** $x^2 - 7x + c$

**43.** $x^2 + 2.4x + c$

**44.** $x^2 - \frac{1}{2}x + c$

**45.** $x^2 + \frac{6}{5}x + c$

Solve each equation by completing the square.

**46.** $x^2 - 6x - 7 = 0$

**47.** $x^2 - 2x + 8 = 0$

**48.** $2x^2 + 4x - 3 = 0$

**49.** $2x^2 + 3x - 5 = 0$

**50.** **FLOOR PLAN** Mario's living room has a length 6 feet wider than the width. The area of the living room is 280 square feet. What are the dimensions of his living room?

### Example 7

**Find the value of *c* that makes $x^2 + 14x + c$ a perfect square. Then write the trinomial as a perfect square.**

**Step 1** Find one half of 14.

**Step 2** Square the result of Step 1.

**Step 3** Add the result of Step 2 to $x^2 + 14x$.

The trinomial $x^2 + 14x + 49$ can be written as $(x + 7)^2$.

### Example 8

**Solve $x^2 + 12x - 13 = 0$ by completing the square.**

$$x^2 + 12x - 13 = 0$$
$$x^2 + 12x = 13$$
$$x^2 + 12x + 36 = 13 + 36$$
$$(x + 6)^2 = 49$$
$$x + 6 = \pm 7$$
$$x + 6 = 7 \quad \text{or} \quad x + 6 = -7$$
$$x = 1 \qquad\qquad x = -13$$

The solution set is $\{-13, 1\}$ or $\{x \mid x = -13, 1\}$.

## 4-6 The Quadratic Formula and the Discriminant

Complete parts **a–c** for each quadratic equation.

**a.** Find the value of the discriminant.

**b.** Describe the number and type of roots.

**c.** Find the exact solutions by using the Quadratic Formula.

**51.** $x^2 - 10x + 25 = 0$

**52.** $x^2 + 4x - 32 = 0$

**53.** $2x^2 + 3x - 18 = 0$

**54.** $2x^2 + 19x - 33 = 0$

**55.** $x^2 - 2x + 9 = 0$

**56.** $4x^2 - 4x + 1 = 0$

**57.** $2x^2 + 5x + 9 = 0$

**58.** **PHYSICAL SCIENCE** Lauren throws a ball with an initial velocity of 40 feet per second. The equation for the height of the ball is $h = -16t^2 + 40t + 5$, where *h* represents the height in feet and *t* represents the time in seconds. When will the ball hit the ground?

### Example 9

**Solve $x^2 - 4x - 45 = 0$ by using the Quadratic Formula.**

In $x^2 - 4x - 45 = 0$, $a = 1$, $b = -4$, and $c = -45$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-45)}}{2(1)}$$

$$= \frac{4 \pm 14}{2}$$

Write as two equations.

$$x = \frac{4 + 14}{2} \quad \text{or} \quad x = \frac{4 - 14}{2}$$
$$= 9 \qquad\qquad = -5$$

The solution set is $\{-5, 9\}$ or $\{x \mid x = -5, 9\}$.

## 4-7 Transformations of Quadratic Graphs

Write each quadratic function in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening. Then graph the function.

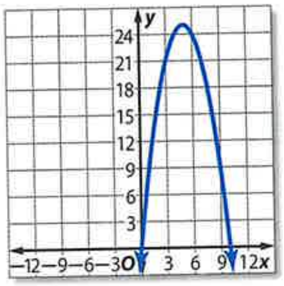
**59.** $y = -3(x-1)^2 + 5$

**60.** $y = 2x^2 + 12x - 8$

**61.** $y = -\frac{1}{2}x^2 - 2x + 12$

**62.** $y = 3x^2 + 36x + 25$

**63.** The graph at the right shows a product of 2 numbers with a sum of 10. Find a function that models this product and use it to determine the two numbers that would give a maximum product.



### Example 10

Write the quadratic function $y = 3x^2 + 24x + 15$ in vertex form. Then identify the vertex, axis of symmetry, and direction of opening.

$y = 3x^2 + 24x + 15$ Original equation

$y = 3(x^2 + 8x) + 15$ Group and factor.

$y = 3(x^2 + 8x + 16) + 15 - 3(16)$ Complete the square.

$y = 3(x + 4)^2 - 33$ Rewrite $x^2 + 8x + 16$ as a perfect square.

So, $a = 3$, $h = -4$, and $k = -33$. The vertex is at $(-4, -33)$ and the axis of symmetry is $x = -4$. Since $a$ is positive, the graph opens up.

## 4-8 Quadratic Inequalities

Graph each quadratic inequality.

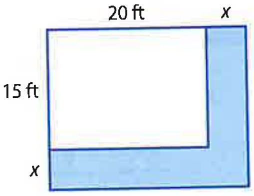
**64.** $y \geq x^2 + 5x + 4$

**65.** $y < -x^2 + 5x - 6$

**66.** $y > x^2 - 6x + 8$

**67.** $y \leq x^2 + 10x - 4$

**68.** Solomon wants to put a deck along two sides of his garden. The deck width will be the same on both sides and the total area of the garden and deck cannot exceed 500 square feet. How wide can the deck be?



Solve each inequality using a graph or algebraically.

**69.** $x^2 + 8x + 12 > 0$

**70.** $6x + x^2 \geq -9$

**71.** $2x^2 + 3x - 20 > 0$

**72.** $4x^2 - 3 < -5x$

**73.** $3x^2 + 4 > 8x$

### Example 11

Graph $y > x^2 + 3x + 2$.

**Step 1** Graph the related function, $y > x^2 + 3x + 2$. Because the inequality symbol $>$ is used, the parabola should be dashed.

**Step 2** Test a point not on the graph of the parabola such as (0, 0).

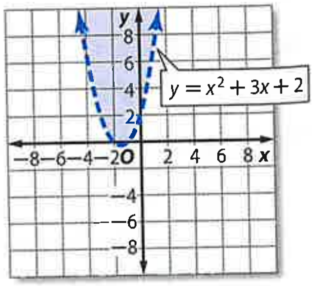
$y > x^2 + 3x + 2$

$(0) \overset{?}{>} (0)^2 + 3(0) + 2$

$0 \not> 2$

So, (0, 0) is not a solution of the inequality.

**Step 3** Shade the region that does not contain the point (0, 0).



# CHAPTER 4 Practice Test

**Complete parts *a–c* for each quadratic function.**

**a.** **Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex.**

**b.** **Make a table of values that includes the vertex.**

**c.** **Use this information to graph the function.**

**1.** $f(x) = x^2 + 4x - 7$

**2.** $f(x) = -2x^2 + 5x$

**3.** $f(x) = -x^2 - 6x - 9$

**Determine whether each function has a maximum or minimum value. State the maximum or minimum value of each function.**

**4.** $f(x) = x^2 + 10x + 25$

**5.** $f(x) = -x^2 + 6x$

**Solve each equation using the method of your choice. Find exact solutions.**

**6.** $x^2 - 8x - 9 = 0$

**7.** $-4.8x^2 + 1.6x + 24 = 0$

**8.** $12x^2 + 15x - 4 = 0$

**9.** $x^2 - 7x - \frac{17}{4} = 0$

**10.** $4x^2 + x = 3$

**11.** $-9x^2 + 40x + 84 = 0$

**12.** **PHYSICAL SCIENCE** Parker throws a ball off the top of a building. The building is 350 feet high and the initial velocity of the ball is 30 feet per second. Find out how long it will take the ball to hit the ground by solving the equation $-16t^2 - 30t + 350 = 0$.

**13.** **MULTIPLE CHOICE** Which equation below has roots at $-6$ and $\frac{1}{5}$?

A $0 = 5x^2 - 29x - 6$

B $0 = 5x^2 + 31x + 6$

C $0 = 5x^2 + 29x - 6$

D $0 = 5x^2 - 31x + 6$

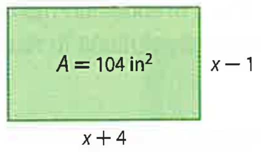
**14.** **PHYSICS** A ball is thrown into the air vertically with a velocity of 112 feet per second. The ball was released 6 feet above the ground. The height above the ground $t$ seconds after release is modeled by $h(t) = -16t^2 + 112t + 6$.

**a.** When will the ball reach 130 feet?

**b.** Will the ball ever reach 250 feet? Explain.

**c.** In how many seconds after its release will the ball hit the ground?

**15.** The rectangle below has an area of 104 square inches. Find the value of $x$ and the dimensions of the rectangle.



**Simplify.**

**16.** $(3 - 4i) - (9 - 5i)$

**17.** $\frac{4i}{4 - i}$

**18.** **MULTIPLE CHOICE** Which value of $c$ makes the trinomial $x^2 - 12x + c$ a perfect square trinomial?

F 6

G 12

H 36

J 144

**Complete parts *a–c* for each quadratic equation.**

**a.** **Find the value of the discriminant.**

**b.** **Describe the number and type of roots.**

**c.** **Find the exact solution by using the Quadratic Formula.**

**19.** $6x^2 + 7x = 0$

**20.** $5x^2 = -6x + 1$

**21.** $2x^2 + 5x - 8 = -13$

**Write each quadratic function in vertex form. Then identify the vertex, axis of symmetry, and direction of opening.**

**22.** $3x^2 + 6x = 2 + y$

**23.** $x^2 + 9x + \frac{81}{4} = y$

**24.** Graph the quadratic inequality $0 < -3x^2 + 4x + 10$.

**Solve each inequality by using a graph or algebraically.**

**25.** $x^2 + 6x > -5$

**26.** $4x^2 - 19x \leq -12$