

# Rational Functions and Equations



## Then

- You simplified expressions involving monomials and polynomials.

## Now

- In this chapter, you will:
  - Identify and graph inverse variations.
  - Identify excluded values of rational functions.
  - Multiply, divide, and add rational expressions.
  - Divide polynomials.
  - Solve rational equations.

## Why? ▲

- HOCKEY** The time it will take for a puck hit from the blue line to reach the goal line is given by the rational expression  $\frac{64}{x}$ , where  $x$  is the speed of the puck in feet per seconds. If a player hits the puck at 100 miles per hour, the puck will reach the goal line in 0.34 second.

Rational Functions and Equations

Activity

A rational function can be described by an equation of the form

$$f(x) = \frac{x^2 + 3}{x}$$

$$f(x) = \frac{1}{x}$$

where  $p$  and  $q$  are polynomials and  $q \neq 0$ .

$$f(x) = \frac{25}{x + 3}$$

$$f(x) = \frac{x^2 - 2x - 1}{x - 1}$$

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Animation



Vocabulary



eGlossary



Personal Tutor



Virtual Manipulatives



Graphing Calculator



Audio



Foldables



Self-Check Practice



Worksheets



# Get Ready for the Chapter

**Diagnose Readiness** | You have two options for checking prerequisite skills.

**1 Textbook Option** Take the Quick Check below. Refer to the Quick Review for help.

## QuickCheck

Solve each proportion.

1.  $\frac{y}{3} = \frac{8}{9}$

2.  $\frac{5}{12} = \frac{x}{36}$

3.  $\frac{7}{2} = \frac{y}{3}$

4.  $\frac{5}{x} = \frac{10}{4}$

5. **DRAWING** Rosie is making a scale drawing. She is using the scale 1 inch = 3 feet. How many inches will represent 10 feet?

Factor each polynomial, if possible. If the polynomial cannot be factored using integers, write *prime*.

6.  $4a - 12$

7.  $2x^2 - 4x$

8.  $6xy + 15x$

9.  $2c^2d - 4c^2d^2$

10.  $d^2 - 12d + 27$

11.  $m^2 - m - 132$

12.  $3x^2 + 2x - 1$

13.  $6x^2 - 5x - 4$

14. **AREA** The area of a rectangle is  $x^2 + 5x + 6$ . What binomial expressions could represent the side lengths of the rectangle?

$$A = x^2 + 5x + 6$$

## QuickReview



### Example 1

Solve  $\frac{3}{5} = \frac{x}{12}$ .

$$\frac{3}{5} = \frac{x}{12}$$

Original equation

$$3 \cdot 12 = 5 \cdot x$$

Cross products

$$36 = 5x$$

Simplify.

$$\frac{36}{5} = \frac{5x}{5}$$

Divide each side by 5.

$$\frac{36}{5} = x$$

Simplify.

### Example 2

Factor  $45x^2 + 27x$ .

Find the GCF of the terms.

$$45x^2 = 3 \cdot 3 \cdot 5 \cdot x \cdot x$$

$$27x = 3 \cdot 3 \cdot 3 \cdot x$$

$$\text{GCF} = 3 \cdot 3 \cdot x \text{ or } 9x$$

$$45x^2 + 27x = 9x(5x) + 9x(3)$$

$$= 9x(5x + 3)$$

**2 Online Option** Take an online self-check Chapter Readiness Quiz at [connectED.mcgraw-hill.com](http://connectED.mcgraw-hill.com).



# Get Started on the Chapter

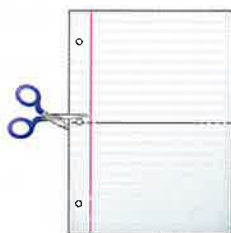
You will learn several new concepts, skills, and vocabulary terms as you study Chapter 11. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

## FOLDABLES Study Organizer



**Rational Functions and Equations** Make this Foldable to help you organize your Chapter 11 notes about rational functions and equations. Begin with 3 sheets of notebook paper.

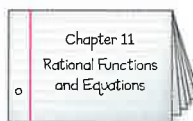
- 1 **Take** one sheet of paper and fold in half along the width. Cut 1 inch slits on each side of the paper.



- 2 **Stack** the two sheets of paper and fold in half along the width. Cut a slit through the center stopping 1 inch from each side.



- 3 **Insert** the first sheet through the second sheets and align the folds to form a booklet. Label the cover with the chapter title.



## New Vocabulary



| English                        |        | Español                        |
|--------------------------------|--------|--------------------------------|
| inverse variation              | p. 676 | variación inversa              |
| product rule                   | p. 677 | regla del producto             |
| excluded value                 | p. 684 | valores excluidos              |
| rational function              | p. 684 | función racional               |
| asymptote                      | p. 685 | asíntota                       |
| rational expression            | p. 690 | expresión racional             |
| least common multiple (LCM)    | p. 713 | mínimo común múltiplo (mcm)    |
| least common denominator (LCD) | p. 714 | mínimo común denominador (mcd) |
| complex fraction               | p. 720 | fracción compleja              |
| mixed expression               | p. 720 | expresión mixta                |
| rational equation              | p. 726 | ecuación racional              |
| extraneous solutions           | p. 727 | soluciones extrañas            |
| work problems                  | p. 728 | problemas de trabajo           |
| rate problems                  | p. 729 | problemas de tasas             |

## Review Vocabulary



**direct variation** **variación directa** an equation of the form  $y = kx$ , where  $k \neq 0$

**Quotient of Powers** **cociente de potencia**  $\frac{a^m}{a^n} = a^{m-n}$

$$\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = x \cdot x \text{ or } x^2$$

$$\frac{x^5}{x^3} = x^{5-3} \text{ or } x^2$$

**Zero Product Property** **propiedad del producto de cero** if the product of two factors is 0, then at least one of the factors must be 0

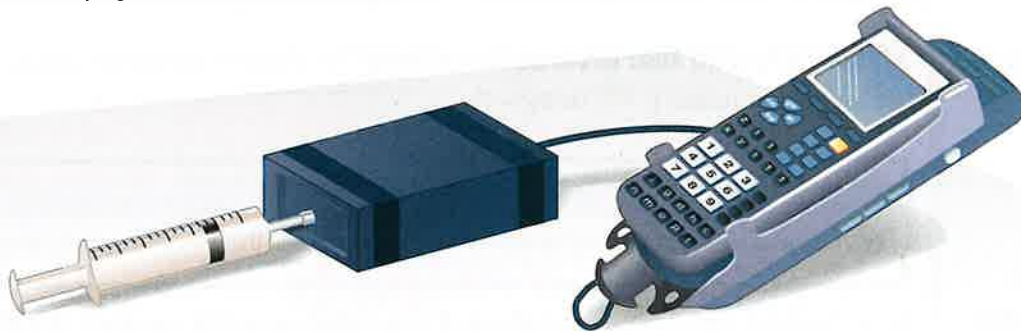
# Graphing Technology Lab Inverse Variation



You can use a data collection device to investigate the relationship between volume and pressure.

## Set Up The Lab

- Connect a syringe to the gas pressure sensor. Then connect the data collection device to both the sensor and the calculator as shown.
- Start the collection program and select the sensor.



### Activity Collect Data

- Step 1** Open the valve between the atmosphere and the syringe. Set the inside ring of the syringe to 20 mL and close the valve. This ensures that the amount of air inside the syringe will be constant throughout the experiment.
- Step 2** Press the plunger of the syringe to the 5 mL mark. Wait for the pressure gauge to stop changing, then take the data reading. Enter 5 as the volume in the calculator. The pressure is measured in atmospheres (atm).
- Step 3** Repeat step 2, pressing the plunger to 7.5 mL, 10.0 mL, 12.5 mL, 15.0 mL, 17.5 mL, and 20.0 mL. Record the volume from each data reading.
- Step 4** After taking the last data reading, use **STAT PLOT** to create a line graph.

### Exercises

1. Does the pressure vary directly as the volume? Explain.
2. As the volume changes from 10 to 20 mL, what happens to the pressure?
3. Predict what the pressure of the gas in the syringe would be if the volume increased to 40 mL.
4. Add a column to the data table to find the product of the volume and the pressure for each data reading. What pattern do you observe?
5. **MAKE A CONJECTURE** The relationship between the pressure and volume of a gas is called Boyle's Law. Write an equation relating the volume  $v$  in milliliters and pressure  $p$  in atmospheres in your experiment. Compare your conjecture to those of two classmates. Formulate mathematical questions about their conjectures.

## Inverse Variation



### Then

- You solved problems involving direct variation.

### Now

- Identify and use inverse variations.
- Graph inverse variations.

### Why?

- The time it takes a runner to finish a race is inversely proportional to the average pace of the runner. The runner's time decreases as the pace of the runner increases. So, these quantities are *inversely proportional*.



### New Vocabulary

inverse variation  
product rule



### Common Core State Standards

#### Mathematical Practices

- 1 Make sense of problems and persevere in solving them.

**1 Identify and Use Inverse Variations** An **inverse variation** can be represented by the equation  $y = \frac{k}{x}$  or  $xy = k$ .

### KeyConcept Inverse Variation

$y$  varies inversely as  $x$  if there is some nonzero constant  $k$  such that  $y = \frac{k}{x}$  or  $xy = k$ , where  $x, y \neq 0$ .

In an inverse variation, the product of two values remains constant. Recall that a relationship of the form  $y = kx$  is a *direct variation*. The constant  $k$  is called the *constant of variation* or the *constant of proportionality*.

### Example 1 Identify Inverse and Direct Variations

Determine whether each table or equation represents an *inverse* or a *direct* variation. Explain.

a.

| $x$ | $y$ |
|-----|-----|
| 1   | 16  |
| 2   | 8   |
| 4   | 4   |

In an inverse variation,  $xy$  equals a constant  $k$ . Find  $xy$  for each ordered pair in the table.

$$1 \cdot 16 = 16 \quad 2 \cdot 8 = 16 \quad 4 \cdot 4 = 16$$

The product is constant, so the table represents an inverse variation.

b.

| $x$ | $y$ |
|-----|-----|
| 1   | 3   |
| 2   | 6   |
| 3   | 9   |

Notice that  $xy$  is not constant. So, the table does not represent an indirect variation.

$$3 = k(1) \quad 6 = k(2) \quad 9 = k(3)$$

$$3 = k \quad 3 = k \quad 3 = k$$

The table of values represents the direct variation  $y = 3x$ .

c.  $x = 2y$

The equation can be written as  $y = \frac{1}{2}x$ . Therefore, it represents a direct variation.

d.  $2xy = 10$

$2xy = 10$  Write the equation.  
 $xy = 5$  Divide each side by 2.

The equation represents an inverse variation.

### Guided Practice

1A.

|     |    |   |   |
|-----|----|---|---|
| $x$ | 1  | 2 | 5 |
| $y$ | 10 | 5 | 2 |

1B.  $-2x = y$



You can use  $xy = k$  to write an inverse variation equation that relates  $x$  and  $y$ .



### ReadingMath

**Variation Equations** For direct variation equations, you say that  $y$  varies directly as  $x$ . For inverse variation equations, you say that  $y$  varies inversely as  $x$ .

### Example 2 Write an Inverse Variation

Assume that  $y$  varies inversely as  $x$ . If  $y = 18$  when  $x = 2$ , write an inverse variation equation that relates  $x$  and  $y$ .

$$\begin{aligned} xy &= k && \text{Inverse variation equation} \\ 2(18) &= k && x = 2 \text{ and } y = 18 \\ 36 &= k && \text{Simplify.} \end{aligned}$$

The constant of variation is 36. So, an equation that relates  $x$  and  $y$  is  $xy = 36$  or  $y = \frac{36}{x}$ .

### GuidedPractice

2. Assume that  $y$  varies inversely as  $x$ . If  $y = 5$  when  $x = -4$ , write an inverse variation equation that relates  $x$  and  $y$ .

If  $(x_1, y_1)$  and  $(x_2, y_2)$  are solutions of an inverse variation, then  $x_1y_1 = k$  and  $x_2y_2 = k$ .

$$\begin{aligned} x_1y_1 &= k \text{ and } x_2y_2 = k \\ x_1y_1 &= x_2y_2 && \text{Substitute } x_2y_2 \text{ for } k. \end{aligned}$$

The equation  $x_1y_1 = x_2y_2$  is called the **product rule** for inverse variations.

### KeyConcept Product Rule for Inverse Variations

**Words** If  $(x_1, y_1)$  and  $(x_2, y_2)$  are solutions of an inverse variation, then the products  $x_1y_1$  and  $x_2y_2$  are equal.

**Symbols**  $x_1y_1 = x_2y_2$  or  $\frac{x_1}{x_2} = \frac{y_2}{y_1}$



### Example 3 Solve for $x$ or $y$

Assume that  $y$  varies inversely as  $x$ . If  $y = 3$  when  $x = 12$ , find  $x$  when  $y = 4$ .

$$\begin{aligned} x_1y_1 &= x_2y_2 && \text{Product rule for inverse variations} \\ 12 \cdot 3 &= x_2 \cdot 4 && x_1 = 12, y_1 = 3, \text{ and } y_2 = 4 \\ 36 &= x_2 \cdot 4 && \text{Simplify.} \\ \frac{36}{4} &= x_2 && \text{Divide each side by 4.} \\ 9 &= x_2 && \text{Simplify.} \end{aligned}$$

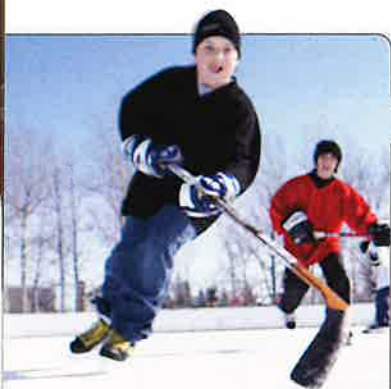
So, when  $y = 4$ ,  $x = 9$ .

### GuidedPractice

3. If  $y$  varies inversely as  $x$  and  $y = 4$  when  $x = -8$ , find  $y$  when  $x = -4$ .

The product rule for inverse variations can be used to write an equation to solve real-world problems.





### Real-WorldLink

A standard hockey puck is 1 inch thick and 3 inches in diameter. Its mass is between approximately 156 and 170 grams.

Source: NHL Rulebook

## Real-World Example 4 Use Inverse Variations

**PHYSICS** The acceleration  $a$  of a hockey puck is inversely proportional to its mass  $m$ . Suppose a hockey puck with a mass of 164 grams is hit so that it accelerates  $122 \text{ m/s}^2$ . Find the acceleration of a 158-gram hockey puck if the same amount of force is applied.

Make a table to organize the information.

Let  $m_1 = 164$ ,  $a_1 = 122$ , and  $m_2 = 158$ . Solve for  $a_2$ .

$$m_1 a_1 = m_2 a_2 \quad \text{Use the product rule to write an equation.}$$

$$164 \cdot 122 = 158 a_2 \quad m_1 = 164, a_1 = 122, \text{ and } m_2 = 158$$

$$20,008 = 158 a_2 \quad \text{Simplify.}$$

$$126.6 \approx a_2 \quad \text{Divide each side by 158 and simplify.}$$

The 158-gram puck has an acceleration of approximately  $126.6 \text{ m/s}^2$ .

| Puck | Mass  | Acceleration        |
|------|-------|---------------------|
| 1    | 164 g | $122 \text{ m/s}^2$ |
| 2    | 158 g | $a_2$               |

### Guided Practice

4. **RACING** Manuel runs an average of 8 miles per hour and finishes a race in 0.39 hour. Dyani finished the race in 0.35 hour. What was her average pace?

## 2 Graph Inverse Variations

The graph of an inverse variation is not a straight line like the graph of a direct variation.



### Example 5 Graph an Inverse Variation

Graph an inverse variation equation in which  $y = 8$  when  $x = 3$ .

**Step 1** Write an inverse variation equation.

$$xy = k \quad \text{Inverse variation equation}$$

$$3(8) = k \quad x = 3, y = 8$$

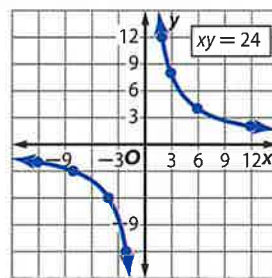
$$24 = k \quad \text{Simplify.}$$

The inverse variation equation is  $xy = 24$  or  $y = \frac{24}{x}$ .

**Step 2** Choose values for  $x$  and  $y$  that have a product of 24.

**Step 3** Plot each point and draw a smooth curve that connects the points.

| $x$ | $y$       |
|-----|-----------|
| -12 | -2        |
| -8  | -3        |
| -4  | -6        |
| -2  | -12       |
| 0   | undefined |
| 2   | 12        |
| 3   | 8         |
| 6   | 4         |
| 12  | 2         |



Notice that since  $y$  is undefined when  $x = 0$ , there is no point on the graph when  $x = 0$ . This graph is called a hyperbola.

### Guided Practice

5. Graph an inverse variation equation in which  $y = 16$  when  $x = 4$ .

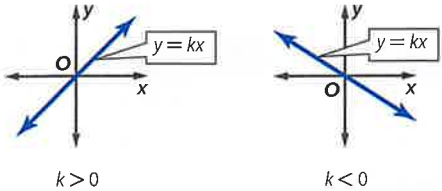
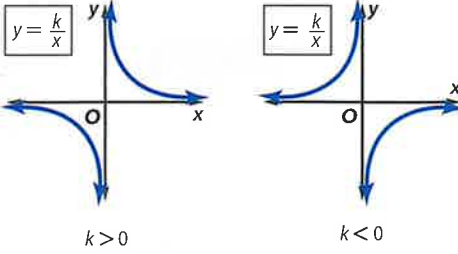
### Problem-Solving Tip

#### CCSS Sense-Making

Sometimes it is necessary to break a problem into parts, solve each part, and then combine them to find the solution to the problem.



## ConceptSummary Direct and Inverse Variations

| Direct Variation  | Inverse Variation  |
|---|--|
|    |    |
| <ul style="list-style-type: none"> <li>• <math>y = kx</math></li> <li>• <math>y</math> varies directly as <math>x</math>.</li> <li>• The ratio <math>\frac{y}{x}</math> is a constant.</li> </ul> | <ul style="list-style-type: none"> <li>• <math>y = \frac{k}{x}</math></li> <li>• <math>y</math> varies inversely as <math>x</math>.</li> <li>• The product <math>xy</math> is a constant.</li> </ul> |

### Check Your Understanding

= Step-by-Step Solutions begin on page R13.



**Example 1** Determine whether each table or equation represents an *inverse* or a *direct* variation. Explain.

1. 

|     |   |   |    |    |
|-----|---|---|----|----|
| $x$ | 1 | 4 | 8  | 12 |
| $y$ | 2 | 8 | 16 | 24 |

2. 

|     |    |    |   |   |
|-----|----|----|---|---|
| $x$ | 1  | 2  | 3 | 4 |
| $y$ | 24 | 12 | 8 | 6 |

3.  $xy = 4$

4.  $y = \frac{x}{10}$

**Examples 2, 5** Assume that  $y$  varies inversely as  $x$ . Write an inverse variation equation that relates  $x$  and  $y$ . Then graph the equation.

5.  $y = 8$  when  $x = 6$

6.  $y = 2$  when  $x = 5$

7.  $y = 3$  when  $x = -10$

8.  $y = -1$  when  $x = -12$

**Example 3** Solve. Assume that  $y$  varies inversely as  $x$ .

9. If  $y = 8$  when  $x = 4$ , find  $x$  when  $y = 2$ .

10. If  $y = 7$  when  $x = 6$ , find  $y$  when  $x = -21$ .

11. If  $y = -5$  when  $x = 9$ , find  $y$  when  $x = 6$ .

**Example 4** 12. **RACING** The time it takes to complete a go-cart race course is inversely proportional to the average speed of the go-cart. One rider has an average speed of 73.3 feet per second and completes the course in 30 seconds. Another rider completes the course in 25 seconds. What was the average speed of the second rider?

13. **OPTOMETRY** When a person does not have clear vision, an optometrist can prescribe lenses to correct the condition. The power  $P$  of a lens, in a unit called diopters, is equal to 1 divided by the focal length  $f$ , in meters, of the lens.

a. Graph the inverse variation  $P = \frac{1}{f}$ .

b. Find the powers of lenses with focal lengths +0.2 to -0.4 meters.



**Example 1** Determine whether each table or equation represents an *inverse* or a *direct* variation. Explain.

14. 

| $x$ | $y$ |
|-----|-----|
| 1   | 30  |
| 2   | 15  |
| 5   | 6   |
| 6   | 5   |

15. 

| $x$ | $y$ |
|-----|-----|
| 2   | -6  |
| 3   | -9  |
| 4   | -12 |
| 5   | -15 |

16. 

| $x$ | $y$ |
|-----|-----|
| -4  | -2  |
| -2  | -1  |
| 2   | 1   |
| 4   | 2   |

17. 

| $x$ | $y$ |
|-----|-----|
| -5  | 8   |
| -2  | 20  |
| 4   | -10 |
| 8   | -5  |

18.  $5x - y = 0$

19.  $xy = \frac{1}{4}$

20.  $x = 14y$

21.  $\frac{y}{x} = 9$

**Examples 2, 5** Assume that  $y$  varies inversely as  $x$ . Write an inverse variation equation that relates  $x$  and  $y$ . Then graph the equation.

22.  $y = 2$  when  $x = 20$

23.  $y = 18$  when  $x = 4$

24.  $y = -6$  when  $x = -3$

25.  $y = -4$  when  $x = -3$

26.  $y = -4$  when  $x = 16$

27.  $y = 12$  when  $x = -9$

**Example 3** Solve. Assume that  $y$  varies inversely as  $x$ .

28. If  $y = 12$  when  $x = 3$ , find  $x$  when  $y = 6$ .

29. If  $y = 5$  when  $x = 6$ , find  $x$  when  $y = 2$ .

30. If  $y = 4$  when  $x = 14$ , find  $x$  when  $y = -5$ .

31. If  $y = 9$  when  $x = 9$ , find  $y$  when  $x = -27$ .

32. If  $y = 15$  when  $x = -2$ , find  $y$  when  $x = 3$ .

33. If  $y = -8$  when  $x = -12$ , find  $y$  when  $x = 10$ .

**Example 4** 34. **EARTH SCIENCE** The water level in a river varies inversely with air temperature. When the air temperature was  $90^\circ$  Fahrenheit, the water level was 11 feet. If the air temperature was  $110^\circ$  Fahrenheit, what was the level of water in the river?

35. **MUSIC** When under equal tension, the frequency of a vibrating string in a piano varies inversely with the string length. If a string that is 420 millimeters in length vibrates at a frequency of 523 cycles a second, at what frequency will a 707-millimeter string vibrate?

Determine whether each situation is an example of an *inverse* or a *direct* variation. Justify your reasoning.

36. The drama club can afford to purchase 10 wigs at \$2 each or 5 wigs at \$4 each.

37. The Spring family buys several lemonades for \$1.50 each.

38. Nicole earns \$14 for babysitting 2 hours, and \$21 for babysitting 3 hours.

39. Thirty video game tokens are divided evenly among a group of friends.

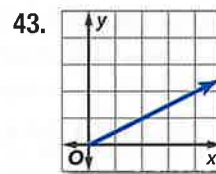
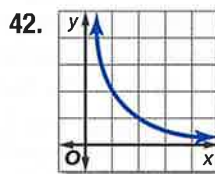
Determine whether each table or graph represents an *inverse* or a *direct* variation. Explain.

40. 

| $x$ | $y$ |
|-----|-----|
| 5   | 1   |
| 8   | 1.6 |
| 11  | 2.2 |

41. 

| $x$ | $y$   |
|-----|-------|
| -3  | -7    |
| -2  | -10.5 |
| 4   | 5.25  |



44. **PHYSICAL SCIENCE** When two people are balanced on a seesaw, their distances from the center of the seesaw are inversely proportional to their weights. If a 118-pound person sits 1.8 meters from the center of the seesaw, how far should a 125-pound person sit from the center to balance the seesaw?

Solve. Assume that  $y$  varies inversely as  $x$ .

45. If  $y = 9.2$  when  $x = 6$ , find  $x$  when  $y = 3$ .
46. If  $y = 3.8$  when  $x = 1.5$ , find  $x$  when  $y = 0.3$ .
47. If  $y = \frac{1}{5}$  when  $x = -20$ , find  $y$  when  $x = -\frac{8}{5}$ .
48. If  $y = -6.3$  when  $x = \frac{2}{3}$ , find  $y$  when  $x = 8$ .
49. **SWIMMING** Logan and Brianna each bought a pool membership. Their average cost per day is inversely proportional to the number of days that they go to the pool. Logan went to the pool 25 days for an average cost per day of \$5.60. Brianna went to the pool 35 days. What was her average cost per day?
50. **PHYSICAL SCIENCE** The amount of force required to do a certain amount of work in moving an object is inversely proportional to the distance that the object is moved. Suppose 90 N of force is required to move an object 10 feet. Find the force needed to move another object 15 feet if the same amount of work is done.
51. **DRIVING** Lina must practice driving 40 hours with a parent or guardian before she is allowed to take the test to get her driver's license. She plans to practice the same number of hours each week.
- Let  $h$  represent the number of hours per week that she practices driving. Make a table showing the number of weeks  $w$  that she will need to practice for the following values of  $h$ : 1, 2, 4, 5, 8, and 10.
  - Describe how the number of weeks changes as the number of hours per week increases.
  - Write and graph an equation that shows the relationship between  $h$  and  $w$ .

### H.O.T. Problems Use Higher-Order Thinking Skills

52. **CCSS CRITIQUE** Christian and Trevor found an equation such that  $x$  and  $y$  vary inversely, and  $y = 10$  when  $x = 5$ . Is either of them correct? Explain.

$$\begin{aligned} \text{Christian} \\ k &= \frac{y}{x} \\ &= \frac{10}{2} \text{ or } 5 \\ y &= 5x \end{aligned}$$

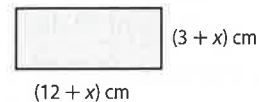
$$\begin{aligned} \text{Trevor} \\ k &= xy \\ &= (5)(10) \text{ or } 50 \\ y &= \frac{50}{x} \end{aligned}$$

53. **CHALLENGE** Suppose  $f$  varies inversely with  $g$ , and  $g$  varies inversely with  $h$ . What is the relationship between  $f$  and  $h$ ?
54. **REASONING** Does  $xy = -k$  represent an inverse variation when  $k \neq 0$ ? Explain.
55. **OPEN ENDED** Give a real-world situation or phenomena that can be modeled by an inverse variation equation. Use the correct terminology to describe your example and explain why this situation is an inverse variation.
56. **WRITING IN MATH** Compare and contrast direct and inverse variation. Include a description of the relationship between slope and the graphs of a direct and inverse variation.



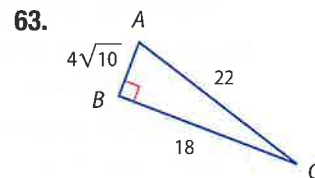
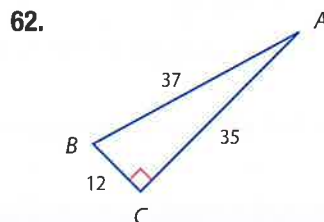
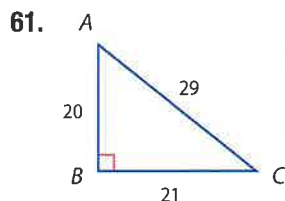
## Standardized Test Practice

57. Given a constant force, the acceleration of an object varies inversely with its mass. Assume that a constant force is acting on an object with a mass of 6 pounds resulting in an acceleration of  $10 \text{ ft/s}^2$ . The same force acts on another object with a mass of 12 pounds. What would be the resulting acceleration?
- A  $4 \text{ ft/s}^2$                       C  $6 \text{ ft/s}^2$   
 B  $5 \text{ ft/s}^2$                       D  $7 \text{ ft/s}^2$
58. Fiona had an average of 56% on her first seven tests. What would she have to make on her eighth test to average 60% on 8 tests?
- F 82%                                  H 98%  
 G 88%                                  J 100%
59. Anthony takes a picture of a 1-meter snake beside a brick wall. When he develops the pictures, the 1-meter snake is 2 centimeters long and the wall is 4.5 centimeters high. What was the actual height of the brick wall?
- A 2.25 cm  
 B 22.5 cm  
 C 225 cm  
 D 2250 cm
60. **SHORT RESPONSE** Find the area of the rectangle.



## Spiral Review

For each triangle, find  $\sin A$ ,  $\cos A$ , and  $\tan A$  to the nearest ten-thousandth. (Lesson 10-6)



64. **CRAFTS** Jane is making a stained glass window using several triangular pieces of glass like the one shown. What is the length of the third side? (Lesson 10-5)



Solve each equation. (Lesson 10-4)

65.  $\sqrt{10c} + 2 = 5$

66.  $\sqrt{9h + 19} = 9$

67.  $\sqrt{7k + 2} + 2 = 5$

68.  $\sqrt{5r - 1} = r - 5$

69.  $6 + \sqrt{2x + 11} = -x$

70.  $4 + \sqrt{4t - 4} = t$

## Skills Review

Simplify. Assume that no denominator is equal to zero.

71.  $\frac{7^8}{7^6}$

72.  $\frac{x^8 y^{12}}{x^2 y^7}$

73.  $\frac{5pq^7}{10p^6 q^3}$

74.  $\left(\frac{2c^3 d}{7z^2}\right)^3$

75.  $\left(\frac{4a^2 b}{2c^3}\right)^2$

76.  $y^0(y^5)(y^{-9})$

77.  $\frac{(4m^{-3}n^5)^0}{mn}$

78.  $\frac{(3x^2 y^5)^0}{(21x^5 y^2)^0}$





You can use a graphing calculator to analyze how changing the parameters  $a$  and  $b$  in  $y = \frac{a}{x-b} + c$  affects the graphs in the family of rational functions.

### Activity Change Parameters

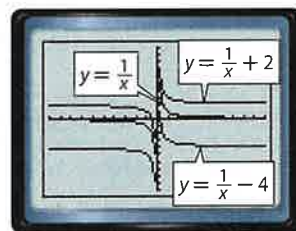


Graph each set of equations on the same screen in the standard viewing window. Describe any similarities and differences among the graphs.

a.  $y = \frac{1}{x}$ ,  $y = \frac{1}{x} + 2$ ,  $y = \frac{1}{x} - 4$

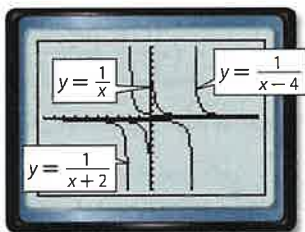
Enter the equations in the Y= list and graph in the standard viewing window.

The graphs have the same shape. Each graph approaches the  $y$ -axis on both sides. However, the graphs have different vertical positions.



$[-10, 10]$  scl: 1 by  $[-10, 10]$  scl: 1

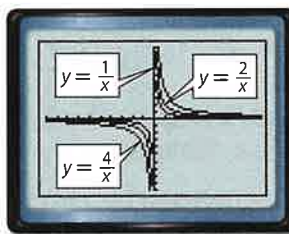
b.  $y = \frac{1}{x}$ ,  $y = \frac{1}{x+2}$ ,  $y = \frac{1}{x-4}$



$[-10, 10]$  scl: 1 by  $[-10, 10]$  scl: 1

The graphs have the same shape, and all approach the  $x$ -axis from both sides. However, the graphs have different horizontal positions.

c.  $y = \frac{1}{x}$ ,  $y = \frac{2}{x}$ ,  $y = \frac{4}{x}$



$[-10, 10]$  scl: 1 by  $[-10, 10]$  scl: 1

The graphs all approach the  $x$ -axis and the  $y$ -axis from both sides. However, the graphs have different shapes.

### Model and Analyze

1. How do  $a$ ,  $b$ , and  $c$  affect the graph of  $y = \frac{a}{x-b} + c$ ? Give examples.

Examine each pair of equations and predict the similarities and differences in their graphs. Use a graphing calculator to confirm your predictions. Write a sentence or two comparing the two graphs.

2.  $y = \frac{1}{x}$ ,  $y = \frac{1}{x} + 2$

3.  $y = \frac{1}{x}$ ,  $y = \frac{1}{x+5}$

4.  $y = \frac{1}{x}$ ,  $y = \frac{3}{x}$



**Then**

- You wrote inverse variation equations.

**Now**

- Identify excluded values.
- Identify and use asymptotes to graph rational functions.

**Why?**

- Trina is reading a 300-page book. The average number of pages she reads each day  $y$  is given by  $y = \frac{300}{x}$ , where  $x$  is the number of days that she reads.



**New Vocabulary**

rational function  
excluded value  
asymptote



**Common Core State Standards**

**Content Standards**  
A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

**Mathematical Practices**

- Construct viable arguments and critique the reasoning of others.
- Look for and make use of structure.

**1 Identify Excluded Values** The function  $y = \frac{300}{x}$  is an example of a **rational function**. This function is nonlinear.

**KeyConcept Rational Functions**

**Words**

A rational function can be described by an equation of the form  $y = \frac{p}{q}$ , where  $p$  and  $q$  are polynomials and  $q \neq 0$ .

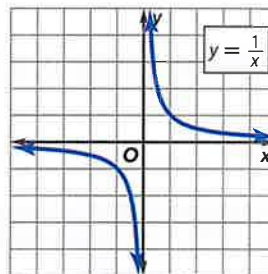
Parent function:  $f(x) = \frac{1}{x}$

Type of graph: hyperbola

Domain:  $\{x \mid x \neq 0\}$

Range:  $\{y \mid y \neq 0\}$

**Graph**



Since division by zero is undefined, any value of a variable that results in a denominator of zero in a rational function is excluded from the domain of the function. These are called **excluded values** for the rational function.



**Example 1 Find Excluded Values**

State the excluded value for each function.

a.  $y = -\frac{2}{x}$

The denominator cannot equal 0. So, the excluded value is  $x = 0$ .

b.  $y = \frac{2}{x+1}$

$x + 1 = 0$

$x = -1$

The excluded value is  $x = -1$ .

Set the denominator equal to 0.

c.  $y = \frac{5}{4x-8}$

$4x - 8 = 0$

$4x = 8$

$x = 2$

The excluded value is  $x = 2$ .

**Guided Practice**

1A.  $y = \frac{5}{2x}$

1B.  $y = \frac{x}{x-7}$

1C.  $y = \frac{4}{3x+9}$



Depending on the real-world situation, in addition to excluding  $x$ -values that make a denominator zero from the domain of a rational function, additional values might have to be excluded from the domain as well.



### Real-WorldLink

As the temperature of the gas inside a hot air balloon increases, the density of the gas decreases. A hot air balloon rises because the density of the air inside it is less than the density of the air outside.

Source: Goddard Space Flight Center

## Real-World Example 2 Graph Real-Life Rational Functions

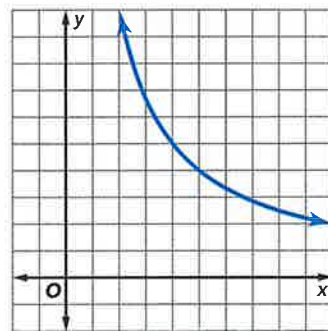


**BALLOONS** If there are  $x$  people in the basket of a hot air balloon, the function  $y = \frac{20}{x}$  represents the number of square feet  $y$  per person. Graph this function.

Since the number of people cannot be zero or less, it is reasonable to exclude negative values and only use positive values for  $x$ .

|                            |    |   |   |    |
|----------------------------|----|---|---|----|
| Number of People $x$       | 2  | 4 | 5 | 10 |
| Square Feet per Person $y$ | 10 | 5 | 4 | 2  |

Notice that as  $x$  increases  $y$  approaches 0. This is reasonable since as the number of people increases, the space per person gets closer to 0.



### Guided Practice

2. **GEOMETRY** A rectangle has an area of 18 square inches. The function  $\ell = \frac{18}{w}$  shows the relationship between the length and width. Graph the function.

**2 Identify and Use Asymptotes** In Example 2, an excluded value is  $x = 0$ . Notice that the graph approaches the vertical line  $x = 0$ , but never touches it.

The graph also approaches but never touches the horizontal line  $y = 0$ . The lines  $x = 0$  and  $y = 0$  are called *asymptotes*. An **asymptote** is a line that the graph of a function approaches.

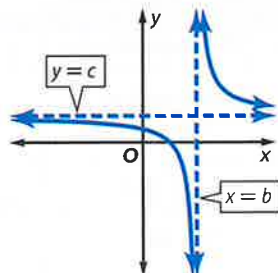
### Study Tip

**Use Asymptotes** Asymptotes are helpful for graphing rational functions. However, they are not part of the graph.

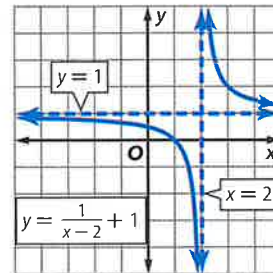
### Key Concept Asymptotes

**Words** A rational function in the form  $y = \frac{a}{x-b} + c$ ,  $a \neq 0$ , has a vertical asymptote at the  $x$ -value that makes the denominator equal zero,  $x = b$ . It has a horizontal asymptote at  $y = c$ .

**Model**



**Example**



The domain of  $y = \frac{a}{x-b} + c$  is all real numbers except  $x = b$ . The range is all real numbers except  $y = c$ . Rational functions cannot be traced with a pencil that never leaves the paper, so choose  $x$ -values on both sides of the vertical asymptote to graph both portions of the function.





### Math HistoryLink

#### Evelyn Boyd Granville

(1924–) Granville majored in mathematics and physics at Smith College in 1945, where she graduated summa cum laude. She earned an M.A. in mathematics and physics and a Ph.D. in mathematics from Yale University. Granville's doctoral work focused on functional analysis.

### Example 3 Identify and Use Asymptotes to Graph Functions

Identify the asymptotes of each function. Then graph the function.

a.  $y = \frac{2}{x} - 4$

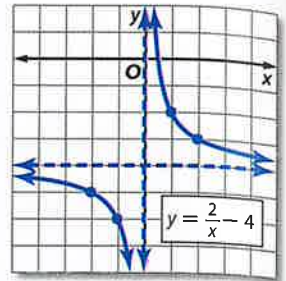
**Step 1** Identify and graph the asymptotes using dashed lines.

vertical asymptote:  $x = 0$

horizontal asymptote:  $y = -4$

**Step 2** Make a table of values and plot the points. Then connect them.

|     |    |    |    |    |
|-----|----|----|----|----|
| $x$ | -2 | -1 | 1  | 2  |
| $y$ | -5 | -6 | -2 | -3 |



b.  $y = \frac{1}{x+1}$

**Step 1** To find the vertical asymptote, find the excluded value.

$x + 1 = 0$       Set the denominator equal to 0.

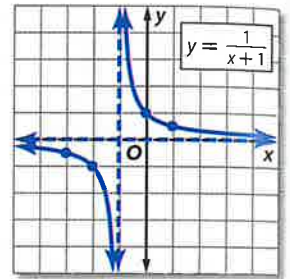
$x = -1$       Subtract 1 from each side.

vertical asymptote:  $x = -1$

horizontal asymptote:  $y = 0$

**Step 2**

|     |      |    |   |     |
|-----|------|----|---|-----|
| $x$ | -3   | -2 | 0 | 1   |
| $y$ | -0.5 | -1 | 1 | 0.5 |



### Guided Practice

3A.  $y = -\frac{6}{x}$

3B.  $y = \frac{1}{x-3}$

3C.  $y = \frac{2}{x+2} + 1$

Four types of nonlinear functions are shown below.

### ConceptSummary Families of Functions

| Quadratic                            | Exponential                 | Radical                               | Rational                                 |
|--------------------------------------|-----------------------------|---------------------------------------|--|
| Parent function:<br>$y = x^2$        | Parent function:<br>varies  | Parent function:<br>$y = \sqrt{x}$    | Parent function:<br>$y = \frac{1}{x}$    |
| General form:<br>$y = ax^2 + bx + c$ | General form:<br>$y = ab^x$ | General form:<br>$y = \sqrt{x-b} + c$ | General form:<br>$y = \frac{a}{x-b} + c$ |
|                                      |                             |                                       |  |





**Example 1** State the excluded value for each function.

1.  $y = \frac{5}{x}$       2.  $y = \frac{1}{x+3}$       3.  $y = \frac{x+2}{x-1}$       4.  $y = \frac{x}{2x-8}$

**Example 2** 5. **PARTY PLANNING** The cost of decorations for a party is \$32. This is split among a group of friends. The amount each person pays  $y$  is given by  $y = \frac{32}{x}$ , where  $x$  is the number of people. Graph the function.

**Example 3** Identify the asymptotes of each function. Then graph the function.

6.  $y = \frac{2}{x}$       7.  $y = \frac{3}{x} - 1$       8.  $y = \frac{1}{x-2}$   
 9.  $y = \frac{-4}{x+2}$       10.  $y = \frac{3}{x-1} + 2$       11.  $y = \frac{2}{x+1} - 5$

Practice and Problem Solving

Extra Practice is on page R11.

**Example 1** State the excluded value for each function.


12.  $y = \frac{-1}{x}$       13.  $y = \frac{8}{x-8}$       14.  $y = \frac{x}{x+2}$       15.  $y = \frac{4}{x+6}$   
 16.  $y = \frac{x+1}{x-3}$       17.  $y = \frac{2x+5}{x+5}$       18.  $y = \frac{7}{5x-10}$       19.  $y = \frac{x}{2x+14}$

**Example 2** 20. **ANTELOPES** A pronghorn antelope can run 40 miles without stopping. The average speed is given by  $y = \frac{40}{x}$ , where  $x$  is the time it takes to run the distance.

- a. Graph  $y = \frac{40}{x}$ .
- b. Describe the asymptotes.


21. **CYCLING** A cyclist rides 10 miles each morning. Her average speed  $y$  is given by  $y = \frac{10}{x}$ , where  $x$  is the time it takes her to ride 10 miles. Graph the function.

**Example 3** Identify the asymptotes of each function. Then graph the function.

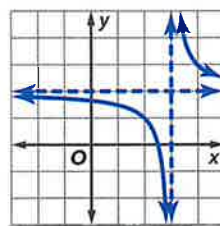
22.  $y = \frac{5}{x}$        23.  $y = \frac{-3}{x}$       24.  $y = \frac{2}{x} + 3$   
 25.  $y = \frac{1}{x} - 2$       26.  $y = \frac{1}{x+3}$       27.  $y = \frac{1}{x-2}$   
 28.  $y = \frac{-2}{x+1}$       29.  $y = \frac{4}{x-1}$       30.  $y = \frac{1}{x-2} + 1$   
 31.  $y = \frac{3}{x-1} - 2$       32.  $y = \frac{2}{x+1} - 4$       33.  $y = \frac{-1}{x+4} + 3$

34. **READING** Refer to the application at the beginning of the lesson.

- a. Graph the function. Interpret key features of the graph in terms of the situation.
- b. Choose a point on the graph, and describe what it means in the context of the situation.

35.  **STRUCTURE** The graph shows a translation of the graph of  $y = \frac{1}{x}$ .

- a. Describe the asymptotes.
- b. Write a possible function for the graph.



36. **BIRDS** A long-tailed jaeger is a sea bird that can migrate 5000 miles or more each year. The average rate in miles per hour  $r$  can be given by the function  $r = \frac{5000}{t}$ , where  $t$  is the time in hours. Use the function to determine the average rate of the bird if it spends 250 hours flying.
37. **CLASS TRIP** The freshmen class is going to a science museum. As part of the trip, each person is also contributing an equal amount of money to name a star.
- Write a verbal description for the cost per person.
  - Write an equation to represent the total cost  $y$  per person if  $p$  people go to the museum.
  - Use a graphing calculator to graph the equation. Interpret key features of the graph in terms of the situation.
  - Estimate the number of people needed for the total cost of the trip to be about \$15.



Graph each function. Identify the asymptotes.

38.  $y = \frac{4x + 3}{2x - 4}$

39.  $y = \frac{x^2}{x^2 - 1}$

40.  $y = \frac{x}{x^2 - 9}$

41. **GEOMETRY** The equation  $h = \frac{2(64)}{b_1 + 8}$  represents the height  $h$  of a trapezoid with an area of 64 square units. The trapezoid has two opposite sides that are parallel and  $h$  units apart; one is  $b_1$  units long and another is 8 units long.
- Describe a reasonable domain and range for the function.
  - Graph the function in the first quadrant.
  - Use the graph to estimate the value of  $h$  when  $b_1 = 10$ .

### H.O.T. Problems Use Higher-Order Thinking Skills

42. **CHALLENGE** Graph  $y = \frac{1}{x^2 - 4}$ . State the domain and the range of the function.
43. **REASONING** Without graphing, describe the transformation that takes place between the graph of  $y = \frac{1}{x}$  and the graph of  $y = \frac{1}{x + 5} - 2$ .
44. **OPEN ENDED** Write a rational function if the asymptotes of the graph are at  $x = 3$  and  $y = 1$ . Explain how you found the function.
45. **CCSS ARGUMENTS** Is the following statement *true* or *false*? If false, give a counterexample.

*The graph of a rational function will have at least one intercept.*

46. **WHICH ONE DOESN'T BELONG** Identify the function that does not belong with the other three. Explain your reasoning.

$$y = \frac{4}{x}$$

$$y = \frac{6}{x + 1}$$

$$y = \frac{8}{x} + 1$$

$$y = \frac{10}{2x}$$

47. **WRITING IN MATH** How are the properties of a rational function reflected in its graph?



## Standardized Test Practice

48. Simplify  $\frac{2a^2d}{3bc} \cdot \frac{9b^2c}{16ad^2}$ .

A  $\frac{abd}{c}$

C  $\frac{6a}{4bd}$

B  $\frac{ab}{d}$

D  $\frac{3ab}{8d}$

49. **SHORT RESPONSE** One day Lola ran 100 meters in 15 seconds, 200 meters in 45 seconds, and 300 meters over low hurdles in one and a half minutes. How many more seconds did it take her to run 300 meters over low hurdles than the 200-meter dash?

50. Scott and Ian started a T-shirt printing business. The total start-up costs were \$450. It costs \$5.50 to print one T-shirt. Write a rational function  $A(x)$  for the average cost of producing  $x$  T-shirts.

F  $A(x) = \frac{450 + 5.5x}{x}$     H  $A(x) = 450x + 5.5$

G  $A(x) = \frac{450}{x} + 5.5$     J  $A(x) = 450 + 5.5x$

51. **GEOMETRY** Which of the following is a quadrilateral with exactly one pair of parallel sides?

A parallelogram

C square

B rectangle

D trapezoid

## Spiral Review

52. **TRAVEL** The Brooks family can drive to the beach, which is 220 miles away, in 4 hours if they drive 55 miles per hour. Kendra says that they would save at least a half an hour if they were to drive 65 miles per hour. Is Kendra correct? Explain.

(Lesson 11-1)

Use a calculator to find the measure of each angle to the nearest degree. (Lesson 10-6)

53.  $\sin C = 0.9781$

54.  $\tan H = 0.6473$

55.  $\cos K = 0.7658$

56.  $\tan Y = 3.6541$

57.  $\cos U = 0.5000$

58.  $\sin N = 0.3832$

If  $c$  is the measure of the hypotenuse of a right triangle, find each missing measure.

If necessary round to the nearest hundredth. (Lesson 10-5)

59.  $a = 15, b = 60, c = ?$

60.  $a = 17, c = 35, b = ?$

61.  $a = \sqrt{110}, b = 1, c = ?$

62.  $a = \sqrt{17}, b = \sqrt{12}, c = ?$

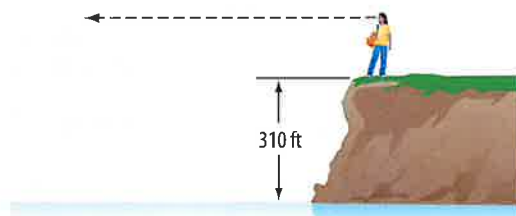
63.  $a = 6, c = 11, b = ?$

64.  $a = 9, b = 6, c = ?$

65. **SIGHT** The formula  $d = \sqrt{\frac{3h}{2}}$  represents the distance  $d$  in miles

that a person  $h$  feet high can see. Irene is standing on a cliff that is 310 feet above sea level. How far can Irene see from the cliff?

Write a simplified radical expression and a decimal approximation. (Lesson 10-2)



## Skills Review

Factor each trinomial.

66.  $x^2 + 11x + 24$

67.  $w^2 + 13w - 48$

68.  $p^2 - 2p - 35$

69.  $72 + 27a + a^2$

70.  $c^2 + 12c + 35$

71.  $d^2 - 7d + 10$

72.  $g^2 - 19g + 60$

73.  $n^2 + 3n - 54$

74.  $5x^2 + 27x + 10$

75.  $24b^2 - 14b - 3$

76.  $12a^2 - 13a - 35$

77.  $6x^2 - 14x - 12$



## Then

- You simplified expressions involving the quotient of monomials.

## Now

- Identify values excluded from the domain of a rational expression.
- Simplify rational expressions.

## Why?

- Big-O is a “hubless” Ferris wheel in Tokyo, Japan. The *centripetal force*, or the force acting toward the center, is given by  $\frac{mv^2}{r}$ , where  $m$  is the mass of the Ferris wheel,  $v$  is the velocity, and  $r$  is the radius.



**New Vocabulary**  
rational expression



**Common Core State Standards**

**Mathematical Practice**  
7 Look for and make use of structure.

**1 Identify Excluded Values** The expression  $\frac{mv^2}{r}$  is an example of a rational expression. A **rational expression** is an algebraic fraction whose numerator and denominator are polynomials. Since division by zero is undefined, the polynomial in the denominator cannot be 0.



**Example 1 Find Excluded Values**

State the excluded values for each rational expression.

a.  $\frac{-8}{r^2 - 36}$

Exclude the values for which  $r^2 - 36 = 0$ .

$$r^2 - 36 = 0$$

The denominator cannot be zero.

$$(r - 6)(r + 6) = 0$$

Factor.

$$r - 6 = 0 \quad \text{or} \quad r + 6 = 0$$

Zero Product Property

$$r = 6$$

$$r = -6$$

Therefore,  $r$  cannot equal 6 or  $-6$ .

b.  $\frac{n^2}{n^2 + 4n - 5}$

Exclude the values for which  $n^2 + 4n - 5 = 0$ .

$$n^2 + 4n - 5 = 0$$

The denominator cannot be zero.

$$(n - 1)(n + 5) = 0$$

Factor.

$$n - 1 = 0 \quad \text{or} \quad n + 5 = 0$$

Zero Product Property

$$n = 1$$

$$n = -5$$

Therefore,  $n$  cannot equal 1 or  $-5$ .

**Guided Practice**

1A.  $\frac{5x}{x^2 - 81}$

1B.  $\frac{3a - 2}{a^2 + 6a + 8}$



### StudyTip

**Cylinder** The volume of a cylinder is  $V = \pi r^2 h$ . The height of a cylinder with volume  $V$  and radius  $r$  is given by  $\frac{V}{\pi r^2}$ .

### Real-World Example 2 Use Rational Expressions



**GEOMETRY** Find the height of a cylinder that has a volume of 821 cubic inches and a radius of 7 inches. Round to the nearest tenth.

**Understand** You have a rational expression with two variables,  $V$  and  $r$ .

**Plan** Substitute 821 for  $V$  and 7 for  $r$  and simplify.

$$\begin{aligned} \text{Solve } \frac{V}{\pi r^2} &= \frac{821}{\pi(7)^2} && \text{Replace } V \text{ with } 821 \text{ and } r \text{ with } 7. \\ &\approx 5.3 && \text{The height of the cylinder is about 5.3 inches.} \end{aligned}$$

**Check** Use estimation to determine whether the answer is reasonable.

$$\frac{800}{3(50)} \approx 5 \quad \checkmark \quad \text{The solution is reasonable.}$$

### GuidedPractice

2. Find the height of the cylinder that has a volume of 710 cubic inches and a diameter of 18 inches.

**2 Simplify Expressions** A rational expression is in simplest form when the numerator and denominator have no common factors except 1. To simplify a rational expression, divide out any common factors of the numerator and denominator.

### KeyConcept Simplifying Rational Expressions



**Words** Let  $a$ ,  $b$ , and  $c$ , be polynomials with  $a \neq 0$  and  $c \neq 0$ .

$$\text{Symbols } \frac{ba}{ca} = \frac{b \cdot a}{c \cdot a} = \frac{b}{c} \quad \text{Example } \frac{3x-9}{4x-12} = \frac{3(x-3)}{4(x-3)} = \frac{3}{4}$$

### Test-TakingTip

#### Eliminate Possibilities

Since there is only one negative factor in the expression in Example 3, the simplified expression should have a negative sign. So, you can eliminate choices A and B.

### Standardized Test Example 3 Use GCF to Simplify an Expression



Which expression is equivalent to  $\frac{(-3x^2)(4x^5)}{9x^6}$ ?

A  $\frac{4}{3}x$       B  $\frac{4}{3x}$       C  $-\frac{4}{3x}$       D  $-\frac{4}{3}x$

**Read the Test Item** The expression represents the product of two monomials and the division of that product by another monomial.

**Solve the Test Item**

**Step 1** Factor the numerator and denominator, using their GCF.

$$\frac{(3x^6)(-4x)}{(3x^6)(3)}$$

**Step 2** Simplify. The correct answer is D.

$$\frac{\cancel{(3x^6)}(-4x)}{\cancel{(3x^6)}(3)} \text{ or } -\frac{4}{3}x$$

### GuidedPractice

3. Which expression is equivalent to  $\frac{16c^2b^4}{8c^3b}$ ?

F  $\frac{2b^3}{c}$       G  $\frac{b^3}{2c}$       H  $\frac{1}{2b^3c}$       J  $2b^3c$



You can use the same procedure to simplify a rational expression in which the numerator and denominator are polynomials.



### Example 4 Simplify Rational Expressions

Simplify  $\frac{2r + 18}{r^2 + 8r - 9}$ . State the excluded values of  $r$ .

$$\frac{2r + 18}{r^2 + 8r - 9} = \frac{2(r + 9)}{(r + 9)(r - 1)} \quad \text{Factor.}$$

$$= \frac{\cancel{2}(\cancel{r+9})}{\cancel{(r+9)}(r-1)} \text{ or } \frac{2}{r-1} \quad \text{Divide the numerator and denominator by the GCF, } r+9.$$

Exclude the values for which  $r^2 + 8r - 9$  equals 0.

$$r^2 + 8r - 9 = 0$$

The denominator cannot equal zero.

$$(r + 9)(r - 1) = 0$$

Factor.

$$r = -9 \text{ or } r = 1$$

Zero Product Property

So,  $r \neq -9$  and  $r \neq 1$ .

#### StudyTip

**CCSS Structure** Determine the excluded values using the original expression rather than the simplified expression.

#### Guided Practice

Simplify each rational expression. State the excluded values of the variables.

4A.  $\frac{n + 3}{n^2 + 10n + 21}$

4B.  $\frac{y^2 + 9y - 10}{2y + 20}$

When simplifying rational expressions, look for binomials that are opposites. For example,  $5 - x$  and  $x - 5$  are opposites because  $5 - x = -1(x - 5)$ . So, you can write  $\frac{x - 5}{5 - x}$  as  $\frac{x - 5}{-1(x - 5)}$ .



### Example 5 Recognize Opposites

Simplify  $\frac{36 - t^2}{5t - 30}$ . State the excluded values of  $t$ .

$$\frac{36 - t^2}{5t - 30} = \frac{(6 - t)(6 + t)}{5(t - 6)} \quad \text{Factor.}$$

$$= \frac{-1(t - 6)(6 + t)}{5(t - 6)} \quad \text{Rewrite } 6 - t \text{ as } -1(t - 6).$$

$$= \frac{-1(\cancel{t-6})(6+t)}{5(\cancel{t-6})} \text{ or } -\frac{6+t}{5} \quad \text{Divide out the common factor, } t - 6.$$

Exclude the values for which  $5t - 30$  equals 0.

$$5t - 30 = 0$$

The denominator cannot equal zero.

$$5t = 30$$

Add 30 to each side.

$$t = 6$$

Divide each side by 5.

So,  $t \neq 6$ .

#### Guided Practice

Simplify each expression. State the excluded values of  $x$ .

5A.  $\frac{12x + 36}{x^2 - x - 12}$

5B.  $\frac{x^2 - 2x - 35}{x^2 - 9x + 14}$



**StudyTip**

**Zeros** Once simplified, a rational function has zeros at the values that make the numerator 0.

Recall that to find the zeros of a quadratic function, you need to find the values of  $x$  when  $f(x) = 0$ . The zeros of a rational function are found in the same way.

**Example 6 Rational Functions**

Find the zeros of  $f(x) = \frac{x^2 + 3x - 18}{x - 3}$ .

$$f(x) = \frac{x^2 + 3x - 18}{x - 3} \quad \text{Original function}$$

$$0 = \frac{x^2 + 3x - 18}{x - 3} \quad f(x) = 0$$

$$0 = \frac{(x + 6)(x - 3)}{x - 3} \quad \text{Factor.}$$

$$0 = \frac{(x + 6)\cancel{(x - 3)}}{\cancel{x - 3}} \quad \text{Divide out common factors.}$$

$$0 = x + 6 \quad \text{Simplify.}$$

When  $x = -6$ , the numerator becomes 0, so  $f(x) = 0$ . Therefore, the zero of the function is  $-6$ .

**GuidedPractice**

Find the zeros of each function.

6A.  $f(x) = \frac{x^2 + 2x - 15}{x + 1}$

6B.  $f(x) = \frac{x^2 + 6x + 8}{x^2 + x - 2}$

**Check Your Understanding**

= Step-by-Step Solutions begin on page R13.



**Example 1** State the excluded values for each rational expression.

1.  $\frac{8}{x^2 - 16}$

2.  $\frac{3m}{m^2 - 6m + 5}$

**Example 2** 3. **PHYSICAL SCIENCE** A 0.16-kilogram ball attached to a string is being spun in a circle 7.26 meters per second. The expression  $\frac{mv^2}{r}$ , where  $m$  is the mass of the ball,  $v$  is the velocity, and  $r$  is the radius, can be used to find the force that keeps the ball spinning in a circle. If the circle has a radius of 0.5 meter, find the force that must be exerted to keep the ball spinning. Round to the nearest tenth.

**Examples 3–5** Simplify each expression. State the excluded values of the variables.

4.  $\frac{28ab^3}{16a^2b}$

5.  $\frac{(-3r)(10r^4)}{6r^5}$

6.  $\frac{5d + 15}{d^2 - d - 12}$

7.  $\frac{x^2 + 11x + 28}{x + 4}$

8.  $\frac{2r - 12}{r^2 - 36}$

9.  $\frac{3y - 27}{81 - y^2}$

**Example 6** Find the zeros of each function.

10.  $f(x) = \frac{x^2 - x - 12}{x - 2}$

11.  $f(x) = \frac{x^2 - x - 6}{x^2 + 8x + 12}$



**Example 1** State the excluded values for each rational expression.

12.  $\frac{-n}{n^2 - 49}$

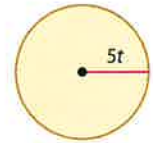
13.  $\frac{5x + 1}{x^2 - 1}$

14.  $\frac{12a}{a^2 - 3a - 10}$

15.  $\frac{k^2 - 4}{k^2 + 5k - 24}$

**Example 2** 16. **GEOMETRY** The volume of a rectangular prism is  $3x^3 + 34x^2 + 72x - 64$ . If the height is  $x + 4$ , what is the area of the base of the prism?

17. **GEOMETRY** Use the circle at the right to write the ratio  $\frac{\text{circumference}}{\text{area}}$ . Then simplify. State the excluded value of the variable.



**Examples 3-5** Simplify each expression. State the excluded values of the variables.

18.  $\frac{15x^4y^2}{40x^3y^3}$

19.  $\frac{32n^2p}{2n^4p}$

20.  $\frac{(4t^3)(2t)}{20t^2}$

21.  $\frac{(7c^2)(-6c^3)}{21c^4}$

22.  $\frac{4x - 24}{x^2 - 12x + 36}$

23.  $\frac{a^2 + 3a}{a^2 - 3a - 18}$

24.  $\frac{n^2 + 7n - 18}{n - 2}$

25.  $\frac{x^2 + 4x - 32}{x + 8}$

26.  $\frac{x^2 - 25}{x^2 + 5x}$

27.  $\frac{2p^2 - 14p}{p^2 - 49}$

28.  $\frac{2x - 10}{25 - x^2}$

29.  $\frac{64 - c^2}{c^2 - 7c - 8}$

**Example 6** Find the zeros of each function.

30.  $f(x) = \frac{x^2 - x - 12}{x^2 + 2x - 35}$

31.  $f(x) = \frac{x^2 + 3x - 4}{x^2 + 9x + 20}$

32.  $f(x) = \frac{2x^2 + 11x - 40}{2x + 5}$

33.  $f(x) = \frac{3x^2 - 18x + 24}{x - 6}$

34.  $f(x) = \frac{x^3 + x^2 - 6x}{x - 1}$

35.  $f(x) = \frac{x^3 - 4x^2 - 12x}{x + 2}$



36. **PYRAMIDS** The perimeter of the base of the Pyramid of the Sun is  $4\pi$  times the height. The perimeter of the base of the Great Pyramid of Giza is  $2\pi$  times the height. Write and simplify each ratio comparing the base perimeters.

- a. Pyramid of the Sun to the Great Pyramid
- b. Great Pyramid to the Pyramid of the Sun

| Pyramid                     | Height (ft) |
|-----------------------------|-------------|
| Pyramid of the Sun (Mexico) | 233.5       |
| Great Pyramid (Egypt)       | 481.4       |

37. **CCSS REASONING** George Ferris built the first Ferris wheel for the World's Columbian Exposition in 1893. It had a diameter of 250 feet.

- a. To find the speed traveled by a car located on the wheel, you can find the circumference of a circle and divide by the time it takes for one rotation. Write a rational expression for the speed of a car rotating in time  $t$ .
- b. Suppose the first Ferris wheel rotated once every 5 minutes. What was the speed of a car on the circumference in feet per minute?

Simplify each expression. State the excluded values of the variables.

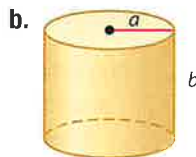
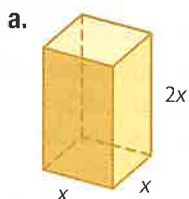
38.  $\frac{3a^2b^4 + 9a^3b - 6a^5b}{3a^2b}$

39.  $\frac{8x^5 - 10xy^2}{2xy^3}$

40.  $\frac{x + 5}{3x^2 + 14x - 5}$



- 41. PACKAGING** To minimize packaging expenses, a company uses packages that have the least surface area to volume ratio. For each figure, write a ratio comparing the surface area to the volume. Then simplify. State the excluded values of the variables.

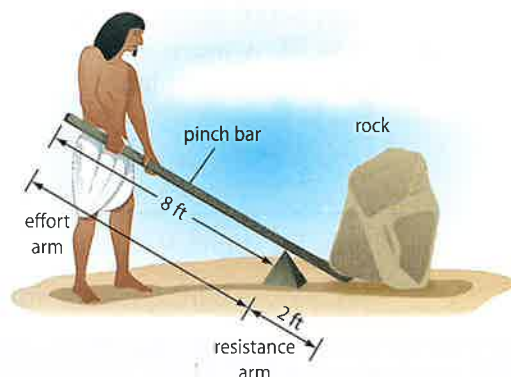


- 42. HISTORY** The diagram shows how a lever may have been used to move blocks.

a. The mechanical advantage of a lever is  $\frac{L_A}{L_R}$ , where  $L_A$  is the length of the effort arm and  $L_R$  is the length of the resistance arm. Find the mechanical advantage of the lever shown.

b. The force placed on the rock is the product of the mechanical advantage and the force applied to the end of the lever. If the Egyptian worker can apply a force of 180 pounds, what is the greatest weight he can lift with the lever?

c. To lift a 535-pound rock using a 7-foot lever with the fulcrum 2 feet from the rock, how much force will have to be used?



### H.O.T. Problems Use Higher-Order Thinking Skills

- 43. ERROR ANALYSIS** Colleen and Sanson examined  $\frac{12x + 36}{x^2 - x - 12}$  and found the excluded value(s). Is either of them correct? Explain.

*Colleen*

$$\frac{12x + 36}{x^2 - x - 12} = \frac{12(x + 3)}{(x - 4)(x + 3)}$$

The excluded values are  
4 and -3.

*Sanson*

$$\begin{aligned} \frac{12x + 36}{x^2 - x - 12} &= \frac{12(x + 3)}{(x - 4)(x + 3)} \\ &= \frac{12(x + 3)}{(x - 4)(x + 3)} \\ &= \frac{12}{x - 4} \end{aligned}$$

The excluded value is 4.

- 44. CCSS PRECISION** Compare and contrast the key features of the graphs of  $y = x - 2$  and  $y = \frac{x^2 + 5x - 14}{x + 7}$ .
- 45. REASONING** Explain why every polynomial is also a rational expression.
- 46. OPEN ENDED** Write a rational expression with excluded values  $-2$  and  $2$ . Explain how you found the expression.
- 47. REASONING** Is  $\frac{2x^2 - 4x}{x - 2}$  in simplest form? Justify your answer.
- 48. WRITING IN MATH** List the steps you would use to simplify  $\frac{x^2 + x - 20}{x + 5}$ . State the excluded value.



## Standardized Test Practice

49. Simplify  $\frac{2x+4}{2}$ .

- A  $x + 1$
- B  $x$
- C  $x + 2$
- D  $\frac{x}{2}$

50. **SHORT RESPONSE** Shiro is buying a car for \$5800. He can pay the full amount in cash, or he can pay \$1000 down and \$230 a month for 24 months. How much more would he pay for the car on the second plan?

51. **GEOMETRY** What is the name of the figure?

- F triangular pyramid
- G triangular prism
- H rectangular prism
- J rectangular pyramid



52. A rectangle has a length of 10 inches and a width of 5 inches. Another rectangle has the same area as the first rectangle but its width is 2 inches. Find the length of the second rectangle.

- A 60 in.
- B 45 in.
- C 30 in.
- D 25 in.

## Spiral Review

State the excluded value for each function. (Lesson 11-2)

53.  $y = \frac{6}{x}$

54.  $y = \frac{2}{x-5}$

55.  $y = \frac{x-4}{x-3}$

56.  $y = \frac{3x}{2x+6}$

Solve. Assume that  $y$  varies inversely as  $x$ . (Lesson 11-1)

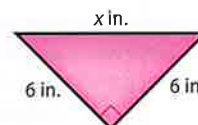
57. If  $y = 10$  when  $x = 4$ ,  
find  $x$  when  $y = 2$ .

58. If  $y = 12$  when  $x = 3$ ,  
find  $x$  when  $y = 6$ .

59. If  $y = -5$  when  $x = 3$ ,  
find  $x$  when  $y = -3$ .

60. If  $y = 21$  when  $x = -6$ ,  
find  $x$  when  $y = 7$ .

61. **CRAFTS** Melinda is working on a quilt using the pattern shown. She has several right triangular pieces of material with two sides that measure 6 inches. What is the length of the third side? (Lesson 10-5)



Simplify. (Lesson 10-2)

62.  $\sqrt{20}$

63.  $\sqrt{18}$

64.  $\sqrt{2} \cdot \sqrt{8}$

65.  $2\sqrt{32}$

66.  $\sqrt{5} \cdot \sqrt{6}$

67.  $\sqrt{40a^2}$

68.  $\sqrt{\frac{t}{8}}$

69.  $\sqrt{\frac{2}{7}} \cdot \sqrt{\frac{7}{3}}$

70. **FINANCIAL LITERACY** Determine the amount of an investment if \$250 is invested at an interest rate of 7.3% compounded quarterly for 40 years. (Lesson 9-3)

## Skills Review

Find the greatest common factor for each set of monomials.

71.  $2x, 8x^2$

72.  $3y^2, 7y^3$

73.  $7g, 10h$

74.  $21c^2d^3, 14cd^2$

75.  $9qt^2, 18q^2t^2, 27qt$

76.  $10ab, 25a^2b^2, 30a^2b$



# 11-3 Graphing Technology Lab

## Simplifying Rational Expressions



When simplifying rational expressions, you can use a graphing calculator to support your answer. If the graphs of the original expression and the simplified expression overlap, they are equivalent. You can also use the graphs to see excluded values.



### Activity Simplify a Rational Expression

Simplify  $\frac{x^2 - 16}{x^2 + 8x + 16}$ .

**Step 1** Factor the numerator and denominator.

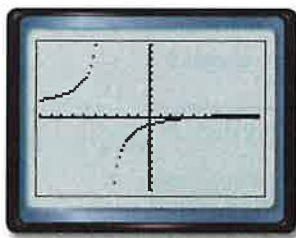
$$\begin{aligned} \frac{x^2 - 16}{x^2 + 8x + 16} &= \frac{(x - 4)(x + 4)}{(x + 4)(x + 4)} \\ &= \frac{(x - 4)}{(x + 4)} \end{aligned}$$

When  $x = -4$ ,  $x + 4 = 0$ . Therefore,  $x$  cannot equal  $-4$  because you cannot divide by zero.

**Step 2** Graph the original expression.

- Set the calculator to Dot mode.
- Enter  $\frac{x^2 - 16}{x^2 + 8x + 16}$  as  $Y_1$  and graph in the standard viewing window.

KEYSTROKES: **MODE**  $\blacktriangledown$   $\blacktriangledown$   $\blacktriangledown$   $\blacktriangledown$   $\blacktriangleright$   
**ENTER** **Y=** **(** **X,T,θ,n** **x<sup>2</sup>**  
**=** **16** **)** **÷** **(** **X,T,θ,n**  
**x<sup>2</sup>** **+** **8** **X,T,θ,n** **+** **16** **)**  
**ZOOM** **6**

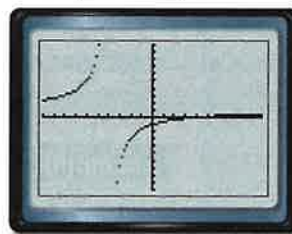


$[-10, 10]$  scl: 1 by  $[-10, 10]$  scl: 1

**Step 3** Graph the simplified expression.

Enter  $\frac{(x - 4)}{(x + 4)}$  as  $Y_2$  and graph.

KEYSTROKES: **Y=**  $\blacktriangledown$  **(** **X,T,θ,n** **=** **4** **)**  
 $\div$  **(** **X,T,θ,n** **+** **4** **)**  
**GRAPH**



$[-10, 10]$  scl: 1 by  $[-10, 10]$  scl: 1

Since the graphs overlap, the two expressions are equivalent.

### Exercises

Simplify each expression. Then verify your answer graphically. Name the excluded values.

1.  $\frac{5x + 15}{x^2 + 10x + 21}$

2.  $\frac{x^2 - 8x + 12}{x^2 + 7x - 18}$

3.  $\frac{2x^2 + 6x + 4}{3x^2 + 9x + 6}$

4. a. Simplify  $\frac{3x - 8}{6x^2 - 16x}$ .

b. How can you use the **TABLE** function to verify that the original expression and the simplified expression are equivalent?

c. How does the **TABLE** function show you that an  $x$ -value is an excluded value?

## Multiplying and Dividing Rational Expressions



### Then

- You multiplied polynomials and divided monomials.

### Now

- Multiply rational expressions.
- Divide rational expressions.

### Why?

- A recent survey showed 10- to 17-year olds talk or text on their cell phones an average of 3.75 hours per day during the summer. The expression below can be used to find the average number of minutes youth talk or text on their phones during summer, approximately 90 days.

$$90 \text{ days} \cdot \frac{3.75 \text{ hours}}{\text{day}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = 20,250 \text{ minutes}$$



### Common Core State Standards

#### Mathematical Practices

- 1 Make sense of problems and persevere in solving them.

**1 Multiply Rational Expressions** To multiply fractions, you multiply numerators and multiply denominators. Use this same method to multiply rational expressions.

### KeyConcept Multiplying Rational Expressions

**Words** Let  $a$ ,  $b$ ,  $c$ , and  $d$  be polynomials with  $b \neq 0$  and  $d \neq 0$ . Then,  $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ .

**Example**  $\frac{x}{2x-3} \cdot \frac{4x^2}{5} = \frac{4x^3}{5(2x-3)}$

### Example 1 Multiply Expressions Involving Monomials



Find each product.

a.  $\frac{r^2x}{9t^3} \cdot \frac{3t^4}{r}$

Divide by the common factors before multiplying.

$$\frac{r^2x}{9t^3} \cdot \frac{3t^4}{r} = \frac{\overset{r}{\cancel{r^2}}x}{\underset{3}{\cancel{9}}t^{\cancel{3}^1}} \cdot \frac{\overset{1}{\cancel{3}}t^{\cancel{4}^1}}{\underset{r}{\cancel{r}}}$$

$$= \frac{rxt}{3}$$

Divide by the common factors 3,  $r$ , and  $t^3$ .

Simplify.

b.  $\frac{a+4}{a^2} \cdot \frac{a}{a^2+2a-8}$

$$\frac{a+4}{a^2} \cdot \frac{a}{a^2+2a-8} = \frac{a+4}{a^2} \cdot \frac{a}{(a+4)(a-2)}$$

Factor the denominator.

$$= \frac{\overset{1}{\cancel{a+4}}}{\underset{a^2}{\cancel{a^2}}} \cdot \frac{\overset{1}{\cancel{a}}}{\underset{1}{\cancel{(a+4)}}(a-2)}$$

The GCF is  $a(a+4)$ .

$$= \frac{1}{a(a-2)} \text{ or } \frac{1}{a^2-2a}$$

Simplify.

### Guided Practice

1A.  $\frac{3x}{16x^2} \cdot \frac{8x^2}{3}$

1B.  $\frac{x+3}{x} \cdot \frac{5}{x^2+7x+12}$

1C.  $\frac{y^2-3y-4}{y+5} \cdot \frac{y+5}{y^2-4y}$



### StudyTip

**Rational Expressions** From this point on, assume that no denominator of a rational expression has a value of zero.

When you multiply fractions that involve units of measure, you can divide by the units in the same way that you divide by variables. Recall that this process is called *dimensional analysis*. You can use dimensional analysis to convert units of measure within a system and between systems.

### Real-World Example 2 Dimensional Analysis



**SKI RACING** Ann Proctor won the 2007 World Waterski Racing Championship race in her category when she finished the 88-kilometer course in 51.23 minutes. What was her average speed in miles per hour?

(Hint: 1 km  $\approx$  0.62 mi)

$$\begin{aligned}\frac{88 \text{ km}}{51.23 \text{ min}} \cdot \frac{0.62 \text{ mi}}{1 \text{ km}} \cdot \frac{60 \text{ min}}{1 \text{ h}} &= \frac{88 \cancel{\text{ km}}}{51.23 \cancel{\text{ min}}} \cdot \frac{0.62 \text{ mi}}{1 \cancel{\text{ km}}} \cdot \frac{60 \cancel{\text{ min}}}{1 \text{ h}} \\ &= \frac{88 \cdot 0.62 \text{ mi} \cdot 60}{51.23 \cdot 1 \cdot 1 \text{ h}} && \text{Simplify.} \\ &= \frac{3273.6 \text{ mi}}{51.23 \text{ h}} && \text{Multiply.} \\ &\approx \frac{63.9 \text{ mi}}{\text{h}} && \text{Divide the numerator and the denominator by 51.23.}\end{aligned}$$

Her average speed was 63.9 miles per hour.

### GuidedPractice

2. **SKI RACING** What was Ann Proctor's speed in feet per second?

**2 Divide Rational Expressions** To divide by a fraction, you multiply by the reciprocal. You can use this same method to divide by a rational expression.

### KeyConcept Dividing Rational Expressions

**Symbols** Let  $a$ ,  $b$ ,  $c$ , and  $d$  be polynomials with  $b \neq 0$ ,  $c \neq 0$ , and  $d \neq 0$ . Then,  $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ .

**Example**  $\frac{x-3}{x} \div \frac{2x^2}{5} = \frac{x-3}{x} \cdot \frac{5}{2x^2} = \frac{5(x-3)}{2x^3}$

### Example 3 Divide by a Rational Expression



Find  $\frac{4}{15n^3} \div \frac{12}{25n}$ .

$$\frac{4}{15n^3} \div \frac{12}{25n} = \frac{4}{15n^3} \cdot \frac{25n}{12} \quad \text{Multiply by } \frac{25n}{12}, \text{ the reciprocal of } \frac{12}{25n}.$$

$$= \frac{\overset{1}{\cancel{4}}}{\underset{3n^2}{\cancel{15n^3}}} \cdot \frac{\overset{5}{\cancel{25n}}}{\underset{3}{\cancel{12}}} \quad \text{Divide by common factors 4, 5, and } n.$$

$$= \frac{5}{9n^2} \quad \text{Simplify.}$$

### GuidedPractice

Find each quotient.

3A.  $\frac{15y^2}{4x} \div \frac{5y}{8x^3}$

3B.  $\frac{12a^2}{5b} \div \frac{25a}{6b^2}$



**ReadingMath**

**Rational Expressions** In a rational expression, the fraction bar acts as a grouping symbol. In Example 4a,  $\frac{2x+6}{x^2}$  is read *the quantity two x plus six divided by x squared*.

**Example 4** Divide by Rational Expressions and Polynomials

Find each quotient.

a.  $\frac{2x+6}{x^2} \div (x+3)$

$$\begin{aligned} \frac{2x+6}{x^2} \div (x+3) &= \frac{2x+6}{x^2} \div \frac{x+3}{1} \\ &= \frac{2x+6}{x^2} \cdot \frac{1}{x+3} \\ &= \frac{2(x+3)}{x^2} \cdot \frac{1}{x+3} \\ &= \frac{\cancel{2(x+3)}}{x^2} \cdot \frac{1}{\cancel{x+3}} \text{ or } \frac{2}{x^2} \end{aligned}$$

Write the binomial as a fraction.

Multiply by the reciprocal of  $x+3$ .

Factor  $4x+6$ .

Divide out the common factor and simplify.

b.  $\frac{a-2}{4a+4} \div \frac{a+5}{a+1}$

$$\begin{aligned} \frac{a-2}{4a+4} \div \frac{a+5}{a+1} &= \frac{a-2}{4a+4} \cdot \frac{a+1}{a+5} \\ &= \frac{a-2}{4(a+1)} \cdot \frac{\cancel{a+1}}{a+5} \\ &= \frac{a-2}{4(a+5)} \end{aligned}$$

Multiply by the reciprocal.

Factor  $4a+4$ .

The GCF is  $a+1$  and simplify.

**Guided Practice**

4A.  $\frac{4d-8}{2d-6} \div \frac{2d-4}{d-4}$

4B.  $\frac{b+4}{3b+2} \div \frac{3b+12}{b+1}$

Sometimes you must factor a quadratic expression before you can simplify the quotient of rational expressions.

**Example 5** Expression Involving Polynomials

Find  $\frac{y-3}{y^2-10y+16} \div \frac{y^2-9}{y-8}$ .

$$\begin{aligned} \frac{y-3}{y^2-10y+16} \div \frac{y^2-9}{y-8} &= \frac{y-3}{y^2-10y+16} \cdot \frac{y-8}{y^2-9} \\ &= \frac{y-3}{(y-2)(y-8)} \cdot \frac{y-8}{(y-3)(y+3)} \\ &= \frac{\cancel{y-3}}{(y-2)(\cancel{y-8})} \cdot \frac{\cancel{y-8}}{\cancel{(y-3)}(y+3)} \\ &= \frac{1}{(y-2)(y+3)} \end{aligned}$$

Multiply by the reciprocal,  $\frac{y-8}{y^2-9}$ .

Factor  $y^2-10y+16$  and  $y^2-9$ .

The GCF is  $(y-3)(y-8)$ .

Simplify.

**StudyTip**

**Canceling** Remember that only factors can be canceled, not individual terms.

**Guided Practice**

Find each quotient.

5A.  $\frac{p^2-4}{5p} \div \frac{p-2}{p+q}$

5B.  $\frac{q^2+3q+2}{12} \div \frac{q+1}{q^2+4}$





**Example 1** Find each product.

1.  $\frac{2x^3}{7x} \cdot \frac{14}{x}$

2.  $\frac{3ab}{4c^4} \cdot \frac{16c^2}{9b}$

3.  $\frac{t^2}{(t-5)(t+5)} \cdot \frac{t+5}{6t}$

4.  $\frac{8}{r+1} \cdot \frac{r^2-1}{2}$

**Example 2** 5. **CCSS PRECISION** The slowest land mammal is the three-toed sloth. It travels 0.07 mile per hour on the ground. What is this speed in feet per minute?

6. **EXERCISE** One hour of moderate inline skating burns approximately 330 Calories. If Nelia plans to do inline skating for 3 hours a week, how many Calories will she burn in a year from the skating?

**Examples 3–5** Find each quotient.

7.  $\frac{8}{3x^2} \div \frac{4}{x}$

8.  $\frac{c^5}{2} \div \frac{c^3}{6d^2}$

9.  $\frac{b^2+6b+5}{6b+6} \div (b+5)$

10.  $\frac{2x+8}{x+3} \div \frac{x+4}{x^2+6x+9}$

Practice and Problem Solving

Extra Practice is on page R11.

**Example 1** Find each product.

11.  $\frac{10n^2}{4} \cdot \frac{2}{n}$

12.  $\frac{12c^3}{21b} \cdot \frac{14b^2}{6c}$

13.  $\frac{x^5y}{2z^3} \cdot \frac{18z^4}{xy}$

14.  $\frac{5c^3d}{c^4d} \cdot \frac{f^2d^3c}{10cf^4}$

15.  $\frac{9}{t-2} \cdot \frac{(t+2)(t-2)}{3}$

16.  $\frac{(a+4)(a-5)}{a^2} \cdot \frac{6a}{a+4}$

17.  $\frac{(k+6)(k-1)}{k+2} \cdot \frac{(k+1)(k+2)}{(k+1)(k-1)}$

18.  $\frac{(r-8)(r+3)}{r} \cdot \frac{2r}{(r+8)(r+3)}$

19.  $\frac{n^2+n-2}{n+2} \cdot \frac{4n}{n-1}$

20.  $\frac{y^2-1}{y^2-49} \cdot \frac{y-7}{y+1}$

**Example 2** 21. **FINANCIAL LITERACY** A scarf bought in Italy cost 18 Euros. The exchange rate at the time was 1 U.S. dollar = 0.73 Euro.

- a. How much did the scarf cost in U.S. dollars?
- b. If the exchange rate at the time was 1 Canadian dollar = 0.69 Euro, how much did the scarf cost in Canadian dollars?

22. **ROLLER COASTERS** A roller coaster has 6 trains. Each train has 3 cars, and each car seats 4 people. Write and simplify an expression including units to find the total number of people that can ride the roller coaster at one time.

**Examples 3–5** Find each quotient.

23.  $\frac{x^5}{y} \div \frac{x}{y^2}$

24.  $\frac{3r^4}{k^2} \div \frac{18r^3}{k}$

25.  $\frac{21b^3}{4c^2} \div \frac{7}{6c^2}$

26.  $\frac{f^4g^2h}{x^2y} \div f^3g$

27.  $\frac{6b-12}{b+5} \div (12b+18)$

28.  $\frac{k+3}{k+2} \div \frac{k}{5k+10}$

29.  $\frac{5x^2}{x^2-5x+4} \div \frac{10x}{x-1}$

30.  $\frac{n^2+7n+12}{16n^2} \div \frac{n+3}{2n}$

31.  $\frac{r+2}{r+1} \div \frac{4}{r^2+3r+2}$

32.  $\frac{3a}{a^2+2a+1} \div \frac{a-1}{a+1}$

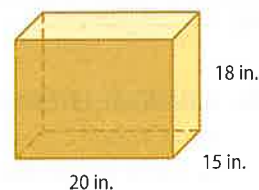


33. **BEARS** A grizzly bear runs 110 feet in 5 seconds. What is the average speed of the bear in miles per hour?
34. **SEWING** The fabric that Megan wants to buy for a costume she is making costs \$7.50 per yard. To the nearest tenth, how many meters can she buy with \$24? (*Hint:* 1 inch = 2.54 centimeters)
35. **TRAVEL** A plane flies 1250 kilometers at an average speed of 540 miles per hour.
- Write a product to find the number of hours it took for the plane to make the trip. Include the units. (*Hint:* 1 mile = 1.609 kilometers).
  - Find the number of hours it took for the plane to make the trip. Round to the nearest tenth.
36. **VOLUNTEERING** Tyree is passing out cups of orange drink from a 3.5-gallon cooler. If each cup of orange drink is 4.25 ounces, about how many cups can he hand out? (*Hint:* There are 128 ounces in a gallon.)
37. **LAND** Louisiana loses about 30 square miles of land each year to coastal erosion, hurricanes, and other natural and human causes. Approximately how many square yards of land are lost per month? (*Hint:* Use 1 square mile = 3,097,600 square yards.)
38. **GEOMETRY** Write an expression to represent the length of the rectangle.

$$A = x^2 + 2x - 24 \quad \frac{x-4}{x-3}$$

Convert each rate. Round to the nearest tenth.

39. 46 feet per second to miles per hour
40. 29.5 meters per second to kilometers per hour
41. 28 milliliters per second to cups per minute. (*Hint:* 1 liter  $\approx$  0.908 quart)
42. 32.4 meters per second to miles per hour. (*Hint:* 1 mile  $\approx$  1.609 kilometers)
43. **LIFE SCIENCE** A human heart pumps about a cup of blood each time it beats. On average, a person's heart beats about 70 times a minute. Write and simplify an expression to find how many gallons of blood are pumped per hour.
44. **GEOMETRY** Refer to the prism at the right.
- Find the volume in cubic inches.
  - Use the ratio  $\frac{1 \text{ foot}^3}{1728 \text{ inches}^3}$  to write a multiplication expression to convert the volume to cubic feet. Then convert the volume.



**CCSS PRECISION** Find each product. Describe what the final answer represents.

45.  $\frac{\$9.80}{1 \text{ hour}} \cdot \frac{15 \text{ hours}}{1 \text{ week}} \cdot \frac{52 \text{ weeks}}{1 \text{ year}}$
46.  $\frac{\$2.85}{1 \text{ gallon of gasoline}} \cdot \frac{15 \text{ gallons of gasoline}}{1 \text{ fill-up}} \cdot \frac{3 \text{ fill-ups}}{1 \text{ month}} \cdot \frac{1 \text{ month}}{30 \text{ days}}$
47.  $\frac{32 \text{ meters}}{1 \text{ second}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{1 \text{ kilometer}}{1000 \text{ meters}} \cdot \frac{1 \text{ mile}}{1.609 \text{ kilometers}}$
48.  $\frac{\$32,000}{1 \text{ year}} \cdot \frac{1 \text{ year}}{52 \text{ weeks}} \cdot \frac{1 \text{ week}}{40 \text{ hours}}$



49. **SPACE** The highest speed at which any spacecraft has ever escaped from Earth is 35,800 miles per hour by the *New Horizons* probe, which was launched in 2006. Convert this speed to feet per second. Round to the nearest tenth.



50. **ELECTRICITY** Simplify the expression below to find the cost of running a 3500-watt air conditioner for one week.

$$3500 \text{ watts} \cdot \frac{1 \text{ kilowatt}}{1000 \text{ watts}} \cdot \frac{168 \text{ hours}}{1 \text{ week}} \cdot \frac{10 \text{ cents}}{1 \text{ kilowatt} \cdot \text{hours}} \cdot \frac{1 \text{ dollar}}{100 \text{ cents}}$$

51. **AMUSEMENT PARKS** In a ride, riders stand along the wall of a circular room with a radius of 3.1 meters. The room completes 27 rotations per minute.
- Write an expression for the number of meters the room moves per second.
  - Simplify the expression you wrote in Part a and describe what it means.
52. **AQUARIUMS** An aquarium is a rectangular prism 30 inches long, 15 inches wide, and 18 inches high.
- Sketch and label a diagram of the aquarium. Then find its volume in cubic inches.
  - Describe how to use the ratio  $\frac{1 \text{ ft}^3}{1728 \text{ in}^3}$  to find the volume of the tank in cubic feet. Then find the volume. Round to the nearest tenth.
  - Pure water weighs 62 pounds per cubic foot. How much would the water in the tank weigh if the tank were filled?
  - A saltwater aquarium requires 2 pounds of minerals for each cubic foot of water. Find the percent concentration by weight of the saltwater by dividing the weight of the minerals by the sum of the weights of the minerals and the water.
  - Kim accidentally added twice the required minerals. Find the percent concentration by weight. Is it twice the recommended concentration? Explain.

### H.O.T. Problems Use Higher-Order Thinking Skills

53. **ERROR ANALYSIS** Mei and Tamika are finding  $\frac{2x+6}{x+5} \div \frac{2}{x+5}$ . Is either of them correct? Explain.

*Mei*

$$\begin{aligned} \frac{2x+6}{x+5} \div \frac{2}{x+5} \\ &= \frac{\cancel{2}x+6}{\cancel{x+5}} \cdot \frac{\cancel{x+5}}{\cancel{2}} \\ &= x+6 \end{aligned}$$

*Tamika*

$$\begin{aligned} \frac{2x+6}{x+5} \div \frac{2}{x+5} \\ &= \frac{2(x+3)}{\cancel{x+5}} \cdot \frac{\cancel{x+5}}{2} \\ &= \frac{4(x+3)}{(x+5)^2} \end{aligned}$$

54. **REASONING** Find the missing term in  $\frac{?}{21} \div \frac{10x^3}{2x} = \frac{3}{2x}$ . Justify your answer.
55. **CCSS PERSEVERANCE** Find  $\frac{x^2 - 3x - 10}{x^2 + 2x - 35} \cdot \frac{x^2 + 4x - 21}{x^2 + 9x + 14}$ . Write in simplest form.
56. **WRITING IN MATH** Give an example and describe how you could use dimensional analysis to solve a real-world problem involving rational expressions.
57. **OPEN ENDED** Give an example of a real-world situation that could be modeled by the quotient of two rational expressions. Provide an example of this quotient.
58. **WRITE A QUESTION** A classmate found that the product of two rational expressions is  $\frac{9x-3}{(x+3)(3x+1)}$ . Write a question to help her find the excluded values.
59. **WRITING IN MATH** Describe how to use dimensional analysis to find the number of hours in one year.



## Standardized Test Practice

**60. GEOMETRY** The perimeter of a rectangle is 30 inches. Its area is 54 square inches. Find the length of the longest side.

- A 6 inches
- B 9 inches
- C 12 inches
- D 30 inches

**61.** Find  $\frac{c^2 - c - 6}{2c - 10} \div \frac{2c + 4}{3c - 15}$ .

- F  $\frac{3(c-3)}{4}$
- G  $\frac{c+5}{c-3}$
- H  $\frac{4(c-3)}{3}$
- J  $\frac{c-3}{c-5}$

**62. EXTENDED RESPONSE** The weekly salaries of six employees at a fast food restaurant are \$140, \$220, \$90, \$180, \$140, \$200.

- a. What is the mean of the six salaries?
- b. What is the median of the six salaries?
- c. What is the mode of the six salaries?

**63.** Tito has three times as many CDs as Dasan. Dasan has two thirds as many CDs as Brant. Brant has 27 CDs. How many CDs does Tito have?

- A 54
- B 32
- C 27
- D 18

## Spiral Review

Simplify each expression. State the excluded values of the variables. (Lesson 11-3)

64.  $\frac{20x^2y}{25xy}$

65.  $\frac{14g^3h^2}{42gh^3}$

66.  $\frac{64qt}{16q^2t^3}$

67.  $\frac{y^2 + 10y + 16}{y + 2}$

68.  $\frac{p^2 - 9}{p^2 - 5p + 6}$

69.  $\frac{z^2 + z - 2}{z^2 - 3z + 2}$

Identify the asymptotes of each function. (Lesson 11-2)

70.  $y = \frac{2}{x}$

71.  $y = \frac{3}{x} + 5$

72.  $y = \frac{1}{x-5} - 4$

73.  $y = \frac{1}{x+3}$

74.  $y = \frac{-1}{x+6} + 7$

75.  $y = \frac{2}{x-8} - 3$

**76. FORESTRY** The number of board feet  $B$  that a log will yield can be estimated by using the formula  $B = \frac{L}{16}(D^2 - 8D + 16)$ , where  $D$  is the diameter in inches and  $L$  is the log length in feet. For logs that are 16 feet long, what diameter will yield approximately 256 board feet? (Lesson 8-9)

Find the degree of each polynomial. (Lesson 8-1)

77. 2

78.  $-3a$

79.  $5x^2 + 3x$

80.  $d^4 - 6c^2$

81.  $2x^3 - 4z + 8xz$

82.  $3d^4 + 5d^3 - 4c^2 + 1$

**83. DRIVING** Tires should be kept within 2 pounds per square inch (psi) of the manufacturer's recommended tire pressure. If the recommendation for a tire is 30 psi, what is the range of acceptable pressures? (Lesson 5-5)

## Skills Review

Factor each polynomial.

84.  $x^2 - 18x - 40$

85.  $x^2 - 5x + 6$

86.  $x^2 - 2x - 24$

87.  $3x^2 + 7x - 20$

88.  $2x^2 + x - 15$

89.  $8x^2 - 4x - 40$



# 11 Mid-Chapter Quiz

## Lessons 11-1 through 11-4

1. Determine whether the table represents an inverse variation. Explain. (Lesson 11-1)

| $x$ | $y$ |
|-----|-----|
| 2   | 8   |
| 4   | 4   |
| 8   | 2   |
| 16  | 1   |

Assume that  $y$  varies inversely as  $x$ . Write an inverse variation equation that relates  $x$  and  $y$ . (Lesson 11-1)

2.  $y = 5$  when  $x = 10$   
 3.  $y = -2$  when  $x = 12$

Solve. Assume that  $y$  varies inversely as  $x$ . (Lesson 11-1)

4. If  $y = 6$  when  $x = 3$ , find  $x$  when  $y = 5$ .  
 5. If  $y = 3$  when  $x = 2$ , find  $y$  when  $x = 4$ .

State the excluded value for each function. (Lesson 11-2)

6.  $y = \frac{2}{x}$   
 7.  $y = \frac{1}{x-6}$

Identify the asymptotes of each function. (Lesson 11-2)

8.  $y = \frac{3}{2x+4}$   
 9.  $y = \frac{2}{x-4}$

10. **MULTIPLE CHOICE** Jorge has  $x^2 + 8x + 15$  square yards of carpet. He wants to carpet rooms that have areas of  $x^2 + 5x + 6$  square yards. Write and simplify an expression to show how many rooms he can carpet.

(Lesson 11-3)

- A  $\frac{x+5}{x+3}$   
 B  $\frac{x+5}{x+2}$   
 C  $\frac{x+3}{x+2}$   
 D  $\frac{x+5}{x+6}$

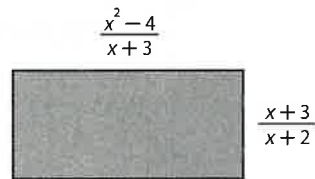
Simplify each expression. State the excluded values of the variables. (Lesson 11-3)

11.  $\frac{16x^2y^3}{8xy}$   
 12.  $\frac{z-5}{z^2-7z+10}$   
 13.  $\frac{3x-15}{x^2-25}$

Find each product. (Lesson 11-4)

14.  $\frac{(x+5)(x-3)}{x^3} \cdot \frac{5x}{x-3}$   
 15.  $\frac{a^2+2a+1}{a+1} \cdot \frac{a-1}{a^2-1}$   
 16.  $\frac{m}{m^2+3m+2} \cdot \frac{m+2}{m^2}$

17. **MULTIPLE CHOICE** Find the area of the rectangle. (Lesson 11-4)



- F  $\frac{x+2}{x-2}$   
 G  $\frac{x+3}{x-2}$   
 H 1  
 J  $x-2$

Find each quotient. (Lesson 11-4)

18.  $\frac{x^4}{y^2} \div \frac{x}{y}$   
 19.  $\frac{x+3}{2x+6} \div \frac{3x-6}{4x-8}$   
 20.  $\frac{x^2+7x+12}{x^2-25} \div \frac{x^2-9}{2x+10}$

21. **ANIMALS** American quarter horses excel at sprinting short distances. A quarter horse named Evening Snow holds the world record for the quarter mile with a time of 20.94 seconds. What is this speed in miles per hour and in meters per second? (Lesson 11-4)

# 11-5 Dividing Polynomials

**Then**

- You divided rational expressions.

**Now**

- 1 Divide a polynomial by a monomial.
- 2 Divide a polynomial by a binomial.

**Why?**

The equation below describes the distance  $d$  a horse travels when its initial velocity is 4 m/s, its final velocity is  $v$  m/s, and its acceleration is  $a$  m/s<sup>2</sup>.

$$d = \frac{v^2 - 4^2}{2a}$$

There are different ways to simplify the expression.

Keep as one fraction.

$$\frac{v^2 - 4^2}{2a} = \frac{v^2 - 16}{2a}$$

Divide each term by  $2a$ .

$$\begin{aligned} \frac{v^2 - 4^2}{2a} &= \frac{v^2}{2a} - \frac{4^2}{2a} \\ &= \frac{v^2}{2a} - \frac{8}{a} \end{aligned}$$



**Common Core State Standards**

**Mathematical Practices**

- 8 Look for and express regularity in repeated reasoning.

**1 Divide Polynomials by Monomials** To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.



**Example 1** Divide Polynomials by Monomials

Find each quotient.

a.  $(2x^2 + 16x) \div 2x$

$$(2x^2 + 16x) \div 2x = \frac{2x^2 + 16x}{2x}$$

Write as a fraction.

$$= \frac{2x^2}{2x} + \frac{16x}{2x}$$

Divide each term by  $2x$ .

$$= \frac{\overset{x}{2x^{\cancel{2}}}}{\underset{1}{\cancel{2x}}} + \frac{\overset{8}{16\cancel{x}}}{\underset{1}{\cancel{2x}}}$$

Divide out common factors.

$$= x + 8$$

Simplify.

b.  $(b^2 + 12b - 14) \div 3b$

$$(b^2 + 12b - 14) \div 3b = \frac{b^2 + 12b - 14}{3b}$$

Write as a fraction.

$$= \frac{b^2}{3b} + \frac{12b}{3b} - \frac{14}{3b}$$

Divide each term by  $3b$ .

$$= \frac{\overset{b}{\cancel{b^2}}}{\underset{3}{\cancel{3b}}} + \frac{\overset{4}{\cancel{12b}}}{\underset{3}{\cancel{3b}}} - \frac{14}{3b}$$

Divide out common factors.

$$= \frac{b}{3} + 4 - \frac{14}{3b}$$

Simplify.

**Guided Practice**

1A.  $(3q^3 - 6q) \div 3q$

1B.  $(4t^5 - 5t^2 - 12) \div 2t^2$

1C.  $(4r^6 + 3r^4 - 2r^2) \div 2r$

1D.  $(6w^3 - 3w) \div 4w^2$



**2 Divide Polynomials by Binomials** You can also divide polynomials by binomials. When a polynomial can be factored and common factors can be divided out, write the division as a rational expression and simplify.



**Example 2** Divide a Polynomial by a Binomial

Find  $(h^2 + 9h + 18) \div (h + 6)$ .

$$\begin{aligned} (h^2 + 9h + 18) \div (h + 6) &= \frac{h^2 + 9h + 18}{h + 6} && \text{Write as a rational expression.} \\ &= \frac{(h + 3)(h + 6)}{h + 6} && \text{Factor the numerator.} \\ &= \frac{(h + 3)\cancel{(h + 6)}}{\cancel{h + 6}} && \text{Divide out common factors.} \\ &= h + 3 && \text{Simplify.} \end{aligned}$$

**Guided Practice** Find each quotient.

2A.  $(b^2 - 2b - 15) \div (b + 3)$

2B.  $(x^2 + 11x + 24) \div (x + 8)$

If the polynomial cannot be factored or if there are no common factors by which to divide, you must use long division.



**Example 3** Use Long Division

Find  $(y^2 + 4y + 12) \div (y + 3)$  by using long division.

**Step 1** Divide the first term of the dividend,  $y^2$ , by the first term of the divisor,  $y$ .

$$\begin{array}{r} y \\ y + 3 \overline{) y^2 + 4y + 12} \\ \underline{(-) y^2 + 3y} \phantom{+ 12} \\ 1y + 12 \end{array} \quad \begin{array}{l} y^2 \div y = y \\ \text{Multiply } y \text{ and } y + 3 \\ \text{Subtract. Bring down the 12.} \end{array}$$

**Step 2** Divide the first term of the partial dividend,  $1y$ , by the first term of the divisor,  $y$ .

$$\begin{array}{r} y + 1 \\ y + 3 \overline{) y^2 + 4y + 12} \\ \underline{(-) y^2 + 3y} \phantom{+ 12} \\ 1y + 12 \\ \underline{(-) y + 3} \\ 9 \end{array} \quad \begin{array}{l} \text{Subtract. Bring down the 12.} \\ \text{Multiply 1 and } y + 3. \\ \text{Subtract.} \end{array}$$

So,  $(y^2 + 4y + 12) \div (y + 3)$  is  $y + 1$  with a remainder of 9. This answer can be written as  $y + 1 + \frac{9}{y + 3}$ .

**Guided Practice** Find each quotient.

3A.  $(3x^2 + 9x - 15) \div (x + 5)$

3B.  $(n^2 + 6n + 2) \div (n - 2)$

**WatchOut!**

**Polynomials** When using long division, be sure the dividend is written in standard form. That is, the terms are written so that the exponents decrease from left to right.

$y^2 + 4y + 12$  yes

$4y + y^2 + 12$  no



### Real-World Example 4 Divide Polynomials to Solve a Problem

**PARTIES** The expression  $5x + 250$  represents the cost of renting a picnic shelter and food for  $x$  people. The total cost is divided evenly among all the people except for the two who bought decorations. Find  $(5x + 250) \div (x - 2)$  to determine how much each person pays.

$$\begin{array}{r} 5 \\ x - 2 \overline{) 5x + 250} \\ \underline{(-) 5x - 10} \\ 260 \end{array}$$

So,  $5 + \frac{260}{x - 2}$  represents the amount each person pays.

#### Guided Practice

4. **GEOMETRY** The area of a rectangle is  $(2x^2 + 10x - 1)$  square units, and the width is  $(x + 1)$  units. What is the length?

When a dividend is written in standard form and a power is missing, add a term of that power with a coefficient of zero.

### Example 5 Insert Missing Terms

Find  $(c^3 + 5c - 6) \div (c - 1)$ .

$$\begin{array}{r} c^2 + c + 6 \\ c - 1 \overline{) c^3 + 0c^2 + 5c - 6} \\ \underline{(-) c^3 - c^2} \\ c^2 + 5c \\ \underline{(-) c^2 - c} \\ 6c - 6 \\ \underline{(-) 6c - 6} \\ 0 \end{array}$$

Insert a  $c^2$ -term that has a coefficient of 0.

Multiply  $c^2$  and  $c - 1$ .

Subtract. Bring down the  $5c$ .

Multiply  $c$  and  $c - 1$ .

Subtract. Bring down the  $-6$ .

Multiply  $6$  and  $c - 1$ .

Subtract.

So,  $(c^3 + 5c - 6) \div (c - 1) = c^2 + c + 6$ .

#### Guided Practice Find each quotient.

5A.  $(2r^3 + 2r^2 - 4) \div (r - 1)$

5B.  $(x^4 + 2x^3 + 6x - 10) \div (x + 2)$

### Check Your Understanding

 = Step-by-Step Solutions begin on page R13.



**Examples 1–2** Find each quotient.

1.  $(8a^2 + 20a) \div 4a$

2.  $(4z^3 + 1) \div 2z$

3.  $(12n^3 - 6n^2 + 15) \div 6n$

4.  $(t^2 + 5t + 4) \div (t + 4)$

5.  $(x^2 + 3x - 28) \div (x + 7)$

6.  $(x^2 + x - 20) \div (x - 4)$

**Example 4** 7. **CHEMISTRY** The number of beakers that can be filled with  $50 + x$  milliliters of a solution is given by  $(400 + 3x) \div (50 + x)$ . How many beakers can be filled.

**Examples 3–5** Find each quotient. Use long division.

8.  $(n^2 + 3n + 10) \div (n - 1)$

9.  $(4y^2 + 8y + 3) \div (y + 2)$

10.  $(4h^3 + 6h^2 - 3) \div (2h + 3)$

11.  $(9n^3 - 13n + 8) \div (3n - 1)$



**Examples 1–2** Find each quotient.

12.  $(14x^2 + 7x) \div 7x$

14.  $(5q^3 + q) \div q$

16.  $(8k^2 - 6) \div 2k$

18.  $(a^2 + a - 12) \div (a - 3)$

20.  $(r^2 - 12r + 11) \div (r - 1)$

22.  $(y^2 - 36) \div (y^2 + 6y)$

24.  $(c^3 - 9c) \div (c - 3)$

26.  $(6x^3 + 15x^2 - 60x + 39) \div (3x^2)$

13.  $(a^3 + 4a^2 - 18a) \div a$

15.  $(6n^2 - 12n + 3) \div 3n$

17.  $(9m^2 + 5m) \div 6m$

19.  $(x^2 - 6x - 16) \div (x + 2)$

21.  $(k^2 - 5k - 24) \div (k - 8)$

23.  $(a^3 - 4a^2) \div (a - 4)$

25.  $(4t^2 - 1) \div (2t + 1)$

27.  $(2h^3 + 8h^2 - 3h - 12) \div (h + 4)$

**Example 4**

28. **GEOMETRY** The area of a rectangle is  $(x^3 - 4x^2)$  square units, and the width is  $(x - 4)$  units. What is the length?

29. **MANUFACTURING** The expression  $-n^2 + 18n + 850$  represents the number of baseball caps produced by  $n$  workers. Find  $(-n^2 + 18n + 850) \div n$  to write an expression for average number of caps produced per person.

**Examples 3, 5** Find each quotient. Use long division.

30.  $(b^2 + 3b - 9) \div (b + 5)$

32.  $(2y^2 - 3y + 1) \div (y - 2)$

34.  $(p^3 - 4p^2 + 9) \div (p - 1)$

36.  $(6x^3 + 5x^2 + 9) \div (2x + 3)$

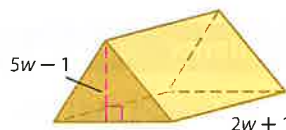
31.  $(a^2 + 4a + 3) \div (a - 1)$

33.  $(4n^2 - 3n + 6) \div (n - 2)$

35.  $(t^3 - 2t - 4) \div (t + 4)$

37.  $(8c^3 + 6c - 5) \div (4c - 2)$

38. **GEOMETRY** The volume of a prism with a triangular base is  $10w^3 + 23w^2 + 5w - 2$ . The height of the prism is  $2w + 1$ , and the height of the triangle is  $5w - 1$ . What is the measure of the base of the triangle? (*Hint:  $V = Bh$* )



Use long division to find the expression that represents the missing length.

39. 
$$\begin{array}{|c|} \hline A = x^2 - 3x - 18 \\ \hline \end{array} \quad ?$$

$$x - 6$$

40. 
$$\begin{array}{|c|} \hline A = \\ \hline \end{array} \quad 2x + 4$$

$$4x^2 + 16x + 16$$

$$?$$

41. Determine the quotient when  $x^3 + 11x + 14$  is divided by  $x + 2$ .

42. What is  $14y^5 + 21y^4 - 6y^3 - 9y^2 + 32y + 48$  divided by  $2y + 3$ ?

43. **CCSS STRUCTURE** Consider  $f(x) = \frac{3x + 4}{x - 1}$ .

- Rewrite the function as a quotient plus a remainder. Then graph the quotient, ignoring the remainder.
- Graph the original function using a graphing calculator.
- How are the graphs of the function and quotient related?
- What happens to the graph near the excluded value of  $x$ ?



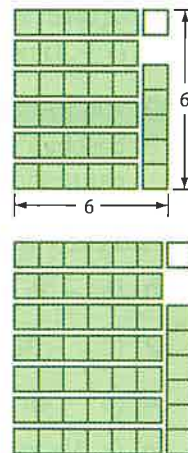
44. **ROAD TRIP** The first Ski Club van has been on the road for 20 minutes, and the second van has been on the road for 35 minutes.
- Write an expression for the amount of time that each van has spent on the road after an additional  $t$  minutes.
  - Write a ratio for the first van's time on the road to the second van's time on the road and use long division to rewrite this ratio as an expression. Then find the ratio of the first van's time on the road to the second van's time on the road after 60 minutes, 200 minutes.

45. **BOILING POINT** The temperature at which water boils decreases by about  $0.9^\circ\text{F}$  for every 500 feet above sea level. The boiling point at sea level is  $212^\circ\text{F}$ .

- Write an equation for the temperature  $T$  at which water boils  $x$  feet above sea level.
- Mount Whitney, the tallest point in California, is 14,494 feet above sea level. At approximately what temperature does water boil on Mount Whitney?

46. **MULTIPLE REPRESENTATIONS** In this problem, you will use picture models to help divide expressions.

- Analytical** The first figure models  $6^2 \div 7$ . Notice that the square is divided into seven equal parts. What are the quotient and the remainder? What division problem does the second figure model?
- Concrete** Draw figures for  $3^2 \div 4$  and  $2^2 \div 3$ .
- Verbal** Do you observe a pattern in the previous exercises? Express this pattern algebraically.
- Analytical** Use long division to find  $x^2 \div (x + 1)$ . Does this result match your expression from part c?



### H.O.T. Problems Use Higher-Order Thinking Skills

47. **ERROR ANALYSIS** Alvin and Andrea are dividing  $c^3 + 6c - 4$  by  $c + 2$ . Is either of them correct? Explain your reasoning.

*Alvin*

$$\begin{array}{r}
 c^2 + 4c - 12 \\
 c + 2 \overline{) c^3 + 6c - 4} \\
 \underline{c^3 + 2c^2} \phantom{- 4} \\
 4c^2 - 4 \\
 \underline{4c^2 + 8c} \phantom{- 4} \\
 -12c - 4 \\
 \underline{-12c - 24} \\
 24
 \end{array}$$

*Andrea*

$$\begin{array}{r}
 c^2 - 2c + 10 \\
 c + 2 \overline{) c^3 + 0c^2 + 6c - 4} \\
 \underline{c^3 + 2c^2} \phantom{- 4} \\
 -2c^2 + 6c - 4 \\
 \underline{-2c^2 - 4c} \phantom{- 4} \\
 10c - 4 \\
 \underline{10c + 20} \\
 -24
 \end{array}$$

48. **CCSS REGULARITY** The quotient of two polynomials is  $4x^2 - x - 7 + \frac{11x + 15}{x^2 + x + 2}$ . What are the polynomials?
49. **OPEN ENDED** Write a division problem involving polynomials that you would solve by using long division. Explain your answer.
50. **WRITING IN MATH** Describe the steps to find  $(w^2 - 2w - 30) \div (w + 7)$ .

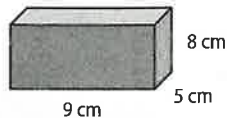


## Standardized Test Practice

51. Simplify  $\frac{21x^3 - 35x^2}{7x}$ .

- A  $3x^2 - 5x$                       C  $3x - 5$   
 B  $4x^2 - 6x$                       D  $5x - 3$

52. **EXTENDED RESPONSE** The box shown is designed to hold rice.



- a. What is the volume of the box?  
 b. What is the area of the label on the box, if the label covers all surfaces?

53. Simplify  $\frac{x^2 + 7x + 12}{x^2 + 5x + 6}$ .

- F  $x + 4$                               H  $x + 2$   
 G  $\frac{x+4}{x+2}$                               J  $\frac{x+2}{x+4}$

54. Susana bought cards at 6 for \$10. She decorated them and sold them at 4 for \$10. She made \$60 in profit. How many cards did she buy and sell if she had none left?

- A 25                                      C 60  
 B 53                                      D 72

## Spiral Review

Find each product. (Lesson 11-4)

55.  $\frac{3x^3}{8x} \cdot \frac{16}{x}$

56.  $\frac{3ad}{4c^4} \cdot \frac{8c^2}{6d}$

57.  $\frac{t^2}{(t-4)(t+4)} \cdot \frac{t-4}{6t}$

58.  $\frac{10}{r-2} \cdot \frac{r^2-4}{2}$

Find the zeros of each function. (Lesson 11-3)

59.  $f(x) = \frac{x+2}{x^2-6x+8}$

60.  $f(x) = \frac{x^2-3x-4}{x^2-x-12}$

61.  $f(x) = \frac{x^2+6x+9}{x^2-9}$

62. **SHADOWS** A flagpole casts a shadow that is 10 feet long when the Sun is at an elevation of  $68^\circ$ . How tall is the flagpole? (Lesson 10-6)

Solve each equation. Check your solution. (Lesson 10-4)

63.  $\sqrt{h} = 9$

64.  $\sqrt{x+3} = -5$

65.  $3 + 5\sqrt{n} = 18$

66.  $\sqrt{x-5} = 2\sqrt{6}$

Solve each equation by using the Quadratic Formula. Round to the nearest tenth if necessary. (Lesson 9-5)

67.  $v^2 + 12v + 20 = 0$

68.  $3t^2 - 7t - 20 = 0$

69.  $5y^2 - y - 4 = 0$

70.  $2x^2 + 98 = 28x$

71.  $2n^2 - 7n - 3 = 0$

72.  $2w^2 = -(7w + 3)$

73. **THEATER** A backdrop for a play uses a series of thin metal arches attached to the stage floor. For each arch the height  $y$ , in feet, is modeled by the equation  $y = -x^2 + 6x$ , where  $x$  is the distance, in feet, across the bottom of the arch. (Lesson 9-2)

- a. Graph the related function and determine the width of the arch at the floor.  
 b. What is the height at the top of the arch?

## Skills Review

Find each sum.

74.  $(3a^2 + 2a - 12) + (8a + 7 - 2a^2)$

75.  $(2c^3 + 3cd - d^2) + (-5cd - 2c^3 + 2d^2)$



## Adding and Subtracting Rational Expressions

### Then

- You added and subtracted polynomials.

### Now

- Add and subtract rational expressions with like denominators.
- Add and subtract rational expressions with unlike denominators.

### Why?



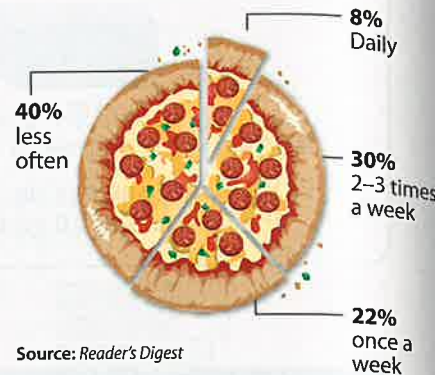
- A survey asked families how often they eat takeout. To determine the fraction of those surveyed who eat takeout more than once a week, you can add. Remember that percents can be written as fractions with denominators of 100.

2-3 times a week plus daily equals more than once a week.

$$\frac{30}{100} + \frac{8}{100} = \frac{38}{100}$$

Thus,  $\frac{38}{100}$  or 38% eat takeout more than once a week.

### How Many Times a Week Families Eat Takeout



**New Vocabulary**  
 least common multiple (LCM)  
 least common denominator (LCD)



**Common Core State Standards**  
 Mathematical Practices  
 6 Attend to precision.

**1 Add and Subtract Rational Expressions with Like Denominators** To add or subtract rational expressions that have the same denominator, add or subtract the numerators and write the sum or difference over the common denominator.

### KeyConcept Add or Subtract Rational Expressions with Like Denominators

Let  $a$ ,  $b$ , and  $c$  be polynomials with  $c \neq 0$ .

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

### Example 1 Add Rational Expressions with Like Denominators

Find  $\frac{5n}{n+3} + \frac{15}{n+3}$ .

$$\frac{5n}{n+3} + \frac{15}{n+3} = \frac{5n+15}{n+3}$$

The common denominator is  $n+3$ .

$$= \frac{5(n+3)}{n+3}$$

Factor the numerator.

$$= \frac{5(\cancel{n+3})}{\cancel{n+3}}$$

Divide by the common factor,  $n+3$ .

$$= \frac{5}{1} \text{ or } 5$$

Simplify.

### Guided Practice

Find each sum.

1A.  $\frac{8c}{6} + \frac{5c}{6}$

1B.  $\frac{4t}{5xy} + \frac{7}{5xy}$

1C.  $\frac{3y}{3+y} + \frac{y^2}{3+y}$





### Example 2 Subtract Rational Expressions with Like Denominators

Find  $\frac{3m-5}{m+4} - \frac{4m+2}{m+4}$ .

$$\begin{aligned} \frac{3m-5}{m+4} - \frac{4m+2}{m+4} &= \frac{(3m-5) - (4m+2)}{m+4} \\ &= \frac{(3m-5) + [-(4m+2)]}{m+4} \\ &= \frac{3m-5-4m-2}{m+4} \\ &= \frac{-m-7}{m+4} \end{aligned}$$

The common denominator is  $m+4$ .

The additive inverse of  $(4m+2)$  is  $-(4m+2)$ .

Distributive Property

Simplify.

#### StudyTip

**Checking Answers** You can check whether you have simplified a rational expression correctly by substituting values, but this does not guarantee that the expressions are always equal. If the results are different, check for an error.

#### GuidedPractice

Find each difference.

2A.  $\frac{2h+4}{h+1} - \frac{5+h}{h+1}$

2B.  $\frac{17h+4}{15h-5} - \frac{2h-6}{15h-5}$

You can sometimes use additive inverses to form like denominators.



### Example 3 Inverse Denominators

Find  $\frac{3n}{n-4} + \frac{6n}{4-n}$ .

$$\begin{aligned} \frac{3n}{n-4} + \frac{6n}{4-n} &= \frac{3n}{n-4} + \frac{6n}{-(n-4)} \\ &= \frac{3n}{n-4} - \frac{6n}{n-4} \\ &= \frac{3n-6n}{n-4} \text{ or } -\frac{3n}{n-4} \end{aligned}$$

Rewrite  $4-n$  as  $-(n-4)$ .

Rewrite so the denominators are the same.

Subtract the numerators and simplify.

#### GuidedPractice

Find each sum or difference.

3A.  $\frac{t^2}{t-3} + \frac{3}{3-t}$

3B.  $\frac{2p}{p-1} - \frac{2p}{1-p}$

**2 Add and Subtract with Unlike Denominators** The **least common multiple (LCM)** is the least number that is a multiple of two or more numbers or polynomials.



### Example 4 LCMs of Polynomials

Find the LCM of each pair of polynomials.

a.  $6x$  and  $4x^3$

**Step 1** Find the prime factors of each expression.

$6x = 2 \cdot 3 \cdot x$                        $4x^3 = 2 \cdot 2 \cdot x \cdot x \cdot x$

**Step 2** Use each prime factor, 2, 3, and  $x$ , the greatest number of times it appears in either of the factorizations.

$6x = 2 \cdot 3 \cdot x$                        $4x^3 = 2 \cdot 2 \cdot x \cdot x \cdot x$

LCM =  $2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot x$  or  $12x^3$



### Review Vocabulary

**Factored Form** A monomial is in factored form when it is expressed as the product of prime numbers and variables, and no variable has an exponent greater than 1.

b.  $n^2 + 5n + 4$  and  $(n + 1)^2$

$$n^2 + 5n + 4 = (n + 1)(n + 4) \quad \text{Factor each expression.}$$

$$(n + 1)^2 = (n + 1)(n + 1)$$

$(n + 1)$  is a factor twice in the second expression.  $(n + 4)$  is a factor once.

$$\text{LCM} = (n + 1)(n + 1)(n + 4) \text{ or } (n + 1)^2(n + 4)$$

### Guided Practice

4A.  $8m^2t$  and  $12m^2t^3$

4B.  $x^2 - 2x - 8$  and  $x^2 - 5x - 14$

To add or subtract fractions with unlike denominators, you need to rename the fractions using the least common multiple of the denominators, called the **least common denominator (LCD)**.

### Key Concept Add or Subtract Rational Expressions with Unlike Denominators

**Step 1** Find the LCD.

**Step 2** Write each rational expression as an equivalent expression with the LCD as the denominator.

**Step 3** Add or subtract the numerators and write the result over the common denominator.

**Step 4** Simplify if possible.

### WatchOut!

**CCSS Precision** Remember that every term of the numerator and the denominator must be multiplied or divided by the same number for the fraction to be equivalent to the original.

### Example 5 Add Rational Expressions with Unlike Denominators

Find  $\frac{3t + 2}{t^2 - 2t - 3} + \frac{t + 1}{t - 3}$ .

Find the LCD. Since  $t^2 - 2t - 3 = (t - 3)(t + 1)$ , the LCD is  $(t - 3)(t + 1)$ .

$$\frac{3t + 2}{t^2 - 2t - 3} + \frac{t + 1}{t - 3} = \frac{3t + 2}{(t - 3)(t + 1)} + \frac{t + 1}{t - 3} \quad \text{Factor } t^2 - 2t - 3.$$

$$= \frac{3t + 2}{(t - 3)(t + 1)} + \frac{t + 1}{t - 3} \left( \frac{t + 1}{t + 1} \right) \quad \text{Write } \frac{t + 1}{t - 3} \text{ using the LCD.}$$

$$= \frac{3t + 2}{(t - 3)(t + 1)} + \frac{t^2 + 2t + 1}{(t - 3)(t + 1)} \quad \text{Simplify.}$$

$$= \frac{3t + 2 + t^2 + 2t + 1}{(t - 3)(t + 1)} \quad \text{Add the numerators.}$$

$$= \frac{t^2 + 5t + 3}{(t - 3)(t + 1)} \quad \text{Simplify.}$$

### Guided Practice

5A.  $\frac{4d^2}{d} + \frac{d + 2}{d^2}$

5B.  $\frac{b + 3}{b} + \frac{b - 5}{b + 1}$



The formula  $\text{time} = \frac{\text{distance}}{\text{rate}}$  is helpful in solving real-world applications.



### Real-World Example 6 Add Rational Expressions

**HANG GLIDING** For the first 5000 meters, a hang glider travels at a rate of  $x$  meters per minute. Then, due to a stronger wind, it travels 6000 meters at a speed that is 3 times as fast.

a. Write an expression to represent how much time the hang glider is flying.

**Understand** For the first 5000 meters, the hang glider's speed is  $x$ . For the last 6000 meters, the hang glider's speed is  $3x$ .

**Plan** Use the formula  $d = r \times t$  or  $t = \frac{d}{r}$  to represent the time  $t$  of each section of the hang glider's trip, with rate  $r$  and distance  $d$ .

**Solve** Time to fly 5000 meters:  $\frac{d}{r} = \frac{5000}{x}$        $d = 5000, r = x$

Time to fly 6000 meters:  $\frac{d}{r} = \frac{6000}{3x}$        $d = 6000, r = 3x$

Total flying time:  $\frac{5000}{x} + \frac{6000}{3x}$

$$\frac{5000}{x} + \frac{6000}{3x} = \frac{5000}{x} \left(\frac{3}{3}\right) + \frac{6000}{3x}$$

The LCD is  $3x$ .

$$= \frac{15,000}{3x} + \frac{6000}{3x}$$

Multiply.

$$= \frac{\overset{7000}{\cancel{21,000}}}{\underset{1}{\cancel{3x}}} \text{ or } \frac{7000}{x}$$

Simplify.

**Check**  $\frac{5000}{x} + \frac{6000}{3x} = \frac{5000}{1} + \frac{6000}{3(1)}$   
 $= 5000 + 2000$  or 7000

Let  $x = 1$  in the original expression.

Simplify.

$$\frac{7000}{x} = \frac{7000}{1} \text{ or } 7000$$

Let  $x = 1$  in the answer expression. Simplify.

Since the expressions have the same value for  $x = 1$ , the answer is reasonable. ✓

b. If the hang glider is flying at a rate of 600 meters per minute for the first 5000 meters, find the total amount of time that the hang glider is flying.

$$\frac{7000}{x} = \frac{7000}{600} \quad \text{Substitute 600 for } x \text{ in the expression.}$$

$$\approx 11.7 \quad \text{Simplify.}$$

So, the hang glider is flying for approximately 11.7 minutes.

c. If the hang glider flew for approximately 15 minutes, find the rate the hang glider flew for the first 5000 meters.

$$\frac{7000}{x} = 15 \quad \text{Set the expression equal to 15.}$$

$$7000 = 15x \quad \text{Multiply each side by } x.$$

$$466.7 \approx x \quad \text{Divide each side by 15 and simplify.}$$

The hang glider was flying at a rate of 466.7 meters per minute.

### Guided Practice

6. **TRAINS** A train travels 5 miles from Lynbrook to Long Beach and then back. The train travels about 1.2 times as fast returning from Long Beach. If  $r$  is the train's speed from Lynbrook to Long Beach, write and simplify an expression for the total time of the round trip.

### Real-WorldLink

The distance a hang glider can travel is determined by its *glide ratio*, or the ratio of the forward distance traveled to the vertical distance dropped.

Source: HowStuffWorks



To subtract rational expressions with unlike denominators, rename the expressions using the LCD. Then subtract the numerators.



### Example 7 Subtract Rational Expressions with Unlike Denominators

Find  $\frac{5}{x} - \frac{2x+1}{4x}$ .

$$\frac{5}{x} - \frac{2x+1}{4x} = \frac{5}{x} \left( \frac{4}{4} \right) - \frac{2x+1}{4x}$$

Write  $\frac{5}{x}$  using the LCD,  $4x$ .

$$= \frac{20}{4x} - \frac{2x+1}{4x}$$

Simplify.

$$= \frac{20 - (2x + 1)}{4x}$$

Subtract the numerators.

$$= \frac{20 - 2x - 1}{4x} \text{ or } \frac{19 - 2x}{4x}$$

Simplify.

#### StudyTip

##### Simplifying Answers

When simplifying a rational expression, you can leave the denominator in factored form, or multiply the terms.

#### Guided Practice

Find each difference.

7A.  $\frac{6}{t+3} - \frac{7}{t}$

7B.  $\frac{y}{y-3} - \frac{2}{y^2+y-12}$

### Check Your Understanding

= Step-by-Step Solutions begin on page R13.



Examples 1–3 Find each sum or difference.

1.  $\frac{3}{7n} + \frac{2}{7n}$

2.  $\frac{x+8}{2} + \frac{x}{2}$

3.  $\frac{14r}{9-r} - \frac{2r}{r-9}$

4.  $\frac{7}{5t} - \frac{3+t}{5t}$

Example 4 Find the LCM of each pair of polynomials.

5.  $3t, 8t^2$

6.  $5m + 15, 2m + 6$

7.  $(x^2 - 8x + 7), (x^2 + x - 2)$

Examples 5–7 Find each sum or difference.

8.  $\frac{6}{n^4} + \frac{2}{n^2}$

9.  $\frac{3}{4x} + \frac{2}{5y}$

10.  $\frac{4}{5n} - \frac{1}{10n^3}$

11.  $\frac{8}{3c} - \frac{-5}{6d}$

12.  $\frac{a}{a+4} + \frac{6}{a+2}$

13.  $\frac{x}{x-3} - \frac{3}{x+2}$

Example 6 14. **EXERCISE** Joseph walks 10 times around the track at a rate of  $x$  laps per hour. He runs 8 times around the track at a rate of  $3x$  laps per hour. Write and simplify an expression for the total time it takes him to go around the track 18 times.

### Practice and Problem Solving

Extra Practice is on page R11.

Examples 1–3 Find each sum or difference.

15.  $\frac{a}{4} + \frac{3a}{4}$

16.  $\frac{1}{6m} + \frac{5m}{6m}$

17.  $\frac{5y}{6} - \frac{y}{6}$

18.  $\frac{11}{4r} - \frac{-1}{4r}$

19.  $\frac{8b}{ab} + \frac{3a}{ab}$

20.  $\frac{t+2}{3} + \frac{t+5}{3}$

21.  $\frac{3c-7}{2c-1} + \frac{2c+1}{1-2c}$

22.  $\frac{15x}{33x-9} + \frac{3}{9-33x}$

23.  $\frac{n+6}{10} - \frac{n+1}{10}$

24.  $\frac{5x+2}{2x+5} - \frac{x-8}{2x+5}$

25.  $\frac{w+2}{8w} - \frac{2w-3}{8w}$

26.  $\frac{3a+1}{a-1} - \frac{a+4}{a-1}$



**Example 4** Find the LCM of each pair of polynomials.

27.  $x^3y, x^2y^2$

28.  $5ab, 10b$

29.  $(3r - 1), (r + 2)$

30.  $2n - 10, 4n - 20$

31.  $(x^2 + 9x + 18), x + 3$

32.  $(k^2 - 2k - 8), (k + 2)^2$

**Examples 5, 7** Find each sum or difference.

33.  $\frac{5}{4x} + \frac{1}{10x}$

34.  $\frac{6}{r} + \frac{2}{r^2}$

35.  $\frac{3}{2a} + \frac{1}{5b}$

36.  $\frac{6g}{g+5} - \frac{g-2}{2g}$

37.  $\frac{7}{4k+8} - \frac{k}{k+2}$

38.  $\frac{5}{2d+2} - \frac{d}{d+5}$

39.  $\frac{-2}{7r} + \frac{4}{t}$

40.  $\frac{n}{n-2} + \frac{n}{n+1}$

41.  $\frac{d}{d+5} + \frac{7}{d-1}$

42.  $\frac{4}{a} - \frac{1}{3a}$

43.  $\frac{6}{5t^2} - \frac{2}{3t}$

44.  $\frac{7}{4r} - \frac{3}{t}$

45.  $\frac{w-3}{w^2-w-20} + \frac{w}{w+4}$

46.  $\frac{n}{2n+10} + \frac{1}{n^2-25}$

47.  $\frac{2x}{x^2+8x+15} - \frac{x+3}{x+5}$

48.  $\frac{r-3}{r^2+6r+9} - \frac{r-9}{r^2-9}$

**Example 6**

49. **TRAVEL** Grace walks to her friend's house 2 miles away and then jogs back home. Her jogging speed is 2.5 times her walking speed  $w$ .

a. Write and simplify an expression to represent the amount of time Grace spends going to and coming from her friend's house.

b. If Grace walks about 3.5 miles per hour, how many minutes did she spend going to and from her friend's house?

50. **BOATS** A boat travels 3 miles downstream and then goes 6 miles upstream. The speed of a boat going downstream is  $x + 2$  miles per hour and going upstream is  $x - 2$  miles per hour, where  $x$  is the speed of the boat in still water.

a. Write and simplify an expression to represent the total time it takes the boat to travel 3 miles downstream and 6 miles upstream.

b. If the rate of the boat in still water  $x$  is 4 miles per hour, how long did it take the boat to travel the 9 miles?

51. **SCHOOL** On Saturday, Mr. Kim graded 2 geometry tests and all his algebra tests  $a$ . On Sunday, he graded 7 more geometry tests. Write an expression to represent the fraction of tests he graded if he had 18 more geometry tests than algebra tests.

52. **CCSS REASONING** A total of 1248 people attended the school play. The same number  $x$  attended each of the two Sunday performances. There were twice as many people at the Saturday performance than at both Sunday performances. Write an expression to represent the fraction of people who attended on Saturday.

Find each sum or difference.

53.  $\frac{x+5}{x^2-4} - \frac{3}{x^2-4}$

55.  $\frac{k^2-26}{k-5} - \frac{1}{5-k}$

57.  $\frac{2}{x-1} + \frac{3}{x+1} - \frac{4x-2}{x^2-1}$

59.  $\frac{a^2-5a}{3a-18} - \frac{7a-36}{3a-18}$

61.  $\frac{x^2-16}{x^3} + \frac{x^3+1}{x^4}$

63.  $\frac{5x}{3x^2+19x-14} - \frac{1}{9x^2-12x+4}$

54.  $\frac{18y}{9y+2} - \frac{-4}{-2-9y}$

56.  $\frac{8}{c-1} + \frac{c}{1-c}$

58.  $\frac{x^2-x-12}{x^2-11x+30} - \frac{x-4}{18-x}$

60.  $\frac{8n-3}{n^2+8n+12} - \frac{5n-9}{n^2+8n+12}$

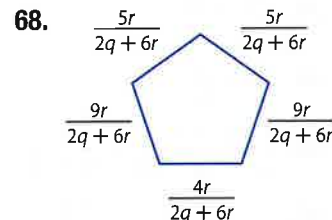
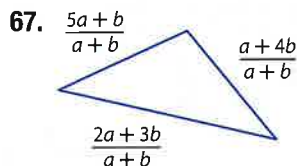
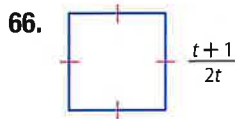
62.  $\frac{x}{7x-3} + \frac{x+2}{15x+30}$

64.  $\frac{2x+7}{x^2-y^2} + \frac{-5}{x^2-2xy+y^2}$

65. **TRIATHLONS** In a sprint triathlon, athletes swim 400 meters, bike 20 kilometers, and run 5 kilometers. An athlete bikes 12 times as fast as she swims and runs 5 times as fast as she swims.

- a. Simplify  $\frac{400}{x} + \frac{20,000}{12x} + \frac{5000}{5x}$ , an expression that represents the time it takes the athlete to complete the sprint triathlon.
- b. If the athlete swims 40 meters per minute, find the total time it takes her to complete the triathlon.

**GEOMETRY** Write an expression for the perimeter of each figure.



69. **BIKES** Marina rides her bike at an average rate of 10 miles per hour. On one day, she rides 9 miles and then rides around a large loop  $x$  miles long. On the second day, she rides 5 miles and then rides around the loop three times.
- a. Write an expression to represent the total time she spent riding her bike on those two days. Then simplify the expression.
- b. If the loop is 2 miles long, how long did Marina ride on those two days?
70. **TRAVEL** The Showalter family drives 80 miles to a college football game. On the trip home, their average speed is about 3 miles per hour slower.
- a. Let  $x$  represent the average speed of the car on the way to the game. Write and simplify an expression to represent the total time it took driving to the game and then back home.
- b. If their average speed on the way to the game was 68 miles per hour, how long did it take the Showalter family to drive to the game and back? Round to the nearest tenth.

**H.O.T. Problems** Use Higher-Order Thinking Skills

71. **CCSS PERSEVERANCE** Find  $\left(\frac{4}{7y-2} + \frac{7y}{2-7y}\right)\left(\frac{y+5}{6} - \frac{y+3}{6}\right)$ .

72. **WRITING IN MATH** Describe in words the steps you use to find the LCD in an addition or subtraction of rational expressions with unlike denominators.

73. **CHALLENGE** Is the following statement *sometimes*, *always*, or *never* true? Explain.

$$\frac{a}{x} + \frac{b}{y} = \frac{ay + bx}{xy}; x \neq 0, y \neq 0$$

74. **OPEN ENDED** Describe a real-life situation that could be expressed by adding two rational expressions that are fractions. Explain what the denominator and numerator represent in both expressions.

75. **WRITING IN MATH** Describe how to add rational expressions with denominators that are additive inverses.



## Standardized Test Practice

**76. SHORT RESPONSE** An object is launched upwards with an initial velocity of 19.6 meters per second from a 58.8-meter-tall platform. The equation for the object's height  $h$ , in meters, at time  $t$  seconds after launch is  $h(t) = -4.9t^2 + 19.6t + 58.8$ . How long after the launch does the object strike the ground?

**77.** Simplify  $\frac{2}{5} + \frac{3}{25} + \frac{1}{10}$ .

A  $\frac{2}{5}$

C  $\frac{31}{50}$

B  $\frac{3}{4}$

D  $\frac{5}{3}$

**78. STATISTICS** Courtney has grades of 84, 65, and 76 on three math tests. What grade must she earn on the next test to have an average of exactly 80 for the four tests?

F 80

H 92

G 84

J 95

**79.** Simplify  $\frac{2}{x} + \frac{3}{x^2} + \frac{1}{2x}$ .

A  $\frac{3x+2}{x^2}$

C  $\frac{5x+6}{2x^2}$

B  $\frac{6}{2x^2}$

D  $\frac{6+x}{x^2}$

## Spiral Review

Find each quotient. (Lesson 11-5)

**80.**  $(6x^2 + 10x) \div 2x$

**81.**  $(15y^3 + 14y) \div 3y$

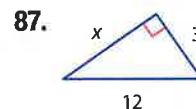
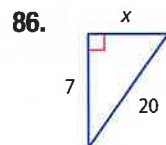
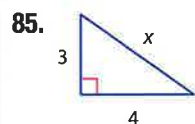
**82.**  $(10a^3 - 20a^2 + 5a) \div 5a$

Convert each rate. Round to the nearest tenth if necessary. (Lesson 11-4)

**83.** 23 feet per second to miles per hour

**84.** 118 milliliters per second to quarts per hour (Hint: 1 liter  $\approx$  1.06 quarts)

Find the length of the missing side. If necessary, round to the nearest hundredth. (Lesson 10-5)



**88. AMUSEMENT RIDE** The height  $h$  in feet of a car above the exit ramp of a free-fall ride can be modeled by  $h(t) = -16t^2 + s$ , where  $t$  is the time in seconds after the car drops, and  $s$  is the starting height of the car in feet. If the designer wants the free fall to last 3 seconds, what should be the starting height in feet? (Lesson 8-9)

Express each number in scientific notation. (Lesson 7-4)

**89.** 12,300

**90.** 0.0000375

**91.** 1,255,000

**92. FINANCIAL LITERACY** Ruben has \$13 to order pizza. The pizza costs \$7.50 plus \$1.25 per topping. He plans to tip 15% of the total cost. Write and solve an inequality to find out how many toppings he can order. (Lesson 5-3)

## Skills Review

Find each quotient.

**93.**  $\frac{12}{3x^2} \div \frac{6}{x}$

**94.**  $\frac{8^4}{2} \div \frac{8^3}{8a^2}$

**95.**  $\frac{4y-8}{y+1} \div (y-2)$



## Mixed Expressions and Complex Fractions

Then

You simplified rational expressions.

Now

- 1 Simplify mixed expressions.
- 2 Simplify complex fractions.

Why?

A Top Fuel dragster can cover  $\frac{1}{4}$  mile in  $4\frac{2}{5}$  seconds. The average speed in miles per second can be described by the expression below. It is called a complex fraction.

$$\frac{\frac{1}{4} \text{ mile}}{4\frac{2}{5} \text{ seconds}}$$



**New Vocabulary**  
mixed expression  
complex fraction



**Common Core State Standards**

**Mathematical Practices**  
4 Model with mathematics.

**1 Simplify Mixed Expressions** An expression like  $2 + \frac{4}{x+1}$  is called a **mixed expression** because it contains the sum of a monomial, 2, and a rational expression,  $\frac{4}{x+1}$ . You can use the LCD to change a mixed expression to a rational expression.

**Example 1** Change Mixed Expressions to Rational Expressions

Write  $2 + \frac{4}{x-1}$  as a rational expression.

$$\begin{aligned} 2 + \frac{4}{x-1} &= \frac{2(x-1)}{x-1} + \frac{4}{x-1} && \text{The LCD is } x-1. \\ &= \frac{2(x-1) + 4}{x-1} && \text{Add the numerators.} \\ &= \frac{2x - 2 + 4}{x-1} && \text{Distributive Property} \\ &= \frac{2x + 2}{x-1} && \text{Simplify.} \end{aligned}$$

**Guided Practice**

Write each mixed expression as a rational expression.

1A.  $2 + \frac{5}{x}$

1B.  $\frac{6y}{4y+8} + 5y$

**2 Simplify Complex Fractions** A **complex fraction** has one or more fractions in the numerator or denominator. You can simplify by using division.

numerical complex fraction

$$\begin{aligned} \frac{\frac{2}{3}}{\frac{5}{8}} &= \frac{2}{3} \div \frac{5}{8} \\ &= \frac{2}{3} \times \frac{8}{5} \\ &= \frac{16}{15} \end{aligned}$$

algebraic complex fraction

$$\begin{aligned} \frac{\frac{a}{b}}{\frac{c}{d}} &= \frac{a}{b} \div \frac{c}{d} \\ &= \frac{a}{b} \times \frac{d}{c} \\ &= \frac{ad}{bc} \end{aligned}$$

To simplify a complex fraction, write it as a division expression. Then find the reciprocal of the second expression and multiply.





### Real-WorldLink

A Jr. Dragster is a half-scale version of a Top Fuel dragster. This car, which can go  $\frac{1}{8}$  mile in  $7\frac{9}{10}$  seconds, is designed to be driven by kids ages 8–17 in the NHRA Jr. Drag Racing League.

Source: NHRA

## Real-World Example 2 Use Complex Fractions to Solve Problems



**RACING** Refer to the application at the beginning of the lesson. Find the average speed of the Top Fuel dragster in miles per minute.

$$\begin{aligned} \frac{\frac{1}{4} \text{ mile}}{4\frac{2}{5} \text{ seconds}} &= \frac{\frac{1}{4} \text{ mile}}{4\frac{2}{5} \cancel{\text{seconds}}} \times \frac{60 \cancel{\text{seconds}}}{1 \text{ minute}} \\ &= \frac{\frac{1}{4} \times 60}{4\frac{2}{5}} \\ &= \frac{60}{\frac{22}{5}} \\ &= \frac{15}{\cancel{4} \times 22} \times \frac{5}{1} \\ &= \frac{75}{22} \text{ or } 3\frac{9}{22} \end{aligned}$$

Convert seconds to minutes. Divide by common units.

Simplify.

Express each term as an improper fraction.

Use the rule  $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$ .

Simplify.

So, the average speed of the Top Fuel dragster is  $3\frac{9}{22}$  miles per minute.

### Guided Practice

2. **RACING** Refer to the information about the Jr. Dragster at the left. What is the average speed of the car in feet per second?

To simplify complex fractions, you can either use the rule as in Example 2, or you can rewrite the fraction as a division expression, as shown below.

## Example 3 Complex Fractions Involving Monomials



Simplify  $\frac{\frac{8t^2}{v}}{\frac{4t}{v^3}}$ .

$$\frac{\frac{8t^2}{v}}{\frac{4t}{v^3}} = \frac{8t^2}{v} \div \frac{4t}{v^3}$$

Write as a division expression.

$$= \frac{8t^2}{v} \times \frac{v^3}{4t}$$

To divide, multiply by the reciprocal.

$$= \frac{\cancel{8}t^2}{\cancel{4}t} \times \frac{\cancel{v^3}}{\cancel{v}} \text{ or } 2tv^2$$

Divide by the common factors  $4t$  and  $v$  and simplify.

### Guided Practice

Simplify each expression.

3A.  $\frac{\frac{g^3h}{b}}{\frac{gh^3}{b^2}}$

3B.  $\frac{\frac{-24m^3t^5}{p^2h}}{\frac{16pm^2}{t^4h}}$





### Real-World Career

#### Lab Technician

Lab technicians work with scientists, running experiments, conducting research projects, and running routine diagnostic samples. Lab technicians in any field need at least a two-year associate degree.

### Study Tip

**Factoring** When simplifying fractions involving polynomials, factor the numerator and the denominator of each expression if possible.

### Example 4 Complex Fractions Involving Polynomials

Simplify each expression.

a. 
$$\frac{\frac{2}{y+3}}{\frac{5}{y^2-9}}$$

$$\frac{\frac{2}{y+3}}{\frac{5}{y^2-9}} = \frac{2}{y+3} \div \frac{5}{y^2-9}$$

Write as a division expression.

$$= \frac{2}{y+3} \times \frac{y^2-9}{5}$$

To divide, multiply by the reciprocal.

$$= \frac{2}{y+3} \times \frac{(y-3)(y+3)}{5}$$

Factor  $y^2 - 9$ .

$$= \frac{2}{\cancel{y+3}} \times \frac{(y-3)\cancel{(y+3)}}{5}$$

Divide by the GCF,  $y + 3$ .

$$= \frac{2(y-3)}{5}$$

Simplify.

b. 
$$\frac{\frac{n^2+7n-18}{n^2-2n+1}}{\frac{n^2-81}{n-1}}$$

$$\frac{\frac{n^2+7n-18}{n^2-2n+1}}{\frac{n^2-81}{n-1}} = \frac{n^2+7n-18}{n^2-2n+1} \div \frac{n^2-81}{n-1}$$

Write as a division expression.

$$= \frac{n^2+7n-18}{n^2-2n+1} \times \frac{n-1}{n^2-81}$$

Multiply by the reciprocal.

$$= \frac{(n-2)(n+9)}{(n-1)(n-1)} \times \frac{n-1}{(n-9)(n+9)}$$

Factor the polynomials.

$$= \frac{(n-2)\cancel{(n+9)}}{(n-1)\cancel{(n-1)}} \times \frac{\cancel{n-1}}{(n-9)\cancel{(n+9)}}$$

Divide out the common factors.

$$= \frac{n-2}{(n-1)(n-9)}$$

Simplify.

### Guided Practice

4A. 
$$\frac{\frac{a+7}{4}}{\frac{a^2-49}{10}}$$

4B. 
$$\frac{\frac{x+4}{x-1}}{\frac{x^2+6x+8}{2x-2}}$$

4C. 
$$\frac{\frac{c-d}{j+p}}{\frac{c^2-d^2}{j^2-p^2}}$$

4D. 
$$\frac{\frac{n^2+4n-21}{n^2-9n+18}}{\frac{n^2+3n-28}{n^2-10n+24}}$$





**Example 1** Write each mixed expression as a rational expression.

1.  $\frac{2}{n} + 4$

2.  $r + \frac{1}{3r}$

3.  $6 + \frac{5}{t+1}$

4.  $\frac{x+7}{2x} - 5x$

**Example 2** 5. **ROWING** Rico rowed a canoe  $2\frac{1}{2}$  miles in  $\frac{1}{3}$  hour.

a. Write an expression to represent his speed in miles per hour.

b. Simplify the expression to find his average speed.

**Examples 3–4** Simplify each expression.

6.  $\frac{2\frac{1}{3}}{1\frac{2}{5}}$

7.  $\frac{\frac{4}{5}}{6\frac{2}{3}}$

8.  $\frac{\frac{a^2}{b^3}}{\frac{b^5}{a}}$

9.  $\frac{\frac{y^4}{x^2}}{\frac{xy^2}{2x^2}}$

10.  $\frac{\frac{6}{x-2}}{\frac{3}{x^2-x-2}}$

11.  $\frac{\frac{r+s}{x^2-y^2}}{\frac{(r+s)^2}{x-y}}$

12.  $\frac{\frac{2+q}{q^2-4}}{\frac{q+4}{q^2-6q+8}}$

13.  $\frac{\frac{p+3}{p^2+p-6}}{\frac{p^2+4p+3}{p^2+6p+9}}$

Practice and Problem Solving

Extra Practice is on page R11.

**Example 1** Write each mixed expression as a rational expression.

14.  $10 + \frac{6}{f}$

15.  $p - \frac{7}{2p}$

16.  $5a - \frac{2a}{b}$

17.  $3h + \frac{1+h}{h}$

18.  $t + \frac{v+w}{v-w}$

19.  $n^2 + \frac{n-1}{n+4}$

20.  $(k+2) + \frac{k-1}{k-2}$

21.  $(d-6) + \frac{d+1}{d-7}$

22.  $\frac{h-3}{h+5} - (h+2)$

**Example 2** 23. **READING** Ebony reads  $6\frac{3}{4}$  pages of a book in 9 minutes. What is her average reading rate in pages per minute?

24. **HORSES** A thoroughbred can run  $\frac{1}{2}$  mile in about  $\frac{3}{4}$  minute. What is the horse's speed in miles per hour?

**Examples 3–4** Simplify each expression.

25.  $\frac{2\frac{2}{9}}{3\frac{1}{3}}$

26.  $\frac{5\frac{3}{5}}{2\frac{1}{7}}$

27.  $\frac{\frac{g^2}{h}}{\frac{g^5}{h^2}}$

28.  $\frac{5n^4}{\frac{p^3}{\frac{6n}{5p}}}$

29.  $\frac{\frac{2}{a}}{\frac{1}{a+6}}$

30.  $\frac{\frac{t+5}{9}}{\frac{t^2-t-30}{12}}$

31.  $\frac{\frac{j^2-16}{j^2+10j+16}}{\frac{15}{j+8}}$

32.  $\frac{\frac{x-3}{x^2+3x+2}}{\frac{x^2-9}{x+1}}$

33. **CCSS MODELING** The Centralville High School Cooking Club has  $12\frac{1}{2}$  pounds of flour with which to make tortillas. There are  $3\frac{3}{4}$  cups of flour in a pound, and it takes about  $\frac{1}{3}$  cup of flour per tortilla. How many tortillas can they make?



34. **SCOOTER** The speed  $v$  of an object spinning in a circle equals the circumference of the circle divided by the time  $T$  it takes the object to complete one revolution.
- Use the variables  $v$ ,  $r$  (the radius of the circle), and  $T$  to write a formula describing the speed of a spinning object.
  - A scooter has tires with a radius of  $3\frac{1}{2}$  inches. The tires make one revolution every  $\frac{1}{10}$  second. Find the speed in miles per hour. Round to the nearest tenth.

35. **SCIENCE** The *density* of an object equals  $\frac{m}{V}$ , where  $m$  is the mass of the object and  $V$  is the volume. The densities of four metals are shown in the table. Identify the metal of each ball described below. (Hint: The volume of a sphere is  $V = \frac{4}{3}\pi r^3$ .)

| Metal  | Density (kg/m <sup>3</sup> ) |
|--------|------------------------------|
| copper | 8900                         |
| gold   | 19,300                       |
| iron   | 7800                         |
| lead   | 11,300                       |

- A metal ball has a mass of 15.6 kilograms and a radius of 0.0748 meter.
  - A metal ball has a mass of 285.3 kilograms and a radius of 0.1819 meter.
36. **SIRENS** As an ambulance approaches, the siren sounds different than if it were sitting still. If the ambulance is moving toward you at  $v$  miles per hour and blowing the siren at a frequency of  $f$ , then you hear the siren as if it were blowing at a frequency  $h$ . This can be described by the equation  $h = \frac{f}{1 - \frac{v}{s}}$ , where  $s$  is the speed of sound, approximately 760 miles per hour.
- Simplify the complex fraction in the formula.
  - Suppose a siren blows at 45 cycles per minute and is moving toward you at 65 miles per hour. Find the frequency of the siren as you hear it.

Simplify each expression.

37.  $15 - \frac{17x + 5}{5x + 10}$

38.  $\frac{\frac{b}{b+3} + 2}{b^2 - 2b - 8}$

39.  $\frac{1 + \frac{2c^2 - 6c - 10}{c + 7}}{2c + 1}$

40.  $\frac{y - \frac{12}{y-4}}{y - \frac{18}{y-3}}$

41.  $\frac{\frac{x^2 - 4x - 32}{x + 1}}{\frac{x^2 + 6x + 8}{x^2 - 1}}$

42.  $\frac{\frac{r^2 - 9r}{r^2 + 7r + 10}}{\frac{r^2 + 5r}{r^2 + r - 2}}$

### H.O.T. Problems Use Higher-Order Thinking Skills

43. **REASONING** Describe the first step to simplify the expression below.

$$\frac{\left(\frac{y}{x} - \frac{x}{y}\right)}{\frac{x+y}{xy}}$$

44. **REASONING** Is  $\frac{n}{1 - \frac{5}{p}} + \frac{n}{\frac{5}{p} - 1}$  sometimes, always, or never equal to 0? Explain.

45. **CCSS PERSEVERANCE** Simplify the rational expression below.

$$\frac{\frac{1}{t-1} + \frac{1}{t+1}}{\frac{1}{t} - \frac{1}{t^2}}$$

46. **OPEN ENDED** Write a complex fraction that, when simplified, results in  $\frac{1}{x}$ .

47. **WRITING IN MATH** Explain how complex fractions can be used to solve a problem involving distance, rate, and time. Give an example.



## Standardized Test Practice

48. A number is between 44 squared and 45 squared. 5 squared is one of its factors, and it is a multiple of 13. Find the number.

- A 1950
- B 2000
- C 2025
- D 2050

49. **SHORT RESPONSE** Bernard is reading a 445-page book. He has already read 157 pages. If he reads 24 pages a day, how long will it take him to finish the book?

50. **GEOMETRY** Angela wanted a round rug to fit her room that is 16 feet wide. The rug should just meet the edges. What is the area of the rug rounded to the nearest tenth?

- F 50.3 ft<sup>2</sup>
- G 100.5 ft<sup>2</sup>
- H 152.2 ft<sup>2</sup>
- J 201.1 ft<sup>2</sup>

51. Simplify  $7x + \frac{10}{2xy}$ .

- A  $\frac{7x + 10}{2xy}$
- B  $\frac{7x^2y + 5}{xy}$
- C  $\frac{17x}{2xy}$
- D  $\frac{7xy + 5}{x^2y}$

## Spiral Review

Find each sum or difference. (Lesson 11-6)

52.  $\frac{6}{7x} - \frac{5+x}{7x}$

53.  $\frac{4}{d-1} + \frac{d}{1-d}$

54.  $\frac{3q+2}{2q+1} + \frac{q-5}{2q+1}$

55.  $\frac{2}{5m} - \frac{1}{15m^3}$

56.  $\frac{10}{3g} - \frac{-3}{4h}$

57.  $\frac{b}{b+3} + \frac{6}{b-2}$

Find each quotient. Use long division. (Lesson 11-5)

58.  $(x^2 - 2x - 30) \div (x + 7)$

59.  $(a^2 + 4a - 22) \div (a - 3)$

60.  $(3q^2 + 20q + 11) \div (q + 6)$

61.  $(3y^3 + 8y^2 + y - 7) \div (y + 2)$

62.  $(6t^3 - 9t^2 + 6) \div (2t - 3)$

63.  $(9h^3 + 5h - 8) \div (3h - 2)$

64. **GEOMETRY** A rectangle has a base of 8 meters and a height of 14 meters. What is the length of the diagonal? (Lesson 10-5)

Graph each function. Determine the domain and range. (Lesson 10-1)

65.  $y = 2\sqrt{x}$

66.  $y = -3\sqrt{x}$

67.  $y = \frac{1}{4}\sqrt{x}$

Factor each polynomial. If the polynomial cannot be factored, write *prime*. (Lesson 8-8)

68.  $x^2 - 81$

69.  $a^2 - 121$

70.  $n^2 + 100$

71.  $-25 + 4y^2$

72.  $p^4 - 16$

73.  $4t^4 - 4$

74. **PARKS** A youth group traveling in two vans visited Mammoth Cave in Kentucky. The number of people in each van and the total cost of the cave are shown. Find the adult price and the student price of the tour. (Lesson 6-3)

| Van | Number of Adults | Number of Students | Total Cost |
|-----|------------------|--------------------|------------|
| A   | 2                | 5                  | \$77       |
| B   | 2                | 7                  | \$95       |

## Skills Review

Solve each equation.

75.  $6x = 24$

76.  $5y - 1 = 19$

77.  $2t + 7 = 21$

78.  $\frac{p}{3} = -4.2$

79.  $\frac{2m+1}{4} = -5.5$

80.  $\frac{3}{4}g = \frac{1}{2}$





### Then

- You solved proportions.

### Now

- Solve rational equations.
- Use rational equations to solve problems.

### Why?

- Oceanic species of dolphins can swim 5 miles per hour faster than coastal species of dolphins. An oceanic dolphin can swim 3 miles in the same time that it takes a coastal dolphin to swim 2 miles.

| Dolphins |          |           |         |
|----------|----------|-----------|---------|
| Species  | Distance | Rate      | Time    |
| coastal  | 2 miles  | x mph     | t hours |
| oceanic  | 3 miles  | x + 5 mph | t hours |

Since  $\text{time} = \frac{\text{distance}}{\text{rate}}$ , the equation below represents this situation.

$$\frac{\text{Time an oceanic dolphin swims}}{\text{distance} \rightarrow \frac{3}{\text{rate} \rightarrow x + 5}} = \frac{\text{time a coastal dolphin swims}}{\text{distance} \rightarrow \frac{2}{\text{rate} \rightarrow x}}$$



### New Vocabulary

- rational equation
- extraneous solution
- work problem
- rate problem



### Common Core State Standards

**Content Standards**  
A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

### Mathematical Practices

- Reason abstractly and quantitatively.
- Model with mathematics.

**1 Solve Rational Equations** A **rational equation** contains one or more rational expressions. When a rational equation is a proportion, you can use cross products to solve it.



### Real-World Example 1 Use Cross Products to Solve Equations

**DOLPHINS** Refer to the information above. Solve  $\frac{3}{x+5} = \frac{2}{x}$  to find the speed of a coastal dolphin. Check the solution.

$$\begin{aligned} \frac{3}{x+5} &= \frac{2}{x} && \text{Original equation} \\ 3x &= 2(x+5) && \text{Find the cross products.} \\ 3x &= 2x + 10 && \text{Distributive Property} \\ x &= 10 && \text{Subtract } 2x \text{ from each side.} \end{aligned}$$

So, a coastal dolphin can swim 10 miles per hour.

**CHECK**

$$\begin{aligned} \frac{3}{x+5} &= \frac{2}{x} && \text{Original equation} \\ \frac{3}{10+5} &\stackrel{?}{=} \frac{2}{10} && \text{Replace } x \text{ with } 10. \\ \frac{3}{15} &\stackrel{?}{=} \frac{1}{5} && \text{Simplify.} \\ \frac{1}{5} &= \frac{1}{5} \checkmark && \text{Simplify.} \end{aligned}$$

### Guided Practice

Solve each equation. Check the solution.

1A.  $\frac{7}{y-3} = \frac{3}{y+1}$

1B.  $\frac{13}{10} = \frac{2f+0.2}{7}$



Another method that can be used to solve any rational equation is to find the LCD of all the fractions in the equation. Then multiply each side of the equation by the LCD to eliminate the fractions.



**Example 2** Use the LCD to Solve Rational Equations

Solve  $\frac{4}{y} + \frac{5y}{y+1} = 5$ . Check the solution.

**Step 1** Find the LCD.

The LCD of  $\frac{4}{y}$  and  $\frac{5y}{y+1}$  is  $y(y+1)$ .

**Step 2** Multiply each side of the equation by the LCD.

$$\frac{4}{y} + \frac{5y}{y+1} = 5$$

Original equation

$$y(y+1) \left( \frac{4}{y} + \frac{5y}{y+1} \right) = y(y+1)(5)$$

Multiply each side by the LCD,  $y(y+1)$ .

$$\left( \frac{\cancel{y}(y+1) \cdot 4}{1 \cdot \cancel{y}} \right) + \left( \frac{y \cdot \cancel{y+1} \cdot 5y}{1 \cdot \cancel{y+1}} \right) = y(y+1)(5)$$

Distributive Property

$$(y+1)4 + y(5y) = y(y+1)(5)$$

Simplify.

$$4y + 4 + 5y^2 = 5y^2 + 5y$$

Multiply.

$$4y + 4 + 5y^2 - 5y^2 = 5y^2 - 5y^2 + 5y$$

Subtract  $5y^2$  from each side.

$$4y + 4 = 5y$$

Simplify.

$$4y - 4y + 4 = 5y - 4y$$

Subtract  $4y$  from each side.

$$4 = y$$

Simplify.

**CHECK**  $\frac{4}{y} + \frac{5y}{y+1} = 5$

Original equation

$$\frac{4}{4} + \frac{5(4)}{4+1} \stackrel{?}{=} 5$$

Replace  $y$  with 4.

$$1 + 4 \stackrel{?}{=} 5$$

Simplify.

$$5 = 5 \checkmark$$

Simplify.

**Guided Practice**

Solve each equation. Check your solutions.

2A.  $\frac{2b-5}{b-2} - 2 = \frac{3}{b+2}$

2B.  $1 + \frac{1}{c+2} = \frac{28}{c^2+2c}$

2C.  $\frac{y+2}{y-2} - \frac{2}{y+2} = \frac{7}{3}$

2D.  $\frac{n}{3n+6} - \frac{n}{5n+10} = \frac{2}{5}$

**StudyTip**

**Solutions** It is important to check the solutions of rational equations to be sure that they satisfy the original equation.

**VocabularyLink**

**extraneous**

**Everyday Use**

irrelevant or unimportant

**extraneous solution**

**Math Use** a result that is not a solution of the original equation

Recall that any value of a variable that makes the denominator of a rational expression zero must be excluded from the domain.

In the same way, when a solution of a rational equation results in a zero in the denominator, that solution must be excluded. Such solutions are called **extraneous solutions**.

$$\frac{4+x}{x-5} + \frac{1}{x} = \frac{2}{x+1}$$

5, 0, and -1 cannot be solutions.





### Example 3 Extraneous Solutions

Solve  $\frac{2n}{n-5} + \frac{4n-30}{n-5} = 5$ . State any extraneous solutions.

$$\frac{2n}{n-5} + \frac{4n-30}{n-5} = 5 \quad \text{Original equation}$$

$$(n-5)\left(\frac{2n}{n-5} + \frac{4n-30}{n-5}\right) = (n-5)5 \quad \text{Multiply each side by the LCD, } n-5.$$

$$\left(\frac{\overset{1}{\cancel{n-5}} \cdot 2n}{1 \cdot \cancel{n-5}}\right) + \left(\frac{\overset{1}{\cancel{n-5}} \cdot (4n-30)}{1 \cdot \cancel{n-5}}\right) = (n-5)5 \quad \text{Distributive Property}$$

$$2n + 4n - 30 = 5n - 25 \quad \text{Simplify.}$$

$$6n - 30 = 5n - 25 \quad \text{Add like terms.}$$

$$6n - 5n - 30 = 5n - 5n - 25 \quad \text{Subtract } 5n \text{ from each side.}$$

$$n - 30 = -25 \quad \text{Simplify.}$$

$$n - 30 + 30 = -25 + 30 \quad \text{Add 30 to each side.}$$

$$n = 5 \quad \text{Simplify.}$$

Since  $n = 5$  results in a zero in the denominator of the original equation, it is an extraneous solution. So, the equation has no solution.

#### StudyTip

**Solutions** It is possible to get both a valid solution and an extraneous solution when solving a rational equation.

#### Guided Practice

3. Solve  $\frac{n^2 - 3n}{n^2 - 4} - \frac{10}{n^2 - 4} = 2$ . State any extraneous solutions.

## 2 Use Rational Equations to Solve Problems

You can use rational equations to solve **work problems**, or problems involving work rates.



### Real-World Example 4 Work Problem

**JOBS** At his part-time job at the zoo, Ping can clean the bird area in 2 hours. Natalie can clean the same area in 1 hour and 15 minutes. How long would it take them if they worked together?

**Understand** It takes Ping 2 hours to complete the job and Natalie  $1\frac{1}{4}$  hours.

You need to find the rate that each person works and the total time  $t$  that it will take them if they work together.

**Plan** Find the fraction of the job that each person can do in an hour.

$$\text{Ping's rate} \rightarrow \frac{1 \text{ job}}{2 \text{ hours}} = \frac{1}{2} \text{ job per hour}$$

$$\text{Natalie's rate} \rightarrow \frac{1 \text{ job}}{1\frac{1}{4} \text{ hours}} \text{ or } \frac{1 \text{ job}}{\frac{5}{4} \text{ hours}} = \frac{4}{5} \text{ job per hour}$$

Since  $\text{rate} \cdot \text{time} = \text{fraction of job done}$ , multiply each rate by the time  $t$  to represent the amount of the job done by each person.



### StudyTip

**CCSS Reasoning** When solving work problems, remember that each term should represent the portion of a job completed in one unit of time.

**Solve** Fraction of job Ping completes plus fraction of job Natalie completes equals 1 job.

$$\frac{1}{2}t + \frac{4}{5}t = 1$$

$$10\left(\frac{1}{2}t + \frac{4}{5}t\right) = 10(1) \quad \text{Multiply each side by the LCD, 10.}$$

$$10\left(\frac{1}{2}t\right) + 10\left(\frac{4}{5}t\right) = 10 \quad \text{Distributive Property}$$

$$5t + 8t = 10 \quad \text{Simplify.}$$

$$t = \frac{10}{13} \quad \text{Add like terms and divide each side by 13.}$$

So, it would take them  $\frac{10}{13}$  hour or about 46 minutes to complete the job if they work together.

**Check** In  $\frac{10}{13}$  hour, Ping would complete  $\frac{1}{2} \cdot \frac{10}{13}$  or  $\frac{5}{13}$  of the job and Natalie would complete  $\frac{4}{5} \cdot \frac{10}{13}$  or  $\frac{8}{13}$  of the job. Together, they complete  $\frac{5}{13} + \frac{8}{13}$  or 1 whole job. So, the answer is reasonable. ✓

### GuidedPractice

4. **RAKING** Jenna can rake the leaves in 2 hours. It takes her brother Benjamin 3 hours. How long would it take them if they worked together?

Rational equations can also be used to solve **rate problems**.



### Real-World Example 5 Rate Problem

**AIRPLANES** An airplane takes off and flies an average of 480 miles per hour. Another plane leaves 15 minutes later and flies to the same city traveling 560 miles per hour. How long will it take the second plane to pass the first plane?

Record the information that you know in a table.

| Plane | Distance  | Rate     | Time                    |
|-------|-----------|----------|-------------------------|
| 1     | $d$ miles | 480 mi/h | $t$ hours               |
| 2     | $d$ miles | 560 mi/h | $t - \frac{1}{4}$ hours |

← Plane 2 took off 15 minutes, or  $\frac{1}{4}$  hour, after Plane 1

Since both planes will have traveled the same distance when Plane 2 passes Plane 1, you can write the following equation.

Distance for Plane 1 = Distance for Plane 2

$$480 \cdot t = 560 \cdot \left(t - \frac{1}{4}\right) \quad \text{distance = rate} \cdot \text{time}$$

$$480t = (560 \cdot t) - \left(560 \cdot \frac{1}{4}\right) \quad \text{Distributive Property}$$

$$480t = 560t - 140 \quad \text{Simplify.}$$

$$-80t = -140 \quad \text{Subtract } 560t \text{ from each side.}$$

$$t = 1.75 \quad \text{Divide each side by } -80.$$

So, the second plane passes the first plane after 1.75 hours.

### GuidedPractice

5. Lenora leaves the house walking at 3 miles per hour. After 10 minutes, her mother leaves the house riding a bicycle at 10 miles per hour. In how many minutes will Lenora's mother catch her?

### Real-WorldLink

The longest nonstop commercial flight was 13,422 miles from Hong Kong Airport in China to London Heathrow in the United Kingdom. It took 22 hours and 42 minutes.

Source: Guinness Book of World Records





Examples 1–3 Solve each equation. State any extraneous solutions.

1.  $\frac{2}{x+1} = \frac{4}{x}$

2.  $\frac{t+3}{5} = \frac{2t+3}{9}$

3.  $\frac{a+3}{a} - \frac{6}{5a} = \frac{1}{a}$

4.  $4 - \frac{p}{p-1} = \frac{2}{p-1}$

5.  $\frac{2t}{t+1} + \frac{4}{t-1} = 2$

6.  $\frac{x+3}{x^2-1} - \frac{2x}{x-1} = 1$

Example 4 7. **WEEDING** Maurice can weed the garden in 45 minutes. Olinda can weed the garden in 50 minutes. How long would it take them to weed the garden if they work together?

Example 5 8. **LANDSCAPING** Hunter is filling a 3.5-gallon bucket to water plants at a faucet that flows at a rate of 1.75 gallons a minute. If he were to add a hose that flows at a rate of 1.45 gallons per minute, how many minutes would it take him to fill the bucket? Round to the nearest tenth.

Practice and Problem Solving

Extra Practice is on page R11.

Examples 1–3 Solve each equation. State any extraneous solutions.

9.  $\frac{8}{n} = \frac{3}{n-5}$

10.  $\frac{6}{t+2} = \frac{4}{t}$

11.  $\frac{3g+2}{12} = \frac{g}{2}$

12.  $\frac{5h}{4} + \frac{1}{2} = \frac{3h}{8}$

13.  $\frac{2}{3w} = \frac{2}{15} + \frac{12}{5w}$

14.  $\frac{c-4}{c+1} = \frac{c}{c-1}$

15.  $\frac{x-1}{x+1} - \frac{2x}{x-1} = -1$

16.  $\frac{y+4}{y-2} + \frac{6}{y-2} = \frac{1}{y+3}$

17.  $\frac{a}{a+3} + \frac{a^2}{a+3} = 2$

18.  $\frac{12}{a+3} + \frac{6}{a^2-9} = \frac{8}{a+3}$

19.  $\frac{3n}{n-1} + \frac{6n-9}{n-1} = 6$

20.  $\frac{n^2-n-6}{n^2-n} - \frac{n-5}{n-1} = \frac{n-3}{n^2-n}$

Example 4 21. **PAINTING** It takes Noah 3 hours to paint one side of a fence. It takes Gilberto 5 hours. How long would it take them if they worked together?

22. **DISHWASHING** Ron works as a dishwasher and can wash 500 plates in two hours and 15 minutes. Chris can finish the 500 plates in 3 hours. About how long would it take them to finish all of the plates if they work together?

Example 5 23. **ICE** A hotel has two ice machines in its kitchen. How many hours would it take both machines to make 60 pounds of ice? Round to the nearest tenth.



24. **CYCLING** Two cyclists travel in opposite directions around a 5.6-mile circular trail. They start at the same time. The first cyclist completes the trail in 22 minutes and the second in 28 minutes. At what time do they pass each other?

**GRAPHING CALCULATOR** For each function, a) describe the shape of the graph, b) use factoring to simplify the function, and c) find the zeros of the function.

25.  $f(x) = \frac{x^2 - x - 30}{x - 6}$

26.  $f(x) = \frac{x^3 + x^2 - 2x}{x + 2}$

27.  $f(x) = \frac{x^3 + 6x^2 + 12x}{x}$

28. **CCSS REASONING** Morgan can paint a standard-sized house in about 5 days. For his latest job, Morgan hires two assistants. At what rate must these assistants work for Morgan to meet a deadline of two days?



29. **AIRPLANES** Headwinds push against a plane and reduce its total speed, while tailwinds push on a plane and increase its total speed. Let  $w$  equal the speed of the wind,  $r$  equal the speed set by the pilot, and  $s$  equal the total speed.
- Write an equation for the total speed with a headwind and an equation for the total speed with a tailwind.
  - Use the rate formula to write an equation for the distance traveled by a plane with a headwind and another equation for the distance traveled by a plane with a tailwind. Then solve each equation for time instead of distance.
30. **MIXTURES** A pitcher of fruit juice has 3 pints of pineapple juice and 2 pints of orange juice. Erin wants to add more orange juice so that the fruit juice mixture is 60% orange juice. Let  $x$  equal the pints of orange juice that she needs to add.
- Copy and complete the table below.

| Juice            | Pints of Orange Juice | Total Pints of Juice | Percent of Orange Juice |
|------------------|-----------------------|----------------------|-------------------------|
| original mixture |                       | 5                    |                         |
| final mixture    | $2 + x$               |                      | 0.6                     |

- Write and solve an equation to find the pints of orange juice to add.
31. **DORMITORIES** The number of hours  $h$  it takes to clean a dormitory varies inversely with the number of people cleaning it  $c$  and directly with the number of people living there  $p$ .
- Write an equation showing how  $h$ ,  $c$ , and  $p$  are related. (*Hint*: Include the constant  $k$ .)
  - It takes 8 hours for 5 people to clean the dormitory when there are 100 people there. How long will it take to clean the dormitory if there are 10 people cleaning and the number of people living in the dorm stays the same?

Solve each equation. State any extraneous solutions.

32.  $\frac{4b+2}{b^2-3b} + \frac{b+2}{b} = \frac{b-1}{b}$

33.  $\frac{x^2-x-3}{x-1} + \frac{x^3+2x^2}{x-1} = 3$

34.  $\frac{y^2+5y-6}{y^3-2y^2} = \frac{5}{y} - \frac{6}{y^3-2y^2}$

35.  $\frac{x-\frac{6}{5}}{x} - \frac{x-10\frac{1}{2}}{x-5} = \frac{x+21}{x^2-5x}$

### H.O.T. Problems Use Higher-Order Thinking Skills

36. **CHALLENGE** Solve  $\frac{2x}{x-2} + \frac{x^2+3x}{(x+1)(x-2)} = \frac{2}{(x+1)(x-2)}$ .
37. **REASONING** How is an excluded value of a rational expression related to an extraneous solution of a corresponding rational equation? Explain.
38. **WRITING IN MATH** Why should you check solutions of rational equations?
39. **CCSS ARGUMENTS** Find a counterexample for the following statement.  
*The solution of a rational equation can never be zero.*
40. **WRITING IN MATH** Describe the steps for solving a rational equation that is not a proportion.



## Standardized Test Practice

41. It takes Cheng 4 hours to build a fence. If he hires Odell to help him, they can do the job in 3 hours. If Odell built the same fence alone, how long would it take him?
- A  $1\frac{5}{7}$  hours                      C 8 hours  
 B  $3\frac{2}{3}$  hours                      D 12 hours
42. In the 1000-meter race, Zoe finished 35 meters ahead of Taryn and 53 meters ahead of Evan. How far was Taryn ahead of Evan?
- F 18 m    G 35 m    H 53 m    J 88 m
43. Twenty gallons of lemonade were poured into two containers of different sizes. Express the amount of lemonade poured into the smaller container in terms of  $g$ , the amount poured into the larger container.
- A  $g + 20$                       C  $g - 20$   
 B  $20 + g$                       D  $20 - g$
44. **GRIDDED RESPONSE** The gym has 2-kilogram and 5-kilogram disks for weight lifting. They have fourteen disks in all. The total weight of the 2-kilogram disks is the same as the total weight of the 5-kilogram disks. How many 2-kilogram disks are there?

## Spiral Review

Simplify each expression. (Lesson 11-7)

45.  $\frac{\frac{c^2}{d}}{\frac{c^3}{d^2}}$

46.  $\frac{\frac{5g^3}{h^2}}{\frac{6g}{5h}}$

47.  $\frac{\frac{2}{b}}{\frac{4}{b-3}}$

48.  $\frac{\frac{q-2}{9}}{\frac{q^2-6q+8}{12}}$

Find the LCM of each pair of polynomials. (Lesson 11-6)

49.  $2h, 4h^2$

50.  $5c^2, 12c^3$

51.  $x - 4, x + 2$

52.  $p - 7, 2(p - 14)$

Look for a pattern in each table of values to determine which kind of model best describes the data. (Lesson 9-6)

53. 

|     |   |   |   |   |   |
|-----|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 |
| $y$ | 4 | 5 | 6 | 7 | 8 |

54. 

|     |   |   |   |    |    |
|-----|---|---|---|----|----|
| $x$ | 1 | 2 | 3 | 4  | 5  |
| $y$ | 2 | 4 | 8 | 16 | 32 |

55. 

|     |    |    |    |   |   |
|-----|----|----|----|---|---|
| $x$ | -3 | -2 | -1 | 0 | 1 |
| $y$ | 14 | 9  | 6  | 5 | 6 |

56. 

|     |   |   |   |   |    |
|-----|---|---|---|---|----|
| $x$ | 3 | 4 | 5 | 6 | 7  |
| $y$ | 3 | 5 | 7 | 9 | 11 |

57. **GENETICS** Brown genes  $B$  are dominant over blue genes  $b$ . A person with genes  $BB$  or  $Bb$  has brown eyes. Someone with genes  $bb$  has blue eyes. Mrs. Dunn has brown eyes with genes  $Bb$ , and Mr. Dunn has blue eyes. Write an expression for the possible eye coloring of their children. Then find the probability that a child would have blue eyes. (Lesson 8-4)

Solve each inequality. Check your solution. (Lesson 5-2)

58.  $\frac{b}{10} \leq 5$

59.  $-7 > \frac{r}{7}$

60.  $\frac{5}{8}y \geq -15$

## Skills Review

Determine the probability of each event if you randomly select a marble from a bag containing 9 red marbles, 6 blue marbles, and 5 yellow marbles.

61.  $P(\text{blue})$

62.  $P(\text{red})$

63.  $P(\text{not yellow})$





You can use TI-Nspire Technology to solve rational equations by graphing, by using tables, and by using a computer algebra system (CAS).

**CCSS Common Core State Standards**  
**Content Standards**

**A.REI.11** Explain why the  $x$ -coordinates of the points where the graphs of the equations  $y = f(x)$  and  $y = g(x)$  intersect are the solutions of the equation  $f(x) = g(x)$ ; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where  $f(x)$  and/or  $g(x)$  are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

**Mathematical Practices**

5 Use appropriate tools strategically.

To solve by graphing, graph both sides of the equation and locate the point(s) of intersection.



### Activity 1 Solve a Rational Equation by Graphing

Solve  $\frac{5}{x+2} = \frac{3}{x}$  by graphing.

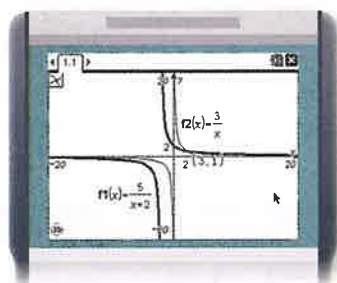
**Step 1** Add a new **Graphs** page.

**Step 2** Use the **Window Settings** option from the **Window/Zoom** menu to adjust the window to  $-20$  to  $20$  for both  $x$  and  $y$ . Set both scales to  $2$ .

**Step 3** Enter  $\frac{5}{x+2}$  into **f1(x)** and  $\frac{3}{x}$  into **f2(x)**.

**Step 4** Change the thickness of the graph of **f1(x)** by selecting the graph of **f1(x)** and the **ctrl** menu **Attributes** option.

**Step 5** Use the **Intersection Point(s)** tool from the **Points & Lines** menu to find the intersection of the two graphs. Select the graph of **f1(x)** enter and then the graph of **f2(x)** enter.



$[-20, 20]$  scl: 2 by  $[-20, 20]$  scl: 2

The graphs intersect at  $(3, 1)$ . This means that  $\frac{5}{x+2}$  and  $\frac{3}{x}$  both equal  $1$  when  $x = 3$ . Thus, the solution of  $\frac{5}{x+2} = \frac{3}{x}$  is  $x = 3$ .

### Exercises

Use a graphing calculator to solve each equation.

1.  $\frac{5}{x} + \frac{4}{x} = 10$

2.  $\frac{12}{x} + \frac{3}{4} = \frac{3}{2}$

3.  $\frac{6}{x} + \frac{3}{2x} = 12$

4.  $\frac{4}{x} + \frac{3}{4x} = \frac{1}{8}$

5.  $\frac{4}{x} + \frac{x-2}{2x} = x$

6.  $\frac{3}{3x-2} + \frac{5}{x} = 0$

7.  $\frac{2x+1}{2} + \frac{3}{2x} = \frac{2}{x}$

8.  $\frac{x}{x+2} + x = \frac{5x+8}{x+2}$

9.  $\frac{1}{2x} + \frac{5}{x} = \frac{3}{x-1}$

10.  $\frac{4x-3}{x-2} + \frac{2x+5}{x-2} = 6$

# Graphing Technology Lab

## Solving Rational Equations *Continued*

### Activity 2 Solve a Rational Equation by Using a Table

Solve  $\frac{2x+1}{3} = \frac{x+2}{2}$  using a table.

**Step 1** Add a new Lists & Spreadsheet page.

**Step 2** Label column A as  $x$ . Enter values from  $-4$  to  $4$  in cells A1 to A9.

**Step 3** In column B in the formula row, enter the left side of the rational equation, with parenthesis around the binomials. In column C in the formula row, enter the right side of the rational equation, with parenthesis around the binomials. Specify **Variable Reference** when prompted.

| A  | B   | C   |
|----|-----|-----|
| -1 | 1/3 | 1/2 |
| 0  | 1/3 | 1   |
| 1  | 1   | 3/2 |
| 2  | 5/2 | 5/2 |
| 3  | 7/3 | 5/2 |
| 4  | 3   | 3   |

Scroll until you see where the values in Columns B and C are equal. This occurs at  $x = 4$ . Therefore the solution of  $\frac{2x+1}{3} = \frac{x+2}{2}$  is 4.

You can also use a computer algebra system (CAS) to solve rational equations.

### Activity 3 Solve a Rational Equation by Using a CAS

Solve  $\frac{x-3}{x} - \frac{x-4}{x-2} = \frac{1}{x}$  using a CAS.

**Step 1** Add a new Calculator page.

**Step 2** To solve, select the **Solve** tool from the **Algebra** menu. Enter the left side of the equation with parenthesis around the binomials. Enter  $=$  and the right side of the equation. Then type a comma, followed by  $x$ , and then **enter**.

The solution of 4 is displayed.



## Exercises

Use a table or CAS to solve each equation.

11.  $\frac{2}{x} + \frac{2+x}{2} = \frac{x+3}{2}$

12.  $\frac{4}{x-2} = -\frac{1}{x+3}$

13.  $\frac{3}{x+2} + \frac{4}{x-1} = 0$

14.  $\frac{1}{x+1} + \frac{2}{x-1} = 0$

15.  $\frac{2}{x+4} + \frac{4}{x-1} = 0$

16.  $\frac{1}{x-2} + \frac{x+2}{4} = 2x$

17.  $\frac{2x}{x+3} + \frac{x+1}{2} = x$

18.  $\frac{2}{x-3} + \frac{3}{x-2} = \frac{4}{x}$

19.  $\frac{x^2}{x+1} + \frac{x}{x-1} = x$

# Study Guide and Review

## Study Guide

### Key Concepts

#### Inverse Variation (Lesson 11-1)

- You can use  $\frac{x_1}{x_2} = \frac{y_2}{y_1}$  to solve problems involving inverse variation.

#### Rational Functions (Lesson 11-2)

- Excluded values are values of a variable that result in a denominator of zero.
- If vertical asymptotes occur, it will be at excluded values.

#### Rational Expressions (Lessons 11-3 and 11-4)

- Multiplying rational expressions is similar to multiplying rational numbers.
- Divide rational expressions by multiplying by the reciprocal of the divisor.

#### Dividing Polynomials (Lesson 11-5)

- To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

#### Adding and Subtracting Rational Expressions

(Lesson 11-6)

- Rewrite rational expressions with unlike denominators using the least common denominator (LCD). Then add or subtract.

#### Complex Fractions (Lesson 11-7)

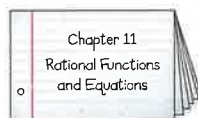
- Simplify complex fractions by writing them as division problems.

#### Solving Rational Equations (Lesson 11-8)

- Use cross products to solve rational equations with a single fraction on each side of the equals sign.

### FOLDABLES Study Organizer

Be sure the Key Concepts are noted in your Foldable.



### Key Vocabulary



- |   |                              |
|---|------------------------------|
| asymptote (p. 685)                      | mixed expression (p. 720)    |
| complex fraction (p. 720)               | product rule (p. 677)        |
| excluded value (p. 684)                 | rate problems (p. 729)       |
| extraneous solution (p. 727)            | rational equation (p. 726)   |
| inverse variation (p. 676)              | rational expression (p. 690) |
| least common denominator (LCD) (p. 714) | rational function (p. 684)   |
| least common multiple (LCM) (p. 713)    | work problems (p. 728)       |

### Vocabulary Check

State whether each sentence is *true* or *false*. If *false*, replace the underlined word, phrase, expression, or number to make a true sentence.

- The least common multiple for  $x^2 - 25$  and  $x - 5$  is  $x - 5$ .
- If the product of two variables is a nonzero constant, the relationship is an inverse variation.
- If the line  $x = a$  is a vertical asymptote of a rational function, then  $a$  is an excluded value.
- A rational expression is a fraction in which the numerator and denominator are fractions.
- The excluded values for  $\frac{x}{x^2 + 5x + 6}$  are  $-2$  and  $-3$ .
- The equation  $\frac{3x}{x-2} = \frac{6}{x-2}$  has an extraneous solution, 2.
- A rational expression has one or more fractions in the numerator and denominator.
- The expression  $\frac{1}{\frac{2}{\frac{3}{4}}}$  can be simplified to  $\frac{2}{3}$ .
- A direct variation can be represented by an equation of the form  $k = xy$ , where  $k$  is a nonzero constant.
- The rational function  $y = \frac{2}{x-1} + 3$  has a horizontal asymptote at  $y = 3$ .



**Lesson-by-Lesson Review** 

**11-1 Inverse Variation**

Solve. Assume that  $y$  varies inversely as  $x$ .

11. If  $y = 4$  when  $x = 1$ , find  $x$  when  $y = 12$
12. If  $y = -1$  when  $x = -3$ , find  $y$  when  $x = -9$
13. If  $y = 1.5$  when  $x = 6$ , find  $y$  when  $x = -16$
14. **PHYSICS** A 135-pound person sits 5 feet from the center of a seesaw. How far from the center should a 108-pound person sit to balance the seesaw?

**Example 1**

If  $y$  varies inversely as  $x$  and  $y = 28$  when  $x = 42$ , find  $y$  when  $x = 56$ .

Let  $x_1 = 42$ ,  $x_2 = 56$ , and  $y_1 = 28$ . Solve for  $y_2$ .

$$\frac{x_1}{x_2} = \frac{y_2}{y_1} \quad \text{Proportion for inverse variation}$$

$$\frac{42}{56} = \frac{y_2}{28} \quad \text{Substitution}$$

$$1176 = 56y_2 \quad \text{Cross multiply.}$$

$$21 = y_2$$

Thus,  $y = 21$  when  $x = 56$ .

**11-2 Rational Functions**

State the excluded value for each function.

15.  $y = \frac{1}{x-3}$
16.  $y = \frac{2}{2x-5}$
17.  $y = \frac{3}{3x-6}$
18.  $y = \frac{-1}{2x+8}$
19. **PIZZA PARTY** Katelyn ordered pizza and soda for her study group for \$38. The cost per person  $y$  is given by  $y = \frac{38}{x}$ , where  $x$  is the number of people in the study group. Graph the function and describe the asymptotes.

**Example 2**

State the excluded value for the function  $y = \frac{1}{4x+16}$ .

Set the denominator equal to zero.

$$4x + 16 = 0$$

$$4x + 16 - 16 = 0 - 16 \quad \text{Subtract 16 from each side.}$$

$$4x = -16 \quad \text{Simplify.}$$

$$x = -4 \quad \text{Divide each side by 4.}$$

**11-3 Simplifying Rational Expressions**

Simplify each expression.

$$20. \frac{2xy^2}{16xyz} \qquad 21. \frac{x+4}{x^2+12x+32}$$

$$22. \frac{x^2+10x+21}{x^3+x^2-42x} \qquad 23. \frac{y^2-25}{y^2+3y-10}$$

$$24. \frac{3x^3}{3x^3+6x^2} \qquad 25. \frac{4y^2}{8y^4+16y^3}$$

State the excluded values for each function.

$$26. y = \frac{x}{x^2+9x+18} \qquad 27. y = \frac{10}{6x^2+7x-3}$$

**Example 3**

Simplify  $\frac{a^2-7a+12}{a^2-13a+36}$ .

Factor and simplify.

$$\frac{a^2-7a+12}{a^2-13a+36} = \frac{(a-3)(a-4)}{(a-9)(a-4)} \quad \text{Factor.}$$

$$= \frac{a-3}{a-9} \quad \text{Simplify.}$$

## 11-4 Multiplying and Dividing Rational Expressions

Find each product or quotient.

28.  $\frac{6x^2y^4}{12} \cdot \frac{3x^3y^2}{xy}$

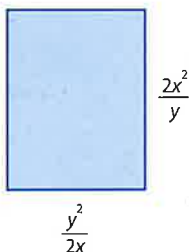
29.  $\frac{3x-6}{x^2-9} \cdot \frac{x+3}{x^2-2x}$

30.  $\frac{x^2}{x+4} \div \frac{3x}{x^2-16}$

31.  $\frac{3b-12}{b+4} \div (b^2-6b+8)$

32.  $\frac{2a^2+7a-15}{a+5} \div \frac{9a^2-4}{3a+2}$

33. **GEOMETRY** Find the area of the rectangle shown. Write the answer in simplest form.



### Example 4

Find  $\frac{7b^2}{9} \cdot \frac{6a^2}{b}$ .

$$\frac{7b^2}{9} \cdot \frac{6a^2}{b} = \frac{42a^2b^2}{9b} \quad \text{Multiply.}$$

$$= \frac{14a^2b}{3} \quad \text{Simplify.}$$

### Example 5

Find  $\frac{x^2-25}{x^2-9} \div \frac{x+5}{x-3}$ .

$$\frac{x^2-25}{x^2-9} \div \frac{x+5}{x-3} = \frac{(x+5)(x-5)}{(x+3)(x-3)} \div \frac{x+5}{x-3} \quad \text{Factor.}$$

$$= \frac{\cancel{(x+5)}(x-5)}{(x+3)\cancel{(x-3)}} \cdot \frac{\cancel{x-3}}{\cancel{x+5}} \quad \text{Multiply by the reciprocal.}$$

$$= \frac{x-5}{x+3} \quad \text{Simplify.}$$

## 11-5 Dividing Polynomials

Find each quotient.

34.  $(x^3 - 2x^2 - 22x + 21) \div (x - 3)$

35.  $(x^3 + 7x^2 + 10x - 6) \div (x + 3)$

36.  $(5x^2y^2 - 10x^2y + 5xy) \div 5xy$

37.  $(48y^2 + 8y + 7) \div (12y - 1)$

38. **GEOMETRY** The area of a rectangle is  $x^2 + 7x + 13$ . If the length is  $(x + 4)$ , what is the width of the rectangle?

### Example 6

Find  $(4x^2 + 17x - 1) \div (4x + 1)$ .

$$\begin{array}{r} x + 4 \\ 4x + 1 \overline{) 4x^2 + 17x - 1} \\ \underline{4x^2 + x} \phantom{- 1} \\ 16x - 1 \\ \underline{16x + 4} \\ -5 \end{array}$$

Multiply  $x$  and  $4x + 1$ .  
Subtract, bring down  $-1$ .  
Multiply  $4$  and  $4x + 1$ .  
Subtract.

The quotient is  $x + 4 - \frac{5}{4x + 1}$ .

## 11-6 Adding and Subtracting Rational Expressions

Find each sum or difference.

39.  $\frac{5a}{b} - \frac{2a}{b}$

40.  $\frac{-3}{2n-3} + \frac{2n}{2n-3}$

41.  $\frac{3}{y+1} - \frac{y}{y-3}$

42.  $\frac{1}{x+1} + \frac{3}{x-2}$

43. **DESIGN** Miguel is decorating a model of a room that is  $\frac{2x}{x+4}$  feet long and  $\frac{8}{x+4}$  feet wide. What is the perimeter of the room?

### Example 7

Find  $\frac{x^2}{x+1} + \frac{2x+1}{x+1}$ .

$$\frac{x^2}{x+1} + \frac{2x+1}{x+1} = \frac{x^2+2x+1}{x+1} \quad \text{Add the numerators.}$$

$$= \frac{(x+1)(x+1)}{x+1} \quad \text{Factor.}$$

$$= x+1 \quad \text{Simplify.}$$

# Study Guide and Review *Continued*

## 11-7 Mixed Expressions and Complex Fractions

Simplify each expression.

44.  $\frac{\frac{a^2b^4}{c}}{\frac{a^3b}{c^2}}$

45.  $\frac{x - \frac{35}{x+2}}{x + \frac{42}{x+13}}$

46.  $\frac{\frac{x^2 - 25}{x+2}}{\frac{x-5}{x^2 - 4}}$

47.  $\frac{y + 9 - \frac{6}{y+4}}{y + 4 + \frac{2}{y+1}}$

48. **FABRICS** Donna makes sets of three different sized tablecloths to sell. A small one takes one-half yard of fabric, a medium one takes five-eighths yard, and a large one takes one and one-quarter yards.

- How many yards of fabric does she need to make a tablecloth of each size?
- If Donna can make a tablecloth of each size in 38 minutes, what is the average rate of fabric she uses in yards per hour?

### Example 8

Simplify  $\frac{\frac{x+3}{6}}{\frac{x^2 - 2x - 15}{x}}$ .

Write as a division expression.

$$\begin{aligned} \frac{\frac{x+3}{6}}{\frac{x^2 - 2x - 15}{x}} &= \frac{x+3}{6} \div \frac{x^2 - 2x - 15}{x} \\ &= \frac{x+3}{6} \cdot \frac{x}{x^2 - 2x - 15} \\ &= \frac{x+3}{6} \cdot \frac{x}{(x+3)(x-5)} \\ &= \frac{x}{6(x-5)} \end{aligned}$$

## 11-8 Rational Equations

Solve each equation. State any extraneous solutions.

49.  $\frac{5n}{6} + \frac{1}{n-2} = \frac{n+1}{3(n-2)}$

50.  $\frac{4x}{3} + \frac{7}{2} = \frac{7x}{12} - 14$

51.  $\frac{11}{2x} + \frac{2}{4x} = \frac{1}{4}$

52.  $\frac{1}{x+4} - \frac{1}{x-1} = \frac{2}{x^2 + 3x - 4}$

53.  $\frac{1}{n-2} = \frac{n}{8}$

54. **PAINTING** Anne can paint a room in 6 hours. Oljay can paint a room in 4 hours. How long will it take them to paint the room working together?

### Example 9

Solve  $\frac{3}{x^2 + 3x} + \frac{x+2}{x+3} = \frac{1}{x}$ .

$$\begin{aligned} \frac{3}{x^2 + 3x} + \frac{x+2}{x+3} &= \frac{1}{x} \\ x(x+3)\left(\frac{3}{x(x+3)}\right) + x(x+3)\left(\frac{x+2}{x+3}\right) &= x(x+3)\left(\frac{1}{x}\right) \\ 3 + x(x+2) &= 1(x+3) \\ 3 + x^2 + 2x &= x + 3 \\ x^2 + x &= 0 \\ x(x+1) &= 0 \\ x = 0 \text{ or } x = -1 \end{aligned}$$

The solution is  $-1$ , and there is an extraneous solution of  $0$ .

# Practice Test

Determine whether each table represents an inverse variation. Explain.

1. 

| x | y  |
|---|----|
| 2 | 10 |
| 4 | 12 |
| 8 | 14 |

2. 

| x | y             |
|---|---------------|
| 2 | 2             |
| 4 | 1             |
| 8 | $\frac{1}{2}$ |

Find each product or quotient.

3.  $\frac{(x+6)(x-2)}{x^3} \cdot \frac{7x^2}{x-3}$

4.  $\frac{(x+3)}{y^2} \div \frac{x^2-9}{y}$

Solve. Assume that  $y$  varies inversely as  $x$ .

5. If  $y = 3$  when  $x = 9$ , find  $x$  when  $y = 1$ .

6. If  $y = 2$  when  $x = 0.5$ , find  $y$  when  $x = 3$ .

Simplify each expression. State the excluded values of the variables.

7.  $\frac{z-6}{z^2-3z-18}$

8.  $\frac{4x-28}{x^2-49}$

9. **MULTIPLE CHOICE** The area of a rectangle is  $x^2 + 5x + 6$  square feet. If the width is  $x + 2$ , what is the length of the rectangle?



- A  $x + 2$
- B  $x + 3$
- C 1
- D 3

Simplify each expression.

10.  $\frac{2\frac{1}{3}}{3\frac{1}{2}}$

11.  $\frac{\frac{x^2-25}{x-2}}{\frac{x-5}{x-2}}$

12.  $\frac{\frac{a-4}{a^2+6a+8}}{\frac{a^2-3a-4}{a^2-a-6}}$

13.  $\frac{\frac{y^2+10y+24}{y^2-9}}{\frac{3y^2+17y-6}{2y^2-11y+15}}$

Find each quotient.

14.  $(2x^2 + 10x) \div 2x$

15.  $(4x^2 - 8x + 5) \div (2x + 1)$

16.  $(3x^2 - 14x - 3) \div (x - 5)$

Assume that  $y$  varies inversely as  $x$ . Write an inverse variation equation that relates  $x$  and  $y$ .

17.  $y = 2$  when  $x = 8$

18.  $y = -3$  when  $x = 1$

Find each sum or difference.

19.  $\frac{15}{x} - \frac{6}{x}$

20.  $\frac{t-5}{t-6} + \frac{t+8}{t-6}$

21.  $\frac{1}{x-6} + \frac{3}{x-2}$

22.  $\frac{2x^2+8x+5}{x^2-2x-24} - \frac{x}{x-6}$

State the excluded value or values for each function.

23.  $y = \frac{6}{x-1}$

24.  $y = \frac{5}{x^2-5x-24}$

Identify the asymptotes of each function.

25.  $y = \frac{2}{(x-4)(x+2)}$

26.  $y = \frac{4}{x^2+3x-28} + 2$

27. **MULTIPLE CHOICE** Lee can shovel the driveway in 3 hours, and Susan can shovel the driveway in 2 hours. How long will it take them working together?

- F 6 hours
- G 5 hours
- H  $\frac{3}{2}$  hours
- J  $\frac{6}{5}$  hours

28. **PAINTING** Sydney can paint a 60-square foot wall in 40 minutes. Working with her friend Cleveland, the two of them can paint the wall in 25 minutes. How long would it take Cleveland to do the job himself?



## Model with an Equation

In order to successfully solve some standardized test questions, you will need to be able to write equations to model different situations. Use this lesson to practice solving these types of problems.

### Strategies for Modeling with Equations

#### Step 1

Read the problem statement carefully.

Ask yourself:

- What am I being asked to solve?
- What information is given in the problem?
- What is the unknown quantity that I need to find?

#### Step 2

Translate the problem statement into an equation.

- Assign a variable to the unknown quantity.
- Write the word sentence as a mathematical number sentence.
- Look for keywords such as *is*, *is the same as*, *is equal to*, or *is identical to* that indicate where to place the equals sign.

#### Step 3

Solve the equation.

- Solve for the unknown in the equation.
- Check your answer to be sure it is reasonable and that it answers the question in the problem statement.



### Standardized Test Example

**Read the problem. Identify what you need to know. Then use the information in the problem to solve.**

It takes Craig 75 minutes to paint a small room. If Delsin can paint the same room in 60 minutes, how long would it take them to paint the room if they work together? Round to the nearest tenth.

- |                      |                      |
|----------------------|----------------------|
| A about 33.3 minutes | C about 45.1 minutes |
| B about 38.4 minutes | D about 50.3 minutes |

Read the problem carefully. You know how long it takes Craig and Delsin to paint a room individually. Model the situation with an equation to find how long it would take them to paint the room if they work together.

Find the rate that each person works when painting individually.

$$\text{Craig's rate: } \frac{1 \text{ job}}{75 \text{ minutes}} = \frac{1}{75} \text{ job per minute}$$

$$\text{Delsin's rate: } \frac{1 \text{ job}}{60 \text{ minutes}} = \frac{1}{60} \text{ job per minute}$$

Let  $t$  represent the number of minutes it would take them to complete the job working together. Multiply each rate by the time  $t$  to represent the portion of the job done by each painter. Add these expressions and set them equal to 1 job. Then solve for  $t$ .

|                                 |      |                                  |        |        |
|---------------------------------|------|----------------------------------|--------|--------|
| Portion that<br>Craig completes | plus | portion that<br>Delsin completes | equals | 1 job. |
| $\frac{1}{75}t$                 | +    | $\frac{1}{60}t$                  | =      | 1      |

Solve for  $t$ :

$$\frac{1}{75}t + \frac{1}{60}t = 1 \quad \text{Original equation}$$

$$300\left(\frac{1}{75}t + \frac{1}{60}t\right) = 300(1) \quad \text{Multiply each side by the LCD, 300.}$$

$$4t + 5t = 300 \quad \text{Simplify.}$$

$$9t = 300 \quad \text{Combine like terms.}$$

$$t \approx 33.3 \quad \text{Divide each side by 9.}$$

So, it would take Craig and Delsin about 33.3 minutes to paint the room working together. The correct answer is A.

## Exercises

Read each problem. Identify what you need to know. Then use the information in the problem to solve.

1. Hana can finish a puzzle in 6 hours, while Eric can finish one in 5 hours. How long would it take them to finish a puzzle together? Round to the nearest tenth.
  - A about 1.8 hours
  - B about 2.4 hours
  - C about 2.5 hours
  - D about 2.7 hours
2. Roberto wants to print 500 flyers for his landscaping business. His printer can complete the job in 35 minutes, and his brother's printer can print them in 45 minutes. How long would it take to print the flyers using both printers? Round to the nearest whole minute.
  - F about 15 minutes
  - G about 18 minutes
  - H about 20 minutes
  - J about 23 minutes

## Standardized Test Practice

Cumulative, Chapters 1 through 11

## Multiple Choice

Read each problem. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. What is the inverse variation equation for the numbers shown in the table?

| $x$ | $y$ |
|-----|-----|
| -8  | 16  |
| -4  | 32  |
| 2   | -64 |
| 8   | -16 |
| 16  | -8  |

- A  $y = -2x$   
 B  $y = 8x$   
 C  $xy = 24$   
 D  $xy = -128$
2. Suppose a square has a side length given by the expression  $\frac{x+5}{8x}$ . What is the perimeter of the square?
- F  $\frac{4x+20}{5x}$   
 G  $\frac{2x+10}{x}$   
 H  $\frac{x+5}{4x}$   
 J  $\frac{x+5}{2x}$
3. Find the length of the hypotenuse if the legs of a right triangle are 2 centimeters and 10 centimeters.

- A 8.1 cm  
 B 8.5 cm  
 C 9.6 cm  
 D 10.2 cm

## Test-Taking Tip

**Question 1** Sometimes you can eliminate answer choices as unreasonable because they do not fit the requirements of the problem. Choices A and B show direct variation equations, so they can be eliminated.

4. In 1990, the population of a country was about 3.66 million people. By 2010, this number had grown to about 4.04 million people. What was the annual rate of change in population from 1990 to 2010?

- F about 15,000 people per year  
 G about 19,000 people per year  
 H about 24,000 people per year  
 J about 38,000 people per year

5. Ricky's Rentals rented 12 more bicycles than scooters last weekend for a total revenue of \$2,125. How many scooters were rented?

| Item    | Rental Fee |
|---------|------------|
| Bicycle | \$20       |
| Scooter | \$45       |

- A 26  
 B 29  
 C 37  
 D 41

6. The table shows the relationship between calories and fat grams contained in orders of french fries from various restaurants.

| Calories | Fat Grams |
|----------|-----------|
| 240      | 14        |
| 280      | 15        |
| 310      | 16        |
| 260      | 12        |
| 340      | 16        |
| 350      | 18        |
| 300      | 13        |

Assuming the data can best be described by a linear model, how many fat grams would be expected to be contained in a 315-calorie order of french fries?

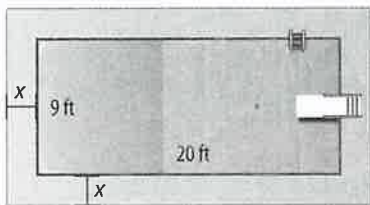
- F 12 fat grams  
 G 13 fat grams  
 H 16 fat grams  
 J 8051 fat grams

## Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

7. Suppose the first term of a geometric sequence is 3 and the fourth term is 192.
- What is the common ratio of the sequence?
  - Write an equation that can be used to find the  $n$ th term of the sequence.
  - What is the sixth term of the sequence?

8. **GRIDDED RESPONSE** Peggy is having a cement walkway installed around the perimeter of her swimming pool with the dimensions shown below. If  $x = 3$  find the area, in square feet, of the pool and walkway.



9. Use the equation  $y = 2(4 + x)$  to answer each question.
- Complete the following table for the different values of  $x$ .
  - Plot the points from the table on a coordinate grid. What do you notice about the points?

| $x$ | $y$ |
|-----|-----|
| 1   |     |
| 2   |     |
| 3   |     |
| 4   |     |
| 5   |     |
| 6   |     |

10. Jason received a \$50 gift certificate for his birthday. He wants to buy a DVD and a poster from a media store. (Assume that sales tax is included in the prices.) Write and solve a linear inequality to show how much he would have left to spend after making these purchases.

**Weekend Blowout Sale**

- \* All DVDs only \$14.95
- \* All CDs only \$11.25
- \* All posters only \$10.99

11. Simplify the complex fraction. Show your work.

$$\frac{\frac{5}{x-3}}{\frac{x-6}{x^2-x-6}}$$

## Extended Response

Record your answers on a sheet of paper. Show your work.

12. Carl's father is building a tool chest that is shaped like a rectangular prism. He wants the tool chest to have a surface area of 62 square feet. The height of the chest will be 1 foot shorter than the width. The length will be 3 feet longer than the height.
- Sketch a model to represent the problem.
  - Write a polynomial that represents the surface area of the tool chest.
  - What are the dimensions of the tool chest?

### Need ExtraHelp?

|                           |      |      |      |     |     |     |     |     |     |     |      |     |
|---------------------------|------|------|------|-----|-----|-----|-----|-----|-----|-----|------|-----|
| If you missed Question... | 1    | 2    | 3    | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11   | 12  |
| Go to Lesson...           | 11-1 | 11-6 | 10-5 | 3-3 | 6-2 | 4-5 | 7-7 | 8-3 | 1-4 | 5-1 | 11-7 | 8-9 |

